SECTION-10
Entropy
(According to Lax)

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Math-280: A Mathematical Introduction to Shock Waves

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Entropy:

Compressible Euler Eqn's

$$\rho_t + \operatorname{div} \rho u = 0 \qquad (MA)$$

$$(\rho u^i)_t + \operatorname{div}(\rho u^i u + p e^i) = 0 \qquad (Mo)$$

$$E_t + \operatorname{div}(E+p)u = 0 \qquad (En)$$

$$\mathcal{S}_t + \operatorname{div} \mathcal{S} u = 0 \qquad (\mathcal{S})$$

$\mathcal{S} = \rho s = \frac{\text{entropy}}{\text{vol}}$, $E \equiv \frac{1}{2}\rho u^2 + \rho e \equiv \frac{\text{energy}}{\text{vol}}$

s = specific entropy, e = specific energy, $v = \frac{1}{\rho} \equiv$ spec vol

- 2nd Law Thermo: $de = T ds - p dv$
- Polytropic (γ-law) gas: $e = c_v \rho^{\gamma-1} \exp s/c_v$

Solve for s: $\frac{e}{c_v \rho^{\gamma-1}} = \exp s/c_v$

$$s = c_v \ln\left(\frac{e}{c_v \rho^{\gamma-1}}\right)$$

$$\mathcal{S} = \rho s = \rho c_v \ln\left(\frac{e}{c_v \rho^{\gamma-1}}\right)$$

Now $e = \frac{E}{\rho} - \frac{1}{2}u^2 \Rightarrow$

$$\mathcal{S} = \mathcal{S}(U) = \rho c_v \ln\left\{\frac{E}{c_v \rho^{\gamma}} - \frac{u^2}{c_v \rho^{\gamma-1}}\right\}$$

$$U = (\rho, m^i, E), \quad m^i = \rho u^i$$

$$u^2 = \frac{m^2}{\rho^2}$$

$$\Rightarrow \mathcal{S} = \mathcal{S}(U) = \mathcal{S}(\rho, m, E) = c_v \rho \ln\left\{\frac{E}{c_v \rho^{\gamma}} - \frac{m^2}{c_v \rho^{\gamma+1}}\right\}$$

Defn: $\mathcal{S}$ is convex if its Hessian is pos definite: $H = \frac{\partial^2 \mathcal{S}}{\partial U_i \partial U_j}$

satisfies

$$v^t H v \geq 0 \quad \forall\, v \in \mathbb{R}^3$$

Thm: $-\mathcal{S}$ is convex (HW)

③

- "Entropy is constant" on smooth soln's

$$\frac{d}{dt}\int_{\Omega(t)} S\, d^3x = \int_{\Omega(t)} S_t + \operatorname{div} S u \, d^3x = 0$$

④

For soln's $u(\underset{\sim}{x},t)$ with shock-waves, we want entropy to increase, so ask

$$\frac{d}{dt}\int_{\Omega(t)} S(\mathbf{U})\, d^3\underset{\sim}{x} \geq 0$$

$$\underset{\substack{\uparrow \\ \text{smooth} \\ \text{solutions}}}{\Leftrightarrow} \int_{\Omega(t)} S_t + \operatorname{div}(S u)\, d^3\underset{\sim}{x} \geq 0 \quad \forall\, \Omega(t)$$

$$\Leftrightarrow \quad S_t + \operatorname{div}(S u) \geq 0. \qquad (*)$$

We look for <u>weak</u> formulation of (*):

"mult by test fn and int. by parts" $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$

Choose $\varphi \geq 0$. Then (*) $\Leftrightarrow$

$$S_t \varphi + \operatorname{div}(S u)\varphi \geq 0 \quad \forall\, \varphi \geq 0$$

$$\Leftrightarrow \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^+} S_t \varphi + \operatorname{div}(S u)\varphi \geq 0 \quad \forall\, \varphi \geq 0$$

(φ compact supp)

$$-\iint_{\mathbb{R}^3\times\mathbb{R}^+} S\varphi_t + Su\cdot\nabla\varphi$$
$$+\iiint (S\varphi)_t + \mathrm{div}(Su\varphi)\, d^3x \ \ge 0$$

