Homework Problems
MAT 280, F14, Diff Geom & GR
Temple

(1) Derive the transformation laws for $\frac{\partial}{\partial x^i}$.

(2) Derive the transformation laws for dx^i.

(3) Prove that $\{dx^0, ..., dx^3\}$ is a basis for the $T^*\mathcal{M}_p$.

(4) Verify the general transformation law for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensor- and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$-tensor-tensors, and write the laws in summation and matrix notation.

(5) Prove that $dx \otimes \frac{\partial}{\partial x^j}|_p$ spans the space of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensor-tensors at p.

(6) Prove that if g_{ij} is defined in each coordinate system so that (at a given $p \in \mathcal{M}$) we have

$$g_{ij}a^i b^j =< X, Y >= \bar{g}_{\alpha\beta}a^\alpha b^\alpha,$$

for all vectors $X = a_i \frac{\partial}{\partial x^i} = \bar{a}_\alpha \frac{\partial}{\partial y^\alpha}, Y = b_j \frac{\partial}{\partial x^j} = \bar{b}_\beta \frac{\partial}{\partial y^\beta} \in T_p\mathcal{M}$, then g_{ij} must transform by

$$g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}.$$

Write the matrix version of this transformation law.

(7) Prove that the dimension of the space of multi-linear functionals on the cross product $V_1 \times V_2$ of two vector spaces, equals the product of the dimensions of the vector spaces V_k.

(8) Prove by counterexample that symmetry is not a covariant (i.e., coordinate independent) property of $(1,1)$-tensors, but is a covariant property of $(0,2)$-tensors.

(9) Let $X_1, ..., X_n$ be a basis of tangent vectors in $T_p\mathcal{M}$. Prove that there exists a coordinate system $\mathbf{x} : \mathcal{U}_\mathbf{x} \to \mathcal{R}$, $p \in \mathcal{U}_\mathbf{x}$ such that

$$X_i = \frac{\partial}{\partial x^i}|_p.$$

(10) Prove that if g is a Riemannian metric and $X \in T_p\mathcal{M}$, then $< X, X >= 0$ if and only if $X = 0$.

(11) Prove that $\Gamma_{ikj} + \Gamma_{jki} = g_{ij,k}$.

(12) Prove that in special relativity, ($g_{ij} \equiv \eta_{ij} = Diag\{-1, 1, 1, 1\}$), the map that takes the components of a vector (with respect to the coordinate basis $\frac{\partial}{\partial x^i}$) in $T_0\mathcal{M}$ to the corresponding point in spacetime named by the Minkowski coordinate system x^i, is exactly the exponential map as we defined it.

(13) For a given vector $X \in T_p\mathcal{M}$, define the orthogonal projection of vector Y onto X by

$$Proj_X Y = \frac{< X, Y >}{< X, X >} X.$$

In the case of 1+1 special relativity $g_{ij} = \eta_{ij}$, show that this is the correct definition by decomposing Y into $Y = aX + bX^{\perp}$ and seeing that $Proj_X Y = aX$ "projects Y onto X along the vector g-orthogonal to Y". (Note that Y can be anything, but explain geometrically why $Proj_X Y$ is undefined when X is lightlike.).

(14) Show that for Lorentz transformation, $L(\theta)L(\bar{\theta}) = L(\theta+\bar{\theta})$, and use this to prove the relativistic velocity addition law

$$\bar{\bar{v}} = \frac{v + \bar{v}}{1 + \frac{v\bar{v}}{c^2}},$$

where v is the velocity of the barred frame as measured in the unbarred frame, and $\bar{v}$ is the velocity of the double barred frame as measured in the barred frame, and $\bar{\bar{v}}$ is the velocity of the double barred frame as measured in the unbarred frame.

(15) Prove that the Proper Lorentz Transformations A are characterized by $A_0^0 \geq 1$ and $Det(A) > 0$. Give a careful proof that if $A(\xi)$ is a family of Lorentz transformations that depend continuously on the parameter $\xi \in \mathbf{R}$ and $A(0)$ is proper, then $A(\xi)$ is proper for every ξ.

(16) Prove that the set of coordinate transformations of the form $x^i = A^i_\alpha y^\alpha + a^i$, where A^i_α, a^i are constants and A^i_α satisfy $\eta_{ij} = A^i_\alpha A^j_\beta \eta_{\alpha\beta}$, forms a group under composition, (the Poincare or *Inhomogeneous Lorentz group*).

(17) Show that in coordinates, $[X, Y] = XY - YX = \left(X^\sigma Y^j_{,\sigma} - Y^\sigma X^j_{,\sigma}\right) \frac{\partial}{\partial x^j}$ where $\left(X^\sigma Y^j_{,\sigma} - Y^\sigma X^j_{,\sigma}\right)$ transforms contra-variantly.

