SECTION-13
The Glimm Scheme

....

Math-280: A Mathematical Introduction to Shock Waves

Blake Temple, UC-Davis

Existence: (Glimm's Method – 1965)

- Consider an $n\times n$ system of cons Laws:

$$u_t + f(u)_x = 0 \qquad \text{(CL)}$$

$$u(x,0) = u_0(x) \qquad \text{(id)}$$

$$\iint_{\substack{t\geq 0\\ -\infty<x<\infty}} u\varphi_t + f(u)\varphi_x \,dx\,dt + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)\,dx = 0 \quad \forall \varphi\in C_0^\infty(t\geq 0) \qquad \text{(WC)}$$

Theorem: (Glimm) Assume that (CL) is an $n\times n$ system of cons. laws that is strictly hyperbolic and (GN) or (LD) in a nbhd $\mathbb{R}^n \supseteq \mathcal{U} \ni \bar{u}$. Then $\exists$ $\varepsilon>0$ and $C>0$ & a nbhd $\bar{u}\in\mathcal{U}_\varepsilon \subseteq \mathcal{U}$ s.t. if

$$TV\{u_0(\cdot)\} < \varepsilon, \quad u(\cdot)\subseteq \mathcal{U}_\varepsilon \qquad (1)$$

then $\exists$ global weak soln of (CL), (id), (WC) satis.

$$TV\{u(\cdot,t)\} \leq C\, TV\{u_0(\cdot)\} \qquad (2)$$

$$\|u(\cdot,t)-u(\cdot,s)\|_{L^1} \leq C|t-s| \qquad (3)$$

- Here $TV\{u_0(\cdot)\}$ denotes the "Total Variation of $u_0(\cdot)$" defined as

$$\sup \sum_{k=1}^{n} |u_0(x_k) - u_0(x_{k-1})| \qquad (4)$$

where sup is taken over all partition sequences

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty. \qquad (5)$$

- Conclude: $TV\{u(\cdot,t)\}$ measures the length of the curve $u(\cdot,t)\subseteq\mathbb{R}^n$ at each fixed t
- Note: (3) says the initial data is taken on in the L^1-sense

$$\lim_{t\to 0} \|u(\cdot,t)-u_0(\cdot)\|_{L^1} = 0 \qquad (6)$$

$$\|u(\cdot,t)-u_0(\cdot)\|_{L^1} \equiv \int_{-\infty}^{\infty} |u(x,t)-u_0(x)|\,dx, \qquad (7)$$

• The total variation bound (2) is the most technical part of Glimm's proof, and uses an (off the wall) non-local functional $Q(t)$ that measures the total potential for wave interaction at time t. The time independent TV bound (2) is proven by showing that if ε is suff small, the $Q(t)$ decreases, and increases in total variation can be bounded by decreases in Q.

• Note: that (2) implies there are no "unbounded spatial oscillations" ≈ the stability of the numerical approximations

• Note: when $n=1$, the TV estimate (2) can be proven when assumption (1) is replaced by only $TV\{u_0\}<\infty$. In this case (2) simplifies to

$$TV\{u(\cdot,t)\} \le TV\{u_0(\cdot)\}. \qquad (8)$$

This follows by maximum principle for scalar cons. laws

③

⊠ The random choice method or Glimm scheme

④

• In Glimm's 1965 proof of Thm ①, Glimm introduces a new numerical method called the random choice method/Glimm scheme by which he defines approximate solutions $u_{\Delta x}(x,t)$ defined on a numerical grid $x_i = i\Delta x$, $t_j = j\Delta t$. The idea is to use Lax's RP solutions as approximations in each grid cell; and then to iterate by using random values of the RP solutions at the end of each time step to re-pose RP's at the subsequent time-step. By randomly sampling, instead of say averaging (Godunov Method), the RCM introduces no new state in the soln at the update, and this eliminates diffusive-type errors that make it very difficult to get estimate (2).

(5)

Glimm's method was considered strikingly original when it came out. Since then:

- Glimm & Lax 1970: soln's of 2×2 G.N. systems decay $\sim \frac{1}{t}$. Not known for comp ful
- Liu ~1977: Deterministic version of Glimm Schem
- large BV: Young/Temple

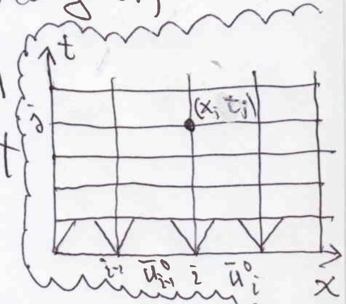
* Bressan ~90's – present

- cont dep on i-data
- limits of vanishing viscosity

All essentially use Glimm's method of estimating the total variation for $n\times n$ systems

(6)

- The approximate solution $u_{\Delta x}(x,t;a)$ generated by RCM: (unstaggered grid)



- Choose mesh lengths $\Delta x, \Delta t \ll 1$
- Define grid $x_i = i\Delta x \quad t_j = j\Delta t$
- Approx i-data by pw const states $u_{\Delta x}(x,0) = \bar{u}^0_i \equiv u_0(x_i)\,,\ x_i < x < x_{i+1}$
- Solve the RP's by Lax method at the discontinuities posed at $t=0$ to get soln

$$u_{\Delta x}(x,t;a) \qquad t < t_1 \tag{9}$$

- Make sure $\Delta t \ll \Delta x$ so that waves don't interact before time t_1. Once we show, $\forall x,t$, $u_{\Delta x}(x,t;a) \in \bar{\mathcal{U}}$, it suffices to take

$$\frac{\Delta x}{\Delta t} = \lambda \geq 2\bar{\Lambda}\,,\ \bar{\Lambda} = \max_{\substack{i=1,\dots,n\\ u\in\bar{\mathcal{U}}}} |\lambda_i(u)| \tag{10}$$

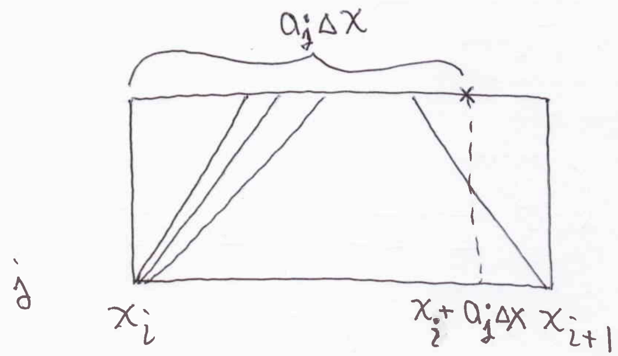
(Courant-Friedrich-Levy CFL condt)

