Math 25 Midterm Soln's

#1 (a) x is an upper bound for $A \subseteq \mathbb{R}$ if

$x \geq a \quad \forall\, a \in A$

(b) x is a LUB for A if $x \not\geq y$ $\forall$ upperbound y of A

(c) Every set bded from above has a LUB

(d) $S_{n+1} \geq S_n \ \forall\, n$. Let $S_0 = \text{LUB}\{S_n\}$. We prove that $S_n \to S_0$. Fix $\varepsilon > 0$. We find N st $n \geq N \Rightarrow |S_n - S_0| < \varepsilon$. But S_0 the $\text{LUB}\{S_n\}$ implies $S_0 \geq S_n \ \forall n$, and $\exists\, N$ st $S_N > S_0 - \varepsilon$. Thus for $n > N$ we have $S_0 - \varepsilon < S_N \leq S_n \leq S_0$

so $|S_n - S_0| < \varepsilon$ ✓

#2 (a) $S_n \to S_0 \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st

$\forall n > N\ \ |S_n - S_0| < \varepsilon$

$S_n \to +\infty$ if $\forall M > 0\ \exists N \in \mathbb{N}$ st

$\forall n > N\ \ S_n > M.$

(b) S_n is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$

st $\forall m, n > N,\ |S_n - S_m| < \varepsilon$

(c) $\neg(S_n \text{ is Cauchy}) \equiv$

$\equiv \neg(\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ st } \forall m, n > N\ |S_n - S_m| < \varepsilon)$

$\equiv \exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists m, n > N\ |S_n - S_m| \geq \varepsilon$

(d) Assume $S_n \longrightarrow S_0 \in \mathbb{R}$. We show S_n is Cauchy. Fix $\varepsilon > 0$. We find N st $m,n > N \Rightarrow |S_n - S_m| < \varepsilon$. Choose N st $n > N \Rightarrow |S_n - S_0| < \frac{\varepsilon}{2}$. Then $m,n > N \Rightarrow$

$$|S_n - S_m| \leq |S_n - S_0 + S_0 - S_m| \leq |S_n - S_0| + |S_m - S_0|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon \checkmark$$

(e) Assume $S_n \longrightarrow +\infty$. Choose $\varepsilon = 1$. We prove that $\forall N \in \mathbb{N}\ \exists\, m,n > N$ st $|S_n - S_m| \geq \varepsilon$. Fix N. Choose any $n > N$. Then since $S_n \longrightarrow +\infty$, setting $M = S_n + 1$ we know $\exists\, m > N$ st $S_m > M = S_n + 1$. Thus $|S_n - S_m| > 1$, proving S_n not Cauchy.

#3 (a) $\overline{S}_N = \text{Sup}\{S_n : n > N\}$

$$\underline{S}_N = \text{Inf}\{S_n : n > N\}$$

(b) $\overline{S} = \lim_{N \to \infty} \overline{S}_N$

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(c) $\underline{S}_N \le \underline{S} \le \overline{S} \le \overline{S}_N$

Prove: $\overline{S} \le \overline{S}_N$

$$\overline{S}_N = \text{Sup}\{S_n : n > N\} \ge \text{Sup}\{S_n : n > M\} = \overline{S}_M$$

if $M \ge N$. $\therefore$ $\overline{S}_N$ is non-increasing sequ

$\Rightarrow$ $\overline{S}_N \ge \overline{S}$ its limit.

#4 $S_n = (1+(-1)^n)e^{1/n}$

(a) $\liminf s_n = 0$, $\limsup s_n = 2$

because $e^{1/n} \longrightarrow 1$

(b) $s_{n_k} \longrightarrow 2$ for $n_k = 2k$

#5 (a) Example: Define s_n as follows –

$$\underbrace{1}_{s_1}, \underbrace{1, \tfrac{1}{2}}_{}, \underbrace{1, \tfrac{1}{2}, \tfrac{1}{3}}_{}, \underbrace{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}}_{}, \underbrace{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}}_{}, \ldots$$

$s_1\ s_2\ s_3\ s_4 \ldots$

(other possible examples exist)

(b) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ cannot be S' for any s_n because it is not closed – i.e., $\frac{1}{n} \longrightarrow 0$ but $0 \notin A$.