0 by div. thm

$$\iiint S\varphi_t + Su\cdot\nabla\varphi\, d^3x \le 0$$

$$\Leftrightarrow \boxed{\iiint S\varphi_t + Su\cdot\nabla\varphi\, d\underset{\sim}{x}^3 \le 0} \qquad (*)$$

$$\forall\ \varphi \in C_0^\infty(\mathbb{R}^3\times\mathbb{R}^+),\ \varphi \ge 0.$$

Defn: Entropy **increases** on a weak solution $u(\underset{\sim}{x},t)$ if $(*)$ holds. We say that $(*)$ expresses that $S_t + \mathrm{div}(Sv) \ge 0$ in the ~~weak sense~~ weak sense, or in the sense of the theory of distributions.

• It turns out that $-S$ is a convex up fn of $(\rho, m, E) \equiv U$.

Defn: Let $u_t + \mathrm{div}\, f = 0$ be an arbitrary system of conservation laws $x \in \mathbb{R}^m$. We say the system possesses a convex entropy if $\exists$ a fn $U(u)$, convex up, such that, on smooth solutions $u(\underset{\sim}{x},t)$ of $u_t + \mathrm{div}\, f = 0$,

$$U(u)_t + \mathrm{div}\, F(u) = 0.$$

Defn: we say entropy increases on a weak soln of $u_t + \mathrm{div}\, f = 0$ if

$$(U) \qquad U_t + \mathrm{div}\, F \le 0 \qquad \iint_{\mathbb{R}^m\times\mathbb{R}^+} U\varphi_t + F\cdot\nabla\varphi \ge 0 \quad \forall\ \varphi \ge 0,\ \varphi \in C_0^\infty$$

in the weak sense. $\Longleftrightarrow$

(ie. $U = -S \Rightarrow S_t + \mathrm{div}\, Su \ge 0$)

⑦

<u>Note</u>: Let U be a convex entropy for system of cons laws $u_t + \operatorname{div} f(u) = 0$, and assume that on weak soln's

$$U_t + \operatorname{div} F \le 0. \qquad (*)$$

Let $u(x,t)$ be a weak soln with compact support in $x \in \mathbb{R}^m$ at each $0 \le t \le T$.

<u>Claim</u>: $\int_{\mathbb{R}^m} U(x,T)\, d^m x - \int_{\mathbb{R}^m} U(x,0)\, d^m x \le 0$

(Total entropy decreases)

Pf. Choose $\varphi = 1$ for $0 < t < T$, $\varphi = 0$ for $t < 0$ and $t > T$. Then $(*)$ implies

$$\iint_{xt} U \underbrace{\varphi_t}_{\delta \text{ fn @ } t=0 \text{ \& } -\delta \text{ @ } t=T} + F \cdot \underbrace{\cancel{\nabla \varphi}}_{0}\, dx\, dt \ge 0$$

$$-\int_{\mathbb{R}^m} U(x,T)\, d^m x + \int_{\mathbb{R}^m} U(x,0)\, d^m x \ge 0 \qquad \checkmark$$

⑧

<u>Expect</u>: weak solutions of a system of cons. laws with convex entropy are <u>unique</u> for given initial data if U holds.

<u>No proof</u> for $n \ge 2$. (n = # of equations) even in 1-space dimension. (Mostly resolved by Bressan)

(U) <u>can</u> serve as a test that numerical schemes converge to the correct physical solution.

- Consider a system of cons. laws with a "convex entropy" in m dimensions $(x^1, \dots, x^m)$

$$u_t + \operatorname{div} f = 0 \qquad \text{(CL)}$$

$$U(u)_t + \operatorname{div} F(u) = 0 \qquad \text{(Ent)}$$

entropy — entropy flux

U convex fn of U.

Then diff entropy:

$$0 = \nabla U \cdot u_t + \nabla F_1(u) u_{x^1} + \cdots + \nabla F_m(u) u_{x^m} \quad (A)$$

Mult (CL) by ∇U & compare:

$$0 = \nabla U \cdot u_t + \nabla U \cdot df_1 \, u_{x^1} + \cdots + \nabla U \cdot df_m \, u_{x^m} \quad (B)$$

Equating (A), (B) we must have

$$\nabla U \cdot df_i = \nabla F_i \quad \forall\, i = 1, \dots, m$$

Defn: A pair of functions $(U(u), F(u))$ is an entropy-entropy flux pair for (CL) if

$$\boxed{\nabla F_i = \nabla U \cdot df_i}$$

In this case

$$U_t + \operatorname{div} F = 0$$

holds on smooth soln's of (CL).