- To continue the approximation, repose RP's at $t=t_1$ in a clever (random) way:

Let $\mathcal{Q} \equiv \prod_{j=1}^{\infty} [0,1]_j$ be a product (measure) space, so that $a \in \mathcal{Q}$ means $a = (a_1, a_2, \dots, a_j, \dots)$, $a_j \in [0,1]$

Now for each $a \in \mathcal{Q}$, construct approximate soln $u_{\Delta x}(x,t;a)$ as follows:

<u>Choose</u>:
$$u_i^1 = u_{\Delta x}(x_i + a_1 \Delta x, t_1 -)$$
$$u_i^j = u_{\Delta x}(x_i + a_j \Delta x, t_j -) \qquad (11)$$



- Continuing by induction we obtain an approximate solution
$$u_{\Delta x}(x,t) \equiv u_{\Delta x}(x,t;a) \qquad (12)$$
defined for each Δx & $a \in \mathcal{Q}$ (so long as $u \in \bar{\mathcal{U}}$ is maintained.)

Steps in Glimm's Proof:

① Fundamental Lemma (stability of $u_{\Delta x}$)

$$TV\{u_{\Delta x}(\cdot,t;a)\} \leq C\,TV\{u_0(\cdot)\} = CV_0$$

for any $a \in \mathcal{A}$. This is used in all subsequent steps

② Prove L^1-Lipschitz continuity in time:

$$\int_{-\infty}^{\infty} |u_{\Delta x}(\cdot,t) - u_{\Delta x}(\cdot,s)|\,dx \leq \underset{\substack{\uparrow \\ \text{depends on } \lambda \text{ \& } V_0}}{C} \{|t-s| + \Delta x\}$$

③ Use a suped-up version of Helly's Thm (Due to Olienik ~1955) to prove that $\forall\, a \in \mathcal{A}$ $\exists$ a subsequence of approx soln's

$$u_{\Delta x_n}(x,t;a) \longrightarrow u_a(x,t)$$

P.W.a.e., and in L^1_{loc} at each time, unif on compact time intervals

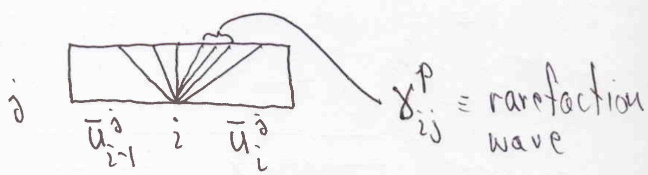
The limit function $u_a(x,t)$ depends on $a \in \mathcal{A}$. It is easy to see that $u_a(x,t)$ will not be a weak soln for ~~any~~ every $a \in \mathcal{A}$. Eg if i-data is a single shock, and $a_j = 1\ \forall j$, then $u_a(x,t)$ will be that same shock propagating at the wrong speed $\frac{dx}{dt} = 0$. Eg

④ Show that $\exists$ a set $\mathcal{N} \subseteq \mathcal{A}$ of measure $\mu(\mathcal{N}) = 0$, such that, if $a \in \mathcal{A} \setminus \mathcal{N}$, then $u_a(x,t)$ is a weak solution of (CL)

<u>Note</u>: TP Liu has proven ≈70's that $\mathcal{A} \setminus \mathcal{N}$ consists of the set of equidistributed sequences - <u>much</u> more technical.

☒ Glimm's Proof:

- For ①, we need a name for every elementary wave that appears in RP's soln's in $u_{\Delta x}(x,t;a)$. Let γ^p_{ij} denote the p-wave that appears in RP $[\bar{u}^j_{i-1}, \bar{u}^j_i](x,t)$ posed @ (x_i, t_j):



Define $|\gamma^p_{ij}| = |u_R - u_L|$ = strength of γ^p_{ij}, where u_L, u_R are the left/right states for γ^p_{ij}

Then $TV\{u_{\Delta x}(\cdot, t; a)\} \equiv \sum_{i,p} |\gamma^p_{ij}|$, $t_j < t < t_{j+1}$

can be taken as the <u>defn</u> of TV, because $|\gamma^p_{ij}| \sim$ total variation across the wave, and TV is linear in waves...

<u>Lemma</u> ① (Fundamental Lemma) $\exists\ \varepsilon > 0$, $C > 0$ & $\mathcal{U}_\varepsilon \ni \bar{u}$ such that, if $u_0(\cdot) \subseteq \mathcal{U}_\varepsilon$ and

$$\sum_{p,i} |\gamma^p_{i0}| < \varepsilon \qquad (13)$$

then

$$\sum_{p,i} |\gamma^p_{ij}| < C\varepsilon. \qquad (14)$$

[It is not hard to show that

$$\frac{1}{C'} TV\{u_{\Delta x}(\cdot, t_j)\} \leq \sum_{p,i} |\gamma^p_i| \leq C' TV\{u_{\Delta x}(\cdot, t_j)\}$$

So (13), (14) is equiv to Step ①.

FIP (Lax's RP soln entails an invertible coord system of wave curves $\forall\ u_L$ ...)

Since the proof of Lemma 1 is the most technical part, we assume (13) & (14) and pospone the proof until the end...

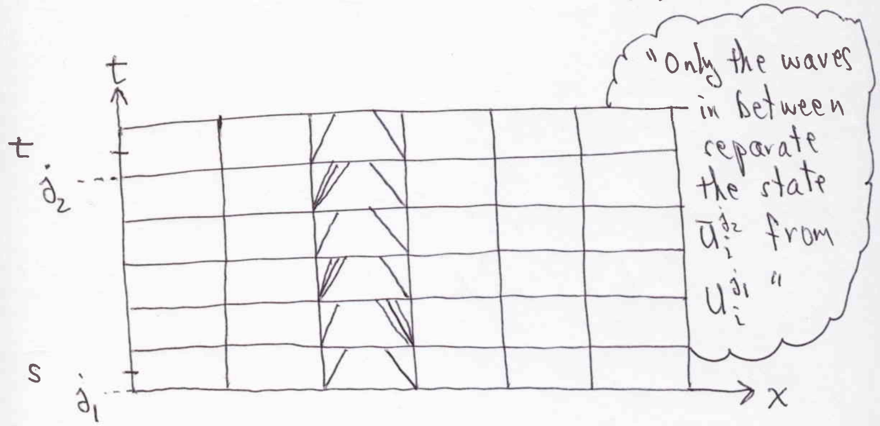
Lemma 2 : $\forall\, a \in \mathcal{Q}$,

$$\|u_{\Delta x}(\cdot,t) - u_{\Delta x}(\cdot,s)\|_{L^1} \le C\,\big|\,|t-s| + \Delta x\,\big|$$

