SECTION-12
Conservative Difference Schemes

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Math-280: A Mathematical Introduction to Shock Waves

Blake Temple, UC-Davis

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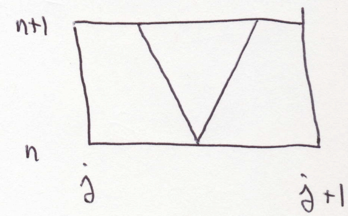
Conservative Difference Schemes:

- Cons Law $u_t + f(u)_x = 0$ says:

"change in u ='s flux thru ∂"

More precisely: If u is an exact soln in R_{jn} (or pw smooth weak soln) then div thm $\Rightarrow$



$$\underset{\text{Top}}{\int u\,dx} - \underset{\text{Bott}}{\int u\,dx} = \underset{\text{RHS}}{\int f(u)\,dt} - \underset{\text{LHS}}{\int f(u)\,dt} = 0$$

$$\approx \text{"} \; U_j^{n+1} - U_j^n = F_{j+1}^n - F_j^n \; \text{"}$$

- Conservative Difference Schemes mimich this —

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Defn A numerical scheme is "conservative" or in "conservation form" if it takes the form:

Cons. Diff Scheme (CDS)

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U_{j-p}^n, \dots, U_{j+q}^n) - F(U_{j-p-1}^n, \dots, U_{j+q-1}^n)\right]$$

p+q+1 entries

Same function of U^n evaluated one step to right

Ex (LaFr)

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{k}{2h}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

written in cons form:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\right]$$

$$F(U_j^n, U_{j+1}^n) = \frac{h}{2k}\left(U_j^n - U_{j+1}^n\right) + \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$$

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Ex Godunov:

$$\bar{U}_j^{n+1} = \bar{U}_j^n - \frac{k}{h}\Big[\underbrace{f(u^*(\bar{U}_j^n, \bar{U}_{j+1}^n))}_{F(\bar{U}_j^n,\, \bar{U}_{j+1}^n)} - f(u^*(\bar{U}_{j-1}^n, \bar{U}_j^n))\Big]$$

$u^*(\bar{U}_L, \bar{U}_R)$ = state prop @ zero speed in soln of RP.

Defn: (CDS) is consistent with (CL) $u_t + f(u)_x = 0$
if

$$F(\bar{u}, \bar{u}, \dots, \bar{u}) = f(\bar{u})$$

Defn: F is Lipschitz cont if $\exists K$ st

$$|F(u_0, u_1, \dots, u_{p+q}) - f(\bar{u})| \le K \max_i |u_i - \bar{u}|$$

Lemma: F Lipschitz cont $\Rightarrow$ F consistent.

(we assume F Lipschitz)

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• Notation: $F(\bar{U}_{j-p}^n, \dots, \bar{U}_{j+q}^n) = F(\bar{U}^n; j)$

Conservation $\Rightarrow$

$$\bar{U}_j^{n+1} = \bar{U}_j^n - \frac{k}{h}\left[F(\bar{U}^n; j) - F(\bar{U}^n; j-1)\right]$$

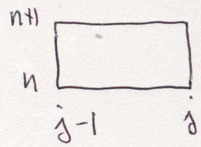
• Note: Cons. Form $\Rightarrow$ collapsing sum on j:

$$\sum_{j=-\infty}^{\infty} F(\bar{U}^n; j) - F(\bar{U}^n; j-1) = f(u_\infty) - f(u_{-\infty})$$

if $\lim_{n \to \pm\infty} \bar{U}_j^n = u_{\pm\infty}$ (+ consistent)

Lax Wendroff Thm (12.1 in Leveque)

Let $\bar{U}_\ell(x,t) = \bar{U}_j^n \quad (x,t)\in R_{nj}$



where $\bar{U}_j^n$ generated by (CDS) with $k_\ell, h_\ell \to 0$. Assume $\bar{U}_\ell \to u$ as follows:

① $\|U_\ell - u\|_{1,\Omega} \to 0$ as $\ell\to\infty \quad \forall\, \Omega=[a,b]\times[0,T]$

$$\int_0^T\int_a^b |U_\ell(x,t)-u(x,t)|\,dx\,dt \to 0$$

② $\forall T\ \exists\, V>0$ st $\displaystyle\sum_{j=-\infty}^{\infty} |U_j^n - U_{j-1}^n| < V$

holds $\forall\ 0\le t_j\le T$, h_ℓ, k_ℓ

Then u is a weak soln of (CL) $u_t+f(u)_x=0$.