(18) If $c(\xi)$ is a curve in manifold $\mathcal{M}^n$ with tangent vector X, and x and y are coordinates systems which overlap, use the chain rule and our conventions to deconstruct the meaning of:

$$\frac{d}{d\xi}\left(\frac{\partial x^i}{\partial y^\alpha}\right) = \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} X^\beta.$$

(19) In the summation convention, when the contraction of two indices expresses the multiplication of a matrix by it's inverse, like

$$\frac{\partial x^i}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^j},$$

we set it equal to $= \delta^i_j$, and then set $i = j$. Assuming $i, j, k, ...$ refer to x-coordinates and $\alpha, \beta, \gamma, ...$ to y-coordinates, use the transformation law for $g^{\gamma\sigma}$ and the summation conventions to directly verify the following identity: (Put in every step, and do not introduce matrix multiplication.)

$$g^{\gamma\sigma} g_{ij} \frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta}\frac{\partial x^i}{\partial y^\gamma} = \frac{\partial y^\gamma}{\partial x^j}\frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta}.$$

(20) Write the following canonical boost to velocity $\mathbf{v} \in \mathbf{R}^3$, $\|\mathbf{v}\| = v$, as a 4×4 matrix, and prove that it is indeed a Lorentz transformation:

$$\begin{pmatrix} \gamma & -\frac{\mathbf{v^t}}{c}\gamma \\ & \\ -\frac{\mathbf{v}}{c}\gamma & I + \frac{\gamma-1}{c^2}\frac{\mathbf{v}\cdot\mathbf{v^t}}{v^2} \end{pmatrix}. \tag{1}$$

(Hint: Lorentz transformations map ON bases to ON bases.)

(21) In the proof that $B = A\bar{A}^{-1}$ is a pure rotation if $A, \bar{A}$ are boosts with the same velocity, we used $\bar{A}^{-1} = \eta\bar{A}^t\eta$ to conclude that $B_0^0 = A_0^\sigma \eta_{\sigma\sigma}\bar{A}_0^\sigma\eta^{00}$, (sum on σ). Verify this using the definition of matrix multiplication, and $\eta^{-1} = \eta = diag\{-1, 1, 1, 1\}$.

(22) Use the summation convention (instead of matrix multiplication) to verify that:

$$g^{\gamma\sigma}\eta_{ij}\frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta}\frac{\partial x^i}{\partial y^\sigma} = \frac{\partial y^\gamma}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = \bar{\Gamma}^\gamma_{\alpha\beta}$$

so long as y is the transformation of an Locally Inertial Frame x, so

$$g^{\gamma\sigma} = \eta^{ij}\frac{\partial y^\gamma}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}$$

(23) In the case of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensors, express the statements that (1) Covariant differentiation ∇_X commutes with contraction (2) Covariant differentiation ∇_X commutes with raising and lowering of indices. The prove both statements.

(24) In the case of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensor-tensors, prove the Liebniz rule for tensor products:

$$\nabla_X (Y \oplus \omega) = \nabla_X Y \oplus \omega + Y \oplus \nabla_X \omega,$$

where

$$Y = Y^i \frac{\partial}{\partial x^i}, \text{ and } \quad \omega = \omega_j \, dx^j.$$

(25) Use that $\omega_{\alpha;i} = \omega_{\alpha,i} - \Gamma^\sigma_{\alpha i}\omega_\sigma$ to prove that in coordinates

$$(\nabla_j \nabla_i \omega)_\alpha - (\nabla_i \nabla_j \omega)_\alpha = R^\sigma_{\alpha ij}\omega_\sigma$$

(26) Prove that if $R^\alpha_{\sigma ij}Z^\sigma$ transforms like a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$-tensor for every vector Z^σ, then $R^\alpha_{\sigma ij}$ transforms as a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$-tensor.

(27) The Riemann curvature tensor for a metric is a "curl" plus a "commutator",

$$R^\alpha_{\beta ij} = \Gamma^\alpha_{\beta j,i} - \Gamma^\alpha_{\beta i,j} + \Gamma^\alpha_{\tau,i}\Gamma^\tau_{\beta,j} - \Gamma^\alpha_{\tau,j}\Gamma^\tau_{\beta,j}$$

Argue that in Riemann normal coordinates, the commutator should in general be a lower order term relative to the curl.

(28) Prove that anti-symmetry and symmetry under pair exchange is a coordinate independent property of tensors.

(29) Complete the argument that the Riemann curvature tensor for a metric has (no more than) twenty independent entries.

(30) Prove that for the Einstein equations $G = \kappa T$, if $T = 0$, then $G_{ij} = 0$ if and only if $R_{ij} = 0$.