Pf. By construction, for $x \in [x_i, x_{i+1}]$,

$$|u(x,t) - u(x,s)| \le 2 \sum_{j=j_1}^{j_2} \sum_{p} \left\{ |\gamma^p_{ij}| + |\gamma^p_{i+1\,j}| \right\}$$

for $t \in (t_{j_2}, t_{j_2+1}]$ & $s \in (t_{j_1}, t_{j_1+1}]$, i.e.,



Therefore we can estimate:

$$\int_{-\infty}^{\infty} |u_{\Delta x}(x,t) - u_{\Delta x}(x,s)|\,dx$$

$$\le \sum_{i=-\infty}^{\infty} \left\{ \int_{x_i}^{x_{i+1}} |u_{\Delta x}(x,t) - u_{\Delta x}(x,s)|\,dx \right\}$$

$$\le \sum_{i=-\infty}^{\infty} \left\{ \Delta x \sum_{j=j_1}^{j_2} \sum_{p} |\gamma^p_{ij}| + |\gamma^p_{i+1,j}| \right\}$$

$$= \Delta x \sum_{j=j_1}^{j_2} \sum_{i,p} \left\{ |\gamma^p_{ij}| + |\gamma^p_{i+1,j}| \right\}$$

$$\underset{\substack{\uparrow \\ \text{Lemma 1}}}{\le} \Delta x \sum_{j=j_1}^{j_2} 2C \sum_{i,p} |\gamma^p_{i0}| \le 2C\varepsilon\Delta x\,(j_2 - j_1)$$

But $\quad j_2 - j_1 \le \left(\frac{t-s}{\Delta t} + \Delta t\right)$,

so

$$\int_{-\infty}^{\infty} |u_{\Delta x}(x,t) - u_{\Delta x}(x,s)|\,dx \le \frac{\Delta x}{\Delta t} 2C\varepsilon|t-s| + 2C\varepsilon\Delta t$$

$$\le \mathrm{Const}(\lambda,\varepsilon)\left\{|t-s| + \Delta t\right\}.$$

Lemma 3 (Oleinik Compactness)

For each $a \in \mathcal{A}$ and $\Delta x \to 0$, $\exists$ a subsequence $\Delta x_k \to 0$ such that

$$u_{\Delta x}(x,t;a) \longrightarrow u_a(x,t)$$

where convergence is in L^1_{loc} at each fixed time, uniformly on compact space intervals. Specifically, $\forall\ M, T, \sigma > 0\ \exists\ \delta$ such that if $\Delta x_k < \delta$, then

$$\int_{-M}^{M} |u_{\Delta x_k}(x,t;a) - u_a(x,t)|\, dx < \sigma$$

for all $0 \le t \le T$.

The Oleinik compactness argument is based on Helly's Theorem:

Helly's Theorem: Let $\{f_n\}_{n=1}^{\infty}$, $f_n : [a,b] \to \mathbb{R}$ denote a sequence of functions satisfying

(A) $\quad TV\{f_n(\cdot)\} < K$,

(B) $\quad \sup\{f_n(\cdot)\} < K$,

uniformly in n. Then $\exists$ subsequence $\{f_{n_k}\}$ which converges at each $x \in [a,b]$ to a function $f(x)$ satisfying

$$TV\{f(\cdot)\} \le K$$

Pf. Handout

Cor. If $TV\{u_0(\cdot)\} < \infty$, then

$$\lim_{x \to -\infty} u_0(x) \equiv u_0(-\infty) \text{ exists}$$

$$\lim_{x \to +\infty} u_0(x) \equiv u_0(+\infty) \text{ exists}$$

and moreover,

$$|u_0(x) - u_0(-\infty)| \leq TV\{u_0(\cdot)\},$$

$\forall\ x \in (-\infty, \infty)$. Moreover, by finite speed of propagation in the construction of $u_{\Delta x}(x,t;a)$, it follows that if $TV\{u_{\Delta x}(\cdot,t;a)\} \leq C\, TV\{u_0(\cdot)\}$, then

$$\lim_{x \to \pm\infty} u_{\Delta x}(x,t;a) = \lim_{x \to \pm} u_0(\cdot).$$

Pf. (Homework)

Proof of Lemma 3: It suffices to show that $\{u_{\Delta x_j}(\cdot,t)\}_{t \in T}$ is uniformly Cauchy in $L^1[-M,M]$ for some subsequence $\Delta x_j \longrightarrow 0$; i.e., we must show that $\forall\ \varepsilon > 0\ \exists\ \delta$ such that, if

$$\Delta x_A < \delta,\ \Delta x_B < \delta \quad (\Delta x_A, \Delta x_B \in \{\Delta x_j\})$$

then

$$\int_{-M}^{M} |u_{\Delta x_B}(x,t) - u_{\Delta x_A}(x,t)|\,dx < \varepsilon.$$

$\forall\ t \in [0,T]$.

Choose a countable dense set of times $\mathcal{T} = \{t_n\}_{n=1}^{\infty}$, $0 \leq t_n \leq T$. Assume $0, T \in \mathcal{T}$.

19

By Helly's Theorem, $\forall\ t_k \in \mathfrak{J}$ $\exists$ a subsequence of Δx_j (call it Δx_j) such that

$$u_{\Delta x_j}(x, t_k) \longrightarrow u(x, t_k)$$

at each $x \in [-M, M]$. By a diagonal argument, $\exists$ a subsequence (call it Δx_j) such that $u_{\Delta x_j}(\cdot, t_k)$ converges pointwise everywhere at each t_k (see pf of Helly Thm).

Note: a diagonal argument applies only to a countable set of times - we wish to extend the convergence to all $t \in [0, T]$: use Lipshitz cont in time - it "ties soln together at each time"

20

Claim: $u_{\Delta x_j}(\cdot, t_k)$ converges to $u(\cdot, t_k)$ in $L^1[-M, M]$, uniformly on every finite set of times in $\mathfrak{J}$. (FIP) - hint: Luzin's Thm

We now show that $u_{\Delta x_j}(\cdot, t)$ converges in $L^1[-M, M]$ to a function $u(\cdot, t)$ uniformly for all $t \in [0, T]$.