In 1-D this is:

$$u_t + f(u)_x = 0$$

$$U(u)_t + F(u)_x = 0$$

$$\nabla F = \nabla U \cdot df \qquad (*)$$

(11)

Theorem (1) Assume that (U, F) is a convex entropy-entropy flux pair for a system of cons. laws

$$u_t + \operatorname{div} f(u) = 0 \qquad \text{(CL)}$$

$u = (u_1, \dots, u_n) \in \mathbb{R}^n$
$f = (f_1, \dots, f_m)$
$f_i(u) \in \mathbb{R}^n$
$x = (x_1, \dots, x_m) \in \mathbb{R}^m$

so that

$$U(u)_t + \operatorname{div} F(u) = 0$$

holds on smooth solutions,

$$\nabla F_i = \nabla U \cdot df_i$$

Then if $u^\varepsilon(x,t)$ is a smooth soln of

$$u_t + \operatorname{div} f(u) = \varepsilon \Delta u$$

$$\left(u^i_t + \operatorname{div} f_i(u) = \varepsilon \Delta u^i \quad \begin{array}{l} i = 1, \dots, n \\ \text{coupled scalar eqn's} \end{array} \right)$$

such that $u^\varepsilon(x,t) \xrightarrow{\varepsilon \to 0} u(x,t)$ boundedly & a.e. at each time t.

Then $u(x,t)$ is a weak soln of (CL) that satisfies $U_t + F_x \le 0$ in weak sense. (U)

(12)

Proof: We do case $m = 1$, $x \in \mathbb{R}$.

Homework: Prove thm in case $m > 1$.

- First assume $\varphi(x,t)$ smooth test fn with compact support $\Rightarrow$

$$\iint_{xt} u^\varepsilon_t \varphi + f(u^\varepsilon)_x \varphi - \varepsilon u^\varepsilon_{xx} \varphi \, dx\,dt$$

$$= -\iint_{xt} u^\varepsilon \varphi_t + f(u^\varepsilon)\varphi_x - \int_{-\infty}^{\infty} u^\varepsilon(x,0)\,\varphi(x,0)\,dx$$

$$- \varepsilon \iint_{xt} u^\varepsilon \varphi_{xx}\,dx\,dt = 0$$

Since $u^\varepsilon \to u$ boundedly & ae at each time t, taking $\varepsilon \to 0$ gives

$$\iint_{xt} u\varphi_t + f(u)\varphi_x = \int_{-\infty}^{\infty} u(x,0)\,\varphi(x,0)\,dx$$

$\Rightarrow u(x,t)$ is a weak soln of CL.

For (U), multiply

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon.$$

by $\nabla U(u^\varepsilon)$, and use $\nabla F(u^\varepsilon) = \nabla U^\varepsilon \cdot df(u^\varepsilon)$

$$\nabla U \cdot u_t^\varepsilon + \nabla U \cdot df\, u_x^\varepsilon = \varepsilon \nabla U \cdot u_{xx}^\varepsilon$$

$$U(u^\varepsilon)_t + F(u^\varepsilon)_x = \varepsilon \nabla U \cdot u_{xx}^\varepsilon.$$

- Consider: $U_x = \nabla U \cdot u_x$ Hessian of U

$$U_{xx} = (\nabla U \cdot u_x)_x = HU[u_x, u_x] + \nabla U\, u_{xx}$$

HU pos def $\Rightarrow$

$$U_{xx} \geq \nabla U \cdot u_{xx}$$

$$\Rightarrow U(u^\varepsilon)_t + F(u^\varepsilon)_x \leq \varepsilon U_{xx}$$

$\therefore$ assuming u^ε smooth and $u^\varepsilon \to u$ boundedly & pw ae, $\phi \geq 0$ $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \{U(u^\varepsilon)_t + F(u^\varepsilon)_x - \varepsilon U_{xx}\}\phi\, dx\, dt \leq 0$$

$$\Rightarrow \quad -\iint \{U(u^\varepsilon)\phi_t + F(u^\varepsilon)\phi_x + \varepsilon U \phi_{xx}\} dx\, dt \leq 0$$

since $u^\varepsilon \to u$ strongly & no deriv's on U, F

$$\xrightarrow{\varepsilon \to 0} \quad -\iint U(u^0)\phi_t + F(u^0)\phi_x \, dx\, dt \leq 0$$

