FINAL EXAM
Math 25
Temple-F06

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (Short Answer: 32pts) Let s_n denote a sequence of real numbers. Give the precise mathematical definitions for the following:

(a) $\lim_{n\to\infty} s_n = -2$.

(b) $\lim_{n\to\infty} s_n = -\infty$

(c) Define what it means for s_n to be Cauchy.

(d) Give the negation of the statement "s_n is Cauchy".

(e) Define $\underline{s}_N$ and $\bar{s}_N$, the approximate $\liminf$ and approximate $\limsup$ of s_n, and use these to define $\underline{s} = \liminf s_n$ and $\bar{s} = \limsup s_n$, respectively.

(f) Use $\leq$ to give the correct inequalities that order the set $\{\underline{s}_N, \bar{s}_N, \underline{s}, \bar{s}\}$.

(g) Define what it means for a sequence to converge in a *metric space* (S, d).

(h) Define the subsequential limit set S of s_n, and identify $Inf\{S\}$ and $Sup\{S\}$.

Problem 2. (20pts) Use the fact that every natural number can be written uniquely as a product of prime factors to prove that $\sqrt{3}$ is not a positive rational number.

Problem 3. (20pts) Use the field axioms for the real numbers to prove that if $a \in \mathbf{R}$, then $a \cdot 0 = 0$. (Give a field axiom reason for every step. Prove any lemma you use.)

Problem 4. (24pts) Assume s_n and s_0 are nonzero. Use the definition of convergence to give a direct proof that if $s_n \to s_0$, then $1/s_n \to 1/s_0$.

Problem 5. (20pts) Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of partial sums for infinite series $\sum_{k=1}^{\infty} a_k$, and let $t_n = \sum_{k=1}^{n} |a_k|$.

(a) Define what it means for the infinite series s_n to converge.

(b) State the Cauchy criterion for convergence of the series s_n.

(c) Prove that if a series converges absolutely, then the series converges.

Problem 6. (20pts) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

Problem 7. (20pts) Let r be a real number such that $|r| < 1$, and let s_n denote the sequence of partial sums

$$s_n = \sum_{k=m}^{n} r^k = r^m + r^{m+1} + r^{m+2} + r^{m+3} + \cdots + r^n.$$

(a) Derive a formula for s_n that does not involve a summation, and use it to evaluate $\lim_{n\to\infty} s_n$.

(b) Prove that the repeating decimal .123123123... is a rational number.

Problem 8. (24pts) Assume that $x_n \to 0$ and $y_n \to 0$ are convergent sequences of real numbers. Prove directly that $\sqrt{x_n^2 + y_n^2} \to 0$ converges.

Problem 9. (20pts) Consider the sequence $s_n = \{(-1)^n n + 1 + n\} \sin n$ of real numbers. Prove that s_n has a convergent subsequence. (You may use any theorem in the book.)

Problem 10. (20pts) (Extra Credit) Let $a_n \geq 0$ be a sequence of positive real numbers, $n = 1, 2, 3...$, and let $p_n = \sum_{k=n}^{2n-1} a_k$. Assume that $p_n \to 0$. Does it follow that $\sum_{k=1}^{\infty} a_k$ converges? That is, does it follow that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ converges to a real number? Prove your assertion, or else give a counterexample.