(21)

If we write A for this E_N, then $mA < \delta$ and

$$\tilde{A} = \{x \varepsilon E: |f_n(x) - f(x)| < \epsilon \text{ for all } n \geq N\}. \blacksquare$$

If, as in the hypothesis of the proposition, we have $f_n(x) \to f(x)$ for each x, we say that the sequence $\langle f_n \rangle$ converges **pointwise** to f on E. If there is a subset B of E with $mB = 0$ such that $f_n \to f$ pointwise on $E \sim B$, we say that $f_n \to f$ a.e. on E. We have the following trivial modification of the last proposition:

24. Proposition: *Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions which converge to a real-valued function f a.e. on E. Then, given $\epsilon > 0$ and $\delta > 0$, there is a set $A \subset E$ with $mA < \delta$, and an N such that for all $x \notin A$ and all $n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon.$$

Ref Royden

Problems

29. Give an example to show that we must require $mE < \infty$ in Proposition 23.

30. Prove *Egoroff's Theorem:* If $\langle f_n \rangle$ is a sequence of measurable functions which converge to a real-valued function f a.e. on a measurable set E of finite measure, then, given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f *uniformly* on $E \sim A$. [Hint: Apply Proposition 24 repeatedly with $\epsilon_n = 1/n$ and $\delta_n = 2^{-n}\eta$.]

31. Prove *Lusin's Theorem:* Let f be a measurable real-valued function on an interval $[a, b]$. Then given $\delta > 0$, there is a continuous function φ on $[a, b]$ such that $m\{x: f(x) \neq \varphi(x)\} < \delta$. Can you do the same on the interval $(-\infty, \infty)$? [Hint: Use Egoroff's theorem, Propositions 15 and 22, and Problem 2.39.]

32. Show that Proposition 23 need not be true if the integer variable n is replaced by a real variable t; that is, construct a family $\langle f_t \rangle$ of measurable real-valued functions on $[0, 1]$ such that for each x we have $\lim_{t \to 0} f_t(x) = 0$, but for some $\delta > 0$ we have $m^*\{x: f_t(x) > \frac{1}{2}\} > \delta$. Hint: Let P_i be the sets in Section 4. For $2^{-i-1} \leq t < 2^{-i}$ define f_t by

(22)

Fix $\varepsilon > 0$. We show $\exists\, \delta > 0$ st

$$\Delta x_A, \Delta x_B < \delta; \quad \Delta x_A, \Delta x_B \in \{\Delta x_j\}$$

implies

$$\int_{-M}^{M} |u_{\Delta x_B}(x,t) - u_{\Delta x_A}(x,t)|\, dx < \varepsilon \quad \text{all } t \leq T.$$

① Choose $0 = \tau_1 < \tau_2 < \cdots < \tau_K = T$ in $\mathcal{J}$ such that

$$|\tau_{k+1} - \tau_k| < \delta_1.$$

Since $u_{\Delta x_j}(\cdot, t)$ converges uniformly in $L^1[-M,M]$ for $t \in \{\tau_1, \ldots, \tau_K\}$, $\exists\, \delta_2$ such that

$$\int_{-M}^{M} |u_{\Delta x_B}(x,\tau_k) - u_{\Delta x_A}(x,\tau_k)|\, dx < \frac{\varepsilon}{3}$$

for $\Delta x_A, \Delta x_B < \delta_2$.

Now choose $t \leq T$, $t \in [\tau_k, \tau_{k+1}]$. We can estimate

$$\int_{-M}^{M} |u_{\Delta x_B}(x,t) - u_{\Delta x_A}(x,t)|\,dx \equiv \| u_{\Delta x_B}(\cdot,t) - u_{\Delta x_A}(\cdot,t)\|_{L^1[-M,M]}$$

$$\leq \| u_{\Delta x_B}(\cdot,t) - u_{\Delta x_B}(\cdot,\tau_k)\|_{L^1[-M,M]} \qquad (\mathrm{I})$$

$$+ \| u_{\Delta x_B}(\cdot,\tau_k) - u_{\Delta x_A}(\cdot,\tau_k)\|_{L^1[-M,M]} \qquad (\mathrm{II})$$

$$+ \| u_{\Delta x_A}(\cdot,\tau_k) - u_{\Delta x_A}(\cdot,t)\|_{L^1[-M,M]} \qquad (\mathrm{III})$$

$$(\mathrm{II}) \leq \frac{\varepsilon}{3}$$

$$(\mathrm{I}) \leq C\{|t-\tau_k| + \Delta x_B\} \leq C\{\delta_1 + \Delta x_B\}$$

$$(\mathrm{III}) \leq C\{|t-\tau_k| + \Delta x_A\} \leq C\{\delta_1 + \Delta x_B\}$$

$$\leq \frac{\varepsilon}{3} + 2C\{\delta_1 + \Delta x_A + \Delta x_B\} \leq \varepsilon$$

if $\delta_1 < \frac{\varepsilon}{6C}$, $\Delta x_A \leq \Delta x_B < \delta < \frac{\varepsilon}{6C}$ ✓

Conclude:

LEMMA 3: The function $u(x,t)$ given by the Oleinik compactness argument satisfies:

$$TV\{u(\cdot,t)\} \leq C\,TV\{u_0(\cdot)\}$$

$$\| u(\cdot,t) - u(\cdot,s)\|_{L^1[-\infty,\infty]} \leq C|t-s|$$

and moreover

$$\| u(\cdot,t) - u_0(\cdot)\|_{L^1[-\infty,\infty]} \leq Ct$$

so that the initial data is taken on in the L^1 sense.

Proof [HOMEWORK].

☒ We now prove that for ae $a \in \mathcal{Q}$, the function $u_a(x,t)$ of Lemma 3 is a weak soln of (CL). Let

$$D(\Delta x, a, \varphi) \equiv \iint_{\substack{-\infty < x < +\infty \\ t \geq 0}} u_{\Delta x} \varphi_t + f(u_{\Delta x}) \varphi_x \, dx dt + \int_{-\infty}^{\infty} u_{\Delta x}(x,0)\varphi(x,0)\,dx \tag{13}$$

<u>Lemma</u> 4: Fix $u_0, \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$. Then $\forall$ sequence $\Delta x \to 0$ $\exists$ a subsequence $\Delta x_k \to 0$ and a set $\mathcal{N} \subseteq \mathcal{Q}$, $\mu(\mathcal{N}) = 0$, such that if $a \in \mathcal{Q} \setminus \mathcal{N}$, then

$$\lim_{\Delta x_k \to 0} D(\Delta x, a, \varphi) = 0.$$

<u>Proof</u>: Recall $\mathcal{Q} \equiv \prod_{j=1}^{\infty} [0,1]_j$ so that $a \in \mathcal{Q}$ means $a = (a_1, a_2, a_3, \dots)$, $a_j \in [0,1]$ and for $E = \prod_{j=1}^{\infty} E_j \subseteq \mathcal{Q}$, $E_j \subseteq [0,1]$, $\mu(E) = \prod_{j=1}^{\infty} \mu(E_j)$ where $\mu(E_j)$ is Lebesgue measure on $[0,1]$. From this μ extends to a measure on the measurable sets of $\mathcal{Q}$, and $\mu(\mathcal{Q}) = 1$.