$\Rightarrow$ u^0 satisfies entropy inequality

$$U_t + F_x \leq 0$$

in weak sense ✓

Cor: Let $u_t + f(u)_x = 0$ be a system of cons. laws that has a convex entropy

$$U(u)_t + F(u)_x = 0$$

on smooth soln's. Assume $u_s(x,t) = [u_L, u_R]$ is a shock that has structure. Then

$$U(u_s)_t + F(u_s)_x \le 0 \qquad (*)$$

in weak sense.

Pf. $u_\varepsilon\left(\frac{x-st}{\varepsilon}\right)$ solves $u_t + f(u)_x = \varepsilon u_{xx}$

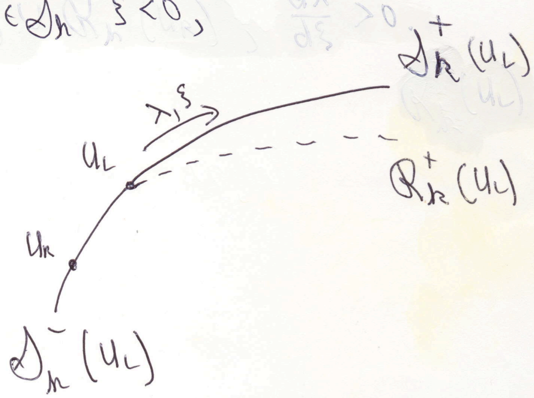
& $u_\varepsilon \longrightarrow u_s$ bdedly & ae at each time $\Rightarrow$

(*) holds ✓

- Assume $u_t + f(u)_x = 0$ is strictly hyperbolic so df has e-pairs (λ_i, R_i), $\lambda_1 < \cdots < \lambda_n$ & assume (U, F) is a convex entropy-entropy flux pair. Assume that $(\lambda, R) \equiv (\lambda_n, R_n)$ is a genuinely nonlinear family, $\nabla\lambda \cdot R > 0$, & let $u(\xi)$ be a arclength parametrization of $\mathcal{S}_n(u_L)$

$u(0) = u_L$, $u(\xi) \in \mathcal{S}_n^-$ $\xi < 0$, $\frac{d\lambda}{d\xi} > 0$

$u(\xi) \in \mathcal{S}_n^+$ $\xi > 0$.



Let $u_R \in \mathcal{S}_n(u_L)$ so $s[u] = [f]$,

$u_R = u(\xi)$ for $\xi < 0$.

Let $u_s(x,t) = [u_L, u_R]$ denote the shock wave soln $\frac{dx}{dt} = s$ $s[u] = [f]$

Defn: we say the shock wave u_s satisfies the entropy condition (EC) if

$$U(u_s)_t + F(u_s)_x < 0$$

in the weak sense.

Theorem: (1) Assume $\nabla\lambda_k \cdot R_k > 0$, and assume U is strictly convex in direction R_k, so that

$$\nabla^2 U(R_k, R_k) > 0.$$

Then for ξ suff small, a k-shock u_s satisfies (EC) iff it satisfies the Lax char condt iff $u_R = u(\xi) \in \mathcal{S}_k^-$, $\xi < 0$.

(2) Assume $\nabla\lambda_k \cdot R_k \equiv 0$. Then we must have U const along $\mathcal{S}_k \equiv \mathcal{R}_k$ & $\nabla^2 U(R_k, R_k) \equiv 0$.

Lemma 1: A k-shock satisfies (EC) iff

$$s[U] > [F] \qquad (\star)$$

$$s\big(U(u_R) - U(u_L)\big) > \big(F(u_R) - F(u_L)\big)$$

(Homework)

Lemma 2 Assume $\nabla\lambda_k \cdot R_k > 0$ & $\nabla^2 U(R_k, R_k) > 0$. Then for ξ suff small, a k-shock u_s satisfies $(\star)$ iff $u_R = u(\xi) \in \mathcal{S}_k^-$, $\xi < 0$.