<u>Proof</u>: We prove u satisfies

$$\int_0^\infty\int_{-\infty}^\infty u\varphi_t + f\varphi_x\,dx\,dt + \int_{-\infty}^\infty u(x,0)\varphi(x,0)\,dx = 0$$

• (CDS) $\bar{U}_j^{n+1} = \bar{U}_{j-1}^n - \frac{k}{h}\left[F(\bar{U}^n;j) - F(\bar{U}^n;j-1)\right]$

mult by $\varphi(x_j,t_n)$ & Sum:

$$0=\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty} \left(\bar{U}_j^{n+1}-\bar{U}_j^n\right)\varphi(x_j,t_n) \qquad (*)$$

$$+\frac{k}{h}\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}\left[F(\bar{U}^n;j)-F(\bar{U}^n;j-1)\right]\varphi(x_j,t_n)$$

(Really finite sums since φ has compact supp)

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Summation by parts:

$$\sum_{n=0}^{\infty}\left(U_j^{n+1}-U_j^{n}\right)\varphi(x_j,t_n) \qquad (A)$$

$$= -U_j^0\varphi(x_j,t_0) - \sum_{n=1}^{\infty} U_j^n\left(\varphi(x_j,t_n)-\varphi(x_j,t_{n-1})\right)$$

$$\sum_{j=-\infty}^{\infty}\left[F(U^n;j)-F(U^n;j-1)\right]\varphi(x_j,t_n) \qquad (B)$$

$$= -\sum_{j=-\infty}^{\infty} F(U^n;j)\left(\varphi(x_{j+1},t_n)-\varphi(x_j,t_n)\right)$$

Putting (A), (B) into (*) gives –

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$$hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty} U_j^n\left(\frac{\varphi(x_j,t_n)-\varphi(x_j,t_{n-1})}{k}\right)$$

$$+hk\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty} F(U^n;j)\left(\frac{\varphi(x_{j+1},t_n)-\varphi(x_j,t_n)}{h}\right) \qquad (\star)$$

$$= -h\sum_{j=-\infty}^{\infty}\varphi(x_j,0)\,U_j^0$$

"Cons form $\Rightarrow$ you can use summation by parts to convert original diff scheme into weak form that looks formally like weak form of CL"

- Key estimate:

$$E = hk \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \left| F(\bar{U}^n; j) - f(\bar{U}^n_j) \right| \left| \frac{\varphi(x_{j+1}, t_n) - \varphi(x_j, t_n)}{h} \right|$$

$$= O(1) h \qquad \to 0 \quad \text{as } \ell \to \infty \qquad (E)$$

which implies that for limit $\ell \to \infty$ we can replace $F(\bar{U}^n; j)$ by $f(\bar{U}^n_j)$ in (☆)

- To see (E), note by Lipshitz cont

$$|F(\bar{U}^n; j) - f(\bar{U}^n_j)| \leq K \left\{ |\bar{U}^n_{j-p+1} - \bar{U}^n_{j-p}| + \cdots + |\bar{U}^n_{j+q} - \bar{U}^n_{j+q-1}| \right\}$$

Using this we can estimate (☆) by

$$E \leq hk \sum_{\substack{n=0 \\ t_n \in \operatorname{Supp}\varphi}}^{\infty} K(p+q+1) TV \, \|\varphi_x\|_{\sup} = O(1) \cdot h$$

since $\exists$ $O(1) k^{-1}$ terms in the sum over n.

- Substituting $f(\bar{U}^n_j)$ for $F(\bar{U}^n; j)$ in (☆) & using $\bar{U}_\ell(x,t) = \bar{U}^n_j$ in R_{nj} together with

$$\frac{\varphi(x_j, t_n) - \varphi(x_j, t_{n-1})}{k} = \varphi_t(x,t) + O(k)$$

$$\frac{\varphi(x_{j+1}, t_n) - \varphi(x_j, t_n)}{h} = \varphi_x(x,t) + O(h)$$

and replacing the sums by integrals gives

$$\iint_{\operatorname{Supp}\varphi} \Big\{ \bar{U}_\ell(x,t) \left[\varphi_t(x,t) + O(h) \right] + f(\bar{U}_\ell(x,t)) \left[\varphi_x(x,t) + O(h) \right] \Big\} dx\,dt + O(1) h$$

$$= -\int_{-\infty}^{\infty} \bar{U}_\ell(x,0) \left(\varphi(x,0) + O(h) \right) dx$$

taking the L^1 limit $\bar{U}_\ell \to u$ gives

$$\iint_{x,\, t>0} u \varphi_t + f \varphi_x \, dx\,dt = -\int_{-\infty}^{\infty} u(x,0) \varphi(x,0)\, dx \quad ✓$$

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Q: Does the limit of a conservative scheme satisfy the entropy condition?

Need: $\eta(U_j^{n+1}) \le \eta(U_j^n) - \frac{k}{h}\left[\Psi(U^n;j) - \Psi(U^n;j-1)\right]$ $*$

We call Ψ a numerical entropy flux for (CFD) if $(*)$ holds & Ψ is consistent with ψ in sense that

① $\Psi(U^n;j) = \Psi(U^n_{j-p}, \dots, U^n_{j+q})$

② $\exists$ const st $\Psi(\bar u, \dots, \bar u) = \psi(\bar u)$ $\forall \bar u \in \mathbb{R}^n$

③ (2) follows from Lip. cont of Ψ:

$$|\Psi(U^n_{j-p}, \dots, U^n_{j+q}) - \psi(\bar u)| \le K \sup |U^n_h - \bar u|$$

Thm: If Ψ consist with ψ, and $U^n_\ell \to u$ converges as above, then $\eta(u)_t + \psi(u)_x \le 0$ in weak sense

Pf. Mult by $\varphi \ge 0$, summation by pts, take limit as above.

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