- Let S_j denote the strip in xt-space

$$S_j \equiv \{(x,t) : -\infty < x < +\infty,\ t_j \leq t < t_{j+1}\}, \tag{14}$$

so that by (13)

$$D(\Delta x, a, \varphi) = \sum_{S_j} \hat{D}_j(\Delta x, a, \varphi) + \int_{-\infty}^{\infty} u_{\Delta x}(x,0)\varphi(x,0)\,dx \tag{15}$$

where

$$\hat{D}_j(\Delta x, a, \varphi) = \iint_{S_j} \{u_{\Delta x}\varphi_t + f(u_{\Delta x})\varphi_x\}\,dx\,dt. \tag{16}$$

Applying the divergencence theorem to each $\hat{D}_j$ in (15) and using the fact that $u_{\Delta x}$ is a piecewise smooth soln w. shock boundaries in each S_j, it follows that (collecting the boundary terms)

$$D(\Delta x, a, \varphi) = -\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} [u_{\Delta x}(x,t_j+) - u_{\Delta x}(x,t_j-)]\varphi(x,t_j)\,dx$$

$$\equiv -\sum_{j=1}^{\infty} D_j(\Delta x, a, \varphi).$$

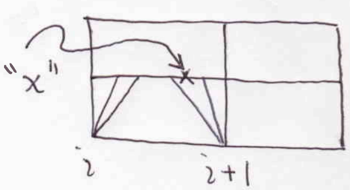
Claim 1 $|D_j(\Delta x, a, \varphi)| \le C\|\varphi\|_\infty TV\{u_0(\cdot)\}\Delta x$

Proof:

$$\left|\int_{-\infty}^{\infty} [u_{\Delta x}(x,t_j+) - u_{\Delta x}(x,t_j-)]\varphi(x,t_j)\,dx\right|$$

$$\le \sum_i \int_{x_i}^{x_{i+1}} \underbrace{\left|[u_{\Delta x}(x,t_j+) - u_{\Delta x}(x,t_j-)]\right|}_{\le \sum_p |\gamma^p_{ij}| + |\gamma^p_{i+1,j}|} |\varphi(x,t_j)|\,dx$$

$u_{\Delta x}(x,t_j+) = u_{\Delta x}(x_i + a_j\Delta x, t_j-) =$ "x"



$$\le \Delta x \|\varphi\|_\infty \sum_{i,p} \{|\gamma^p_{ij}| + |\gamma^p_{i+1,j}|\}$$

$$\le C\|\varphi\|_\infty TV\{u_0(\cdot)\}\Delta x$$

• <u>Note</u>: $D_j = 0$ if t_j is outside the support of φ. Thus the # of times such that $D_j \neq 0$ is $\leq \frac{T_\varphi}{\Delta t} \leq \text{Const} \cdot \frac{1}{\Delta x}$ (T_φ beyond supp of φ), thus

$$|D(\Delta x, a, \varphi)| = \left|\sum_{j=1}^{\infty} D_j\right| \leq \sum_{j=1}^{\infty} |D_j|$$

$$\leq \sum_j C \, \|\varphi\|_\infty TV\{u_0(\cdot)\} \Delta x$$

$$\leq \frac{\text{Const}}{\cancel{\Delta x}} C \, \|\varphi\|_\infty TV\{u_0(\cdot)\} \cancel{\Delta x}$$

$$\leq \text{Const depending on } \varphi$$

Conclude: the estimate of Claim 1 is too crude to conclude $D \to 0$ with $\Delta x \to 0$.

• Consider now

$$\int_{\mathcal{Q}} D(\Delta x, a, \varphi)^2 \, da = \int_{\mathcal{Q}} \left\{ \sum_{j=1}^{\infty} D_j(\Delta x, a, \varphi) \right\}^2 da$$

$$= \sum_{j=1}^{\infty} \int_{\mathcal{Q}} D_j(\Delta x, a, \varphi)^2 \, da + \sum_{j \neq k} D_j(\Delta x, a, \varphi) D_k(\Delta x, a, \varphi) \, da \qquad (17)$$

<u>Claim 2</u>: Assume φ is constant on the grid rectangles $R_{ij} \equiv \{(x,t) : x_i \leq x < x_{i+1}, t_j \leq t < t_{j+1}\}$, say $\varphi = \varphi_{ij} \equiv \text{const}$ for $(x,t) \in R_{ij}$. Then

$$\int_{\mathcal{Q}} D_j(\Delta x, a, \varphi) D_k(\Delta x, a, \varphi) \, da = 0$$

for all $j \neq k$.

Corollary: If $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$, then

$$\left|\int_{\mathcal{Q}} D_j(\Delta x, a, \varphi) D_k(\Delta x, a, \varphi)\, da\right| \le C_\varphi\, \Delta x^3 \qquad (18)$$

where C_φ is a const depending only on the C^1-norm of φ, $C_\varphi \equiv C_\varphi(\|\varphi\|_\infty, \|\nabla\varphi\|_\infty)$.

Pf

Assuming the Corollary of Claim ②, we prove Lemma 4 as follows: By (17) & the fact that $\varphi \equiv 0$ for $t \ge t_{j_\varphi} = T_\varphi$, we have

$$\int_{\mathcal{Q}} D(\Delta x, a, \varphi)^2\, da \le \sum_{j=1}^{j_\varphi} \int_{\mathcal{Q}} D_j(\Delta x, a, \varphi)^2\, da + \left|\frac{j_\varphi}{\Delta t}\right|^2 \Delta x^3,$$

so using Claim ① in the first term & $\frac{\Delta x}{\Delta t} = \lambda$ gives

$$\le \frac{j_\varphi}{\Delta t} C^2 \|\varphi\|^2 TV\{u_0(\cdot)\}^2 \Delta x^2 + \left|\frac{j_\varphi}{\Delta t}\right|^2 \Delta x^2 \qquad (19)$$

$$\le \left\{ j_\varphi^2 \lambda^2 C^2 \|\varphi\|^2 TV\{u_0(\cdot)\} + 1 \right\} \Delta x$$

or

$$\int_{\mathcal{Q}} D(\Delta x, a, \varphi)^2\, da = O(1)\,\Delta x \longrightarrow 0 \text{ as } \Delta x \to 0. \qquad (20)$$