Proof of Lemma 2:

The pt: until 3rd deriv, everything that doesn't have $[u]$ on it cancels out!

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1st: $[U] = U(u(\xi)) - U(u_L)$

$$(s[U]-[F])^{\circ} \underset{①}{=} \dot{s}[U] + s\dot{U} - \dot{F}$$

But $u(\xi)$, $s(\xi)$ satisfy (R-H) identically:

$$0 = (s[u]-[f])^{\circ} = \dot{s}[u] + s\dot{u} - \dot{f}$$

$$\dot{F} = \nabla F \cdot \dot{u}, \qquad \nabla U\, df = \nabla F$$

$$\Rightarrow \quad \dot{F} = \nabla U \dot{f} = \nabla U\{\dot{s}[u] + s\dot{u}\}$$

$$\Rightarrow \quad ① = \dot{s}[U] + \cancel{s\dot{U}} - \nabla U\,\dot{s}[u] - \cancel{s\dot{U}}$$

$$= \dot{s}[U] - \dot{s}\nabla U[u] \qquad (=0 \text{ at } \xi = 0)$$

$$(s[U]-[F])^{\circ\circ} = \ddot{s}[U] + \cancel{\dot{s}\dot{U}} - \ddot{s}\nabla U[u] - \dot{s}\nabla^2 U\cdot\dot{u}[u] \underbrace{- \cancel{\dot{s}\nabla U\cdot\dot{u}}}_{\dot{U}}$$

$$= \ddot{s}[U] - \ddot{s}\nabla U[u] - \dot{s}\nabla^2 U\,\dot{u}[u]$$

$(=0$ when $\xi = 0)$

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$$(s[U]-[F])^{\circ\circ\circ} = \ddot{s}\dot{U} - \ddot{s}\nabla U\cdot\dot{u} - \dot{s}\nabla^2 U[\dot{u},\dot{u}]$$
$$+\{\ \}_{I}[U] + \{\ \}_{II}[u]$$

Set $\xi = 0$:

$$(s[U]-[F])^{\circ\circ\circ}\Big|_{\xi=0} = \cancel{\ddot{s}\dot{U}} - \cancel{\ddot{s}\dot{U}} - \dot{s}\nabla^2 U[\dot{u},\dot{u}]$$

$$< 0 \quad (\text{since } \dot{u} = R_k)$$

$\Rightarrow$ for ξ suff small, $s[U]-[F] \leq 0$ iff $\xi \leq 0$

& $s[U]-[F] < 0$ iff $\xi < 0$. ✓

<u>Note</u>: In case of a linearly degenerate field, $s = \lambda \equiv$ const fn of ξ $\Rightarrow (s[U]-[F])^{\circ} \equiv 0$

$\Rightarrow$ (EC) holds on soln & its time reversal.

$= \dot{s}[U] - \dot{s}\nabla U[u] \equiv 0$

Note: in 1-d, ($m=1, x \in \mathbb{R}$), condt on U, F to be an entropy-entropy flux is

$$\nabla F = \nabla U \cdot df$$

This is n equations in 2 unknowns U, F with indept var's u.

$\Rightarrow$ Expect: can solve if $n=2$, but $n>2$

$\Rightarrow$ only special systems have entropies

Thm: (Lax) $n=2 \Rightarrow \exists$ infinite family of entropy-entropy flux pairs

(Used by Diperna in method of compensated compactness)

Thm: Lax, Harten - ref Young: an entropy-entropy flux pair exists iff the system of conservation laws is symmetrizable: ie

$\exists$ mapping $u \rightarrow v$ st

the system in v-variables is symmetric.

$$u_t + f_x = 0$$

$$\underbrace{\frac{\partial u}{\partial v}}_{\text{symmetric}} v_t + \underbrace{df \frac{\partial u}{\partial v}}_{\text{symmetric}} v_x = 0$$

Conclusions:

① For shocks, (U) is equivalent to the condition that char. impinge on shock when $|u_L - u_R| \ll 1$.

② Every solution $u(x,t)$ that is a limit, boundedly a.e. of $u_t + f(u)_x = \varepsilon u_{xx}$ as $\varepsilon \to 0$ satisfies (U)

③ Solutions generated by Glimm's Godunov method satisfy (U).