That is, for fixed $\varphi \in C_0^1$, $D(\Delta x, \cdot, \varphi) \to 0$ in $L^2(\mathcal{Q})$ as $\Delta x \to 0$. It follows from L^2-convergence that, on a subsequence, $D(\Delta x, a, \varphi) \to 0$ pointwise off a set of measure zero in $\mathcal{Q}$; ie., for every sequence $\Delta x \to 0$, $\exists$ subsequence $\Delta x_k \to 0$ and a set $\mathcal{N} \subseteq \mathcal{Q}$, $\mu(\mathcal{N}) = 0$ such that, if $a \in \mathcal{Q}\setminus\mathcal{N}$, then

$$D(\Delta x_k, a, \varphi) \longrightarrow 0 \quad \text{as } \Delta x_k \longrightarrow 0.$$

This proves Lemma 4, once we give pf of Claim ② & Cor. ✓

• Proof of Claim 2 : Recall that

$$D_k(\Delta x, a, \varphi) = \sum_{i=-\infty}^{\infty} \int_{x_i}^{x_i+\Delta x} [U_{\Delta x}(x_i + a_k \Delta x, t_k) - U_{\Delta x}(x, t_k)] \varphi(x, t_k)\, dx. \quad (21)$$

Now note that $D_k(\Delta x, a, \varphi)$ depends only on values a_ℓ for $\ell \le k$. That is, only the choices for sampling before time t_k affect the solution at t_k. Thus for example,

$$\int_{\mathcal{A}} D_k(\Delta x, a_k, \varphi)\, da = \underbrace{\int_0^1 \cdots \int_0^1}_{\text{all others integrate to } 1} \int_0^1 D_k\, da_k\, da_{k-1} \cdots da_1 \quad (22)$$

(Fubinis Theorem)

Now assume $j < k$. Then

$$\int_{\mathcal{A}} D_j D_k = \int_0^1 \cdots \int_0^1 \int_0^1 D_j D_k\, da_k \cdots da_1 \ldots \quad (23)$$

But D_j does not depend on a_k $\Rightarrow$

$$= \int_0^1 \cdots \int_0^1 \left(\int_0^1 D_k\, da_k \right) D_j\, da_{k-1} \cdots da_1. \quad (24)$$

We now show that if $\varphi = \varphi_{ij}$ on R_{ij} is constant, then

$$\int_0^1 D_k\, a_k = 0, \quad (25)$$

thereby proving Claim 2.

So assume $\varphi \equiv \varphi_{ij} \equiv \text{const}$ on R_{ij}. Then

$$\int_0^1 D_k \, da_k = \int_0^1 \sum_{i=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} [u_{\Delta x}(x_i + a_k \Delta x, t_k^-) - u(x, t_k^-)] \varphi_{ik} \, da_k$$

$$= \sum_{i=-\infty}^{\infty} \varphi_{ik} \int_0^1 \int_{x_i}^{x_{i+1}} [u_{\Delta x}(x_i + a_k \Delta x, t_k^-) - u(x, t_k^-)] \, dx \, da_k$$

chg var's: $x = x_i + z\Delta x$, $0 \le z \le 1$

$$= \sum_{i=1}^{\infty} \varphi_{ik} \int_0^1 \int_0^1 [u_{\Delta x}(x_i + a_k \Delta x, t_k^-) - u(x_i + z\Delta x, t_k^-)] \, dz \, da_k$$

$= 0$!

same integrals !

This proves Claim ② ✓

Pf of Corollary to Claim ②:

Since $\varphi \in C_0^1$, it follows by Taylor's Thm, expanding about a value in each R_{ij}, that

$$\varphi(x,t) = \varphi_{ij} + \psi(x,t)\Delta x, \quad (x,t) \in R_{ij}$$

where $\psi(x,t)$ is a bounded function of compact support. Thus

$$\int_0^1 D_k \, da_k = \int_0^1 \sum_{i=-\infty}^{+\infty} \int_{x_i}^{x_{i+1}} [u_{\Delta x}(x_i + a_k \Delta x, t_k^-) - u_{\Delta x}(x, t_k^-)] \varphi_{ik} \overset{dx}{\downarrow} da_k$$

$$+ O(1) \Delta x \int_0^1 \sum_{i=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} [\quad] \psi \, dx \, da_k$$

$$= 0 + O(1) \|\psi\|_\infty \Delta x \sum_{i=-\infty}^{\infty} \Delta x \Big(\sum_p |\gamma_{ik}^p| + |\gamma_{i+1,k}^p| \Big)$$

$$\le O(1) \Delta x^2 \qquad (26)$$

But by Claim ①,

$$|D_j| = O(1)\Delta x \tag{27}$$

Applying (26) & (27) in (24),

$$\left|\int_Q D_j D_k\right| = \left|\int_0^1 \cdots \int_0^1 \left(\int_0^1 D_k a_k\right) D_j \, da_{k-1} \cdots da_1\right|$$

$$\leq \int_0^1 \cdots \int_0^1 \left|\int_0^1 D_k a_k\right| |D_j| \, da_{k-1} \cdots da_1$$

$$\leq [O(1)\Delta x^2][O(1)\Delta x]\, \mu(Q)$$

$$\leq O(1)\Delta x^3$$

as claimed in Corollary ✓

<u>Theorem</u>: Fix $u_0(\cdot)$. Then $\forall$ sequence $\Delta x \to 0$ $\exists$ subsequence $\Delta x_k \to 0$ and a set $\mathcal{N} \subseteq Q$, $\mu(\mathcal{N}) = 0$, such that if $a \in Q \setminus \mathcal{N}$, then

$$\lim_{\Delta x_k \to 0} D(\Delta x_k, a, \varphi) = 0 \quad \forall \varphi \in C_0^1$$

(uniform in φ)

<u>Proof</u>: Choose $\{\varphi_m\}_{m=0}^{\infty}$ dense in $C_0^1(\mathbb{R} \times \mathbb{R}^+)$, and choose $\mathcal{N}_m$ and $\Delta x_{k_m} \to 0$ such that Lemma 4 holds $\forall m$. Taking $\mathcal{N} = \bigcup_{m=0}^{\infty} \mathcal{N}_m$ and taking the "diagonal subsequence" Δx_k we have $\mu(\mathcal{N}) = 0$, and $a \in \mathcal{N} \Rightarrow$

$$\lim_{\Delta x_k \to 0} D(\Delta x_k, a, \varphi_m) = 0 \quad \forall m.$$

Claim: $\lim_{\Delta x_k \to 0} D(\Delta x_k, a, \varphi) = 0 \ \forall \varphi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$.

I.e., choose $\varphi_m \to \varphi$ in C_0^1. Then

$$D(\Delta x_k, a, \varphi) = D(\Delta x_k, a, \varphi - \varphi_m) + \underset{\downarrow\, 0}{D(\Delta x_k, a, \varphi_m)}$$

So

$$|D(\Delta x_k; a, \varphi - \varphi_m)| \le \left| \iint_{x,t} u_{\Delta x}(\varphi - \varphi_m)_t + f(u_{\Delta x})(\varphi - \varphi_m)_x \, dx\,dt \right| + \left| \int_{-\infty}^{\infty} u_0(x)[\varphi(x,0) - \varphi_m(x,0)]\, dx \right|$$

$$\le O(1)\|\varphi - \varphi_m\|_{C^1}$$

$\Rightarrow \forall \varepsilon > 0$, $|D(\Delta x_k, a, \varphi)| < \varepsilon$ for $\Delta x_k < h$

by choosing φ_m st $|D(\Delta x_k, a, \varphi - \varphi_m)| < \frac{\varepsilon}{2}$

and $\Delta x_k < h$ so that $|D(\Delta x_k, a, \varphi_m)| < \frac{\varepsilon}{2}$)

and by this we conclude

$$D(\Delta x_k, a, \varphi) \to 0 \text{ as } \Delta x_k \to 0 \checkmark$$

Proof of Glimm's Thm :

Let u_0 be given i-data satisfying

$$TV\{u_0(\cdot)\} < \varepsilon.$$

Then by Theorem, $\exists \Delta x_k \to 0$, $\mathcal{N} \subseteq \mathcal{Q}$, $\mu(\mathcal{N}) = 0$ st

$$\lim_{\Delta x_k \to 0} D(\Delta x, a, \varphi) = 0 \quad \forall a \in \mathcal{Q}\setminus\mathcal{N}.$$

By Lemma 3, for each a $\exists$ subsequence st $u_{\Delta x_k} \to u_a \equiv u$ p.w. a.e. But

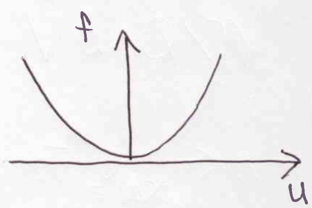
$$0 = \lim_{\Delta x_k \to 0} D(\Delta x_k, a, \varphi) = \lim_{\Delta x_k \to 0} \left\{ \iint_{x,t} u_{\Delta x}\varphi_t + f(u_{\Delta x})\varphi_x + \int_{-\infty}^{\infty} u_{\Delta x}(x,0)\varphi(x,0)\,dx \right\}$$

$$\underset{\text{LDC}}{=} \iint_{x,t} \lim_{\Delta x_k \to 0} \{u_{\Delta x}\varphi_t + f(u_{\Delta x})\varphi_x\} + \int_{-\infty}^{\infty} \lim_{\Delta x_k \to 0} \{u_{\Delta x}(x,0)\varphi(x,0)\}\,dx$$

$$= \iint_{x,t} u\varphi_t + f(u)\varphi_x + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)\,dx \iff u(x,t) \text{ solves (CL) \& satisfies (WC)} \checkmark$$

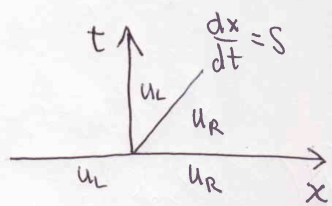
It remains to prove the Fundamental Total Variation estimate Lemma 1.
The proof is not technical when $n=1$, the case of a scalar (CL). It is instructive to first prove Lemma 1 in this simpler case.

- Consider $u_t + f(u)_x = 0$ when u, f scalars and $f(u)$ is convex up, e.g. Burgers $f(u) = \frac{1}{2}u^2$.

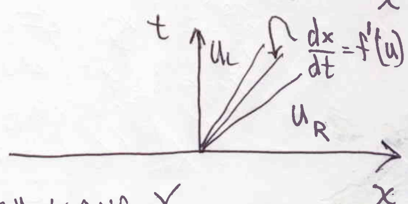


- Elementary waves solve RP

Shock wave $u_L > u_R$ $\quad s[u] = [f]$



Rarefaction wave $u_L < u_R$



<u>Lemma A</u>: For an elementary wave γ,

$$TV\{u(\cdot,t)\} = |u_L - u_R| \equiv |\gamma|$$

- <u>Lemma B</u>: If $|u_L| < M$ & $|u_R| < M$, then the wave speed is bounded by

$$\Sigma \equiv \mathrm{Max}\{|f'(u_L)|, |f'(u_R)|\}$$

(This follows from $s = \frac{[f]}{[u]}$)

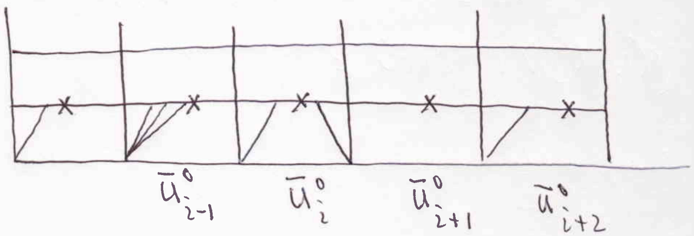
- <u>Lemma C</u>: If $|u_0(x)| < M$ and $\frac{\Delta x}{\Delta t} = \lambda \geq \Sigma$, then the Glimm approx soln $u_{\Delta x}(x,t;a)$ is defined $\forall\, x, t \geq 0$, and all $a \in \mathcal{A}$

43

• <u>Lemma</u> 1 (Case $n=1$): Assume $|u_0(x)|<M$, $\frac{\Delta x}{\Delta t} = \lambda \geq \underline{\Lambda}$ and $a \in \mathcal{A}$. Then

$$TV\{u_{\Delta x}(\cdot,t;a)\} \leq TV\{u_{\Delta x}(\cdot,0;a)\} \leq TV\{u_0(\cdot)\}$$

<u>Proof</u>:



We have: $TV\{u_{\Delta x}(\cdot,0)\} = \sum_{i=-\infty}^{\infty} |\bar{u}^0_{i+1} - \bar{u}^0_i|$

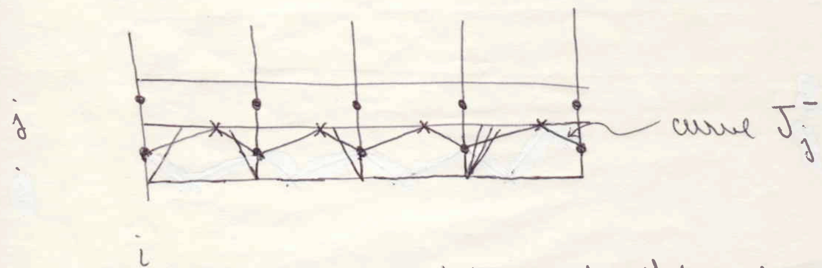
$$= \sum_{i=\infty}^{\infty} |u_0(x_{i+1}) - u_0(x_i)| \leq TV\{u_0(\cdot)\}.$$

This extends to $t<t_1$ because the jump in u across a wave is monotone. Thus it only remains to show that $TV\{u_{\Delta x}(\cdot,t;a)\}$ jumps <u>down</u> between t_1- & t_1+; ie it suffices to show that

44

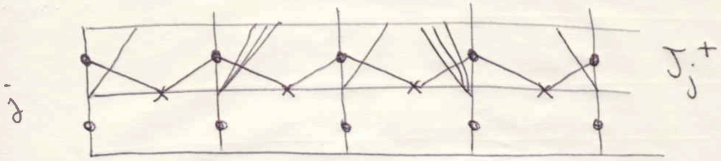
<u>CLAIM</u>: $\sum_{i=-\infty}^{\infty} |\bar{u}^{j+1}_{i+1} - \bar{u}^{j+1}_i| \leq \sum_{i=-\infty}^{\infty} |\bar{u}^j_{i+1} - \bar{u}^j_i|$

We prove this using the following construction of Glimm (the construction is more useful in the case $n>1$)



- curve J_j^- connects "sample pts" on t_j to pts $(x_i, t_j - \frac{1}{2}\Delta t)$.
- J_j^- constructed so that it crosses all the waves in the solution at t_{j-1}.
- <u>DEFN</u>: $TV\{J_j^-\} \equiv \sum_{J_j^-} |\gamma_\sigma| = \sum_{i=-\infty}^{\infty} |\bar{u}^{j-1}_{i+1} - \bar{u}^{j-1}_i| = TV\{u_{\Delta x}(\cdot,t)\}$

  sum over all waves which cross J_j^- $\qquad$ $t_{j-1} \leq t < t_j$



- curve J_j^+ connects sample pt on t_j to pts $(x_i, t_j + \frac{1}{2}\Delta t)$.

- J_j^+ constructed so that it crosses all the waves in the solution at t_j

- DEFN: $TV\{J_j^+\} \equiv \sum_{J_j^+} |\gamma_\sigma| = TV\{u_{\Delta x}(\cdot, t)\},\ t_j \le t < t_{j+1}$

↑ sum over all wave crossing J_j^+

- Thus, the claim is proved once we show

$$TV\{J_j^+\} \le TV\{J_j^-\}.$$

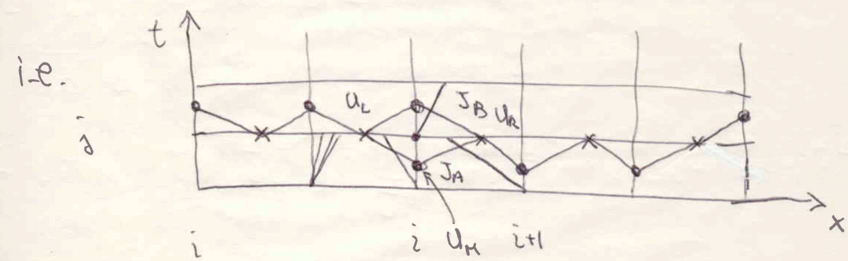
Defn: An I-curve is any spacelike p.w. linear curve connecting sample pts on t_j to pts on $(x_i, t_j \pm \frac{1}{2}\Delta t)$. $TV\{J\} = \sum_J |\gamma_\sigma|$ ← sum over all waves which cross J.

- By induction, it suffices to show that

$$TV\{J_B\} \le TV\{J_A\}$$

whenever J_A and J_B agree except at i where "$J_B > J_A$"

i.e.



$$TV\{J_B\} - TV\{J_A\} = \sum_{J_B} |\gamma_\sigma| - \sum_{J_A} |\gamma_\sigma|$$

$$= |\gamma_{i,j}| - |\gamma_{i,j-1}| - |\gamma_{i+1,j+1}|$$

$$= |u_L - u_R| - |u_L - u_M| - |u_M - u_R| \le 0$$

↑ Δ-inequality

HARDEST PART $n > 1$ since Δ-inequality fails!

The " " term was zero because $u_{\Delta x}$ was a p.w. const soln of (WC) for $t_{j-1} < t \le t_j$. Adding gave

$$D(\Delta x) = \sum_j \int_{-\infty}^{\infty} [u(x,t_j+) - u(x,t_j-)]\,\varphi(x,t_j)\,dx$$

Similarly,

$$\iint_{x,t} \bar{U}\varphi_t + F\varphi_x = \sum_j \iint_{S_j} \bar{U}\varphi_t + F\varphi_x$$

$$= \sum_j \left\{ \text{"}\iint_{S_j} \bar{U}_t\varphi + F_x\varphi\text{"} + \int_{-\infty}^{\infty} \bar{U}(x,t_j-)\varphi - \bar{U}(x,t_{j-1}+)\varphi \right.$$

$\text{"} \ge$ because u is an ~~ent~~ p.w. const entropy satis soln for $(x,t)\in S_j$

$$\le \sum_j \int_{-\infty}^{\infty} \{[\bar{U}(x,t_j+) - \bar{U}(x,t_j-)]\,\varphi(x,t_j)\}\,dx$$

$\downarrow 0$ by same analysis as for error term from equiv

THM: Solutions generated by Glimm's method satisfy (EC)

Proof: Let $u_{\Delta x}(x,t)$ denote an approximate solution generated by $a \in \mathcal{Q}\setminus N$, and assume

$$u_{\Delta x}(x,t) \Rightarrow u(x,t)$$

a soln of (WC) as $\Delta x \to 0$ within some subsequence. Recall:

$$D(\Delta x) = \iint_{x,t} \{u_{\Delta x}\varphi_t + f(u_{\Delta x})\varphi_x\} + \int_{-\infty}^{\infty} u_{\Delta x}(x,0)\varphi(x,0)\,dx$$

is the error, and $D(\Delta x) \to 0$ for our choice of a.

Recall:

$$\iint_{x,t} \{\ \} = \sum_j \iint_{S_j} \{\ \}$$

$$\iint_{S_j} u_{\Delta x}\varphi_t + f(u_{\Delta x})\varphi_x = \text{"}\int_{S_j} (u_{\Delta x})_t\varphi + f(u_{\Delta x})_x\varphi\text{"}$$

$$+ \int_{-\infty}^{\infty} u(x,t_j-)\varphi - u(x,t_{j-1}+)\varphi$$