SECTION-14
Glimm's Total Variation Estimate

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Math-280: A Mathematical Introduction to Shock Waves

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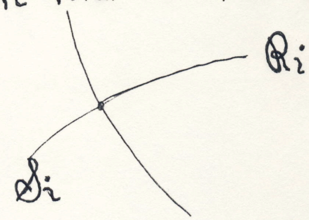
Glimm's Method: (The total Variation estimate)



• Let γ_i denote the signed strength of a wave

ie. $\gamma_i \equiv$ arclength along wave curve from u_L to u_R.

• Main estimate: $\alpha_1, \dots, \alpha_n$ $\beta_1, \dots, \beta_n$ enter a diamond, and $\epsilon_1, \dots, \epsilon_n$ come out, then

$$(*) \qquad \epsilon_i = \alpha_i + \beta_i + O(1)\, D$$

$$D = \sum_{APP(i,j)} |\alpha_i||\beta_j|$$

$O(1)$ depends only on values of f in $\mathcal{U}$

$$|O(1)| \le G_0 \frac{1}{n} \left(\Rightarrow \sum_i |\epsilon_i - \alpha_i - \beta_i| \le G_0 D \right)$$

• Defn: $$Q(J) = \sum_{App(J)} |\gamma_i||\gamma_j|$$

$$L(J) = \sum_J |\gamma_i|$$

$$F(J) = L(J) + G\, Q(J)$$

where G is a constant to be chosen

We estimate:

$$F(J_2) - F(J_1) = L(J_2) - L(J_1) + G\left(Q(J_2) - Q(J_1)\right)$$

$$L(J_2) - L(J_1) = \sum_i |\epsilon_i| - |\alpha_i| - |\beta_i| \overset{*}{\le} G_0 D$$

$$Q(J_2) - Q(J_1) = \sum_{App\, J_2} |\gamma_i||\gamma_j| - \sum_{App\, J_1} |\gamma_i||\gamma_j|$$

$$(*) \quad |\epsilon_i| - |\alpha_i| - |\beta_i| \le |\alpha_i - \beta_i - \gamma_i|$$

③

Now: if $\gamma_i, \gamma_j \in \underbrace{J_1 \cap J_2}_{=\Delta}$, then the corresponding terms cancel. ∴

$$Q(J_2) - Q(J_1) = \sum_{\substack{App\ J_2 \\ \gamma_i \notin \Delta}} |\gamma_i||\varepsilon_j| - \sum_{\substack{App\ J_1 \\ \gamma_i \notin \Delta}} |\gamma_i||\alpha_j| - \sum_{\substack{App\ J_1 \\ \gamma_i \notin \Delta}} |\gamma_i||\beta_j|$$

$$- \underbrace{\sum_{App} |\alpha_i||\beta_j|}_{D}$$

④

Claim:

$$\sum_{\substack{App\ J_2 \\ \gamma_i \in J_1 \cap J_2}} |\gamma_i||\varepsilon_j| - \sum_{\substack{App\ J_1 \\ \gamma_i \in J_1 \cap J_2}} |\gamma_i||\alpha_j| - \sum_{\substack{App\ J_1 \\ \gamma_i \in J_1 \cap J_2}} |\gamma_i||\beta_j|$$

$$\leq G_0 L(J_1) D \frac{1}{n}$$

Proof: If sign ε_j = sign α_j = sign β_j, then $\varepsilon_j\ \alpha_j\ \beta_j$ all approach the same wave γ_i in $J_1 \cap J_2 \Rightarrow$

$$LHS = \sum_{\substack{App\ J_2 \\ \gamma_i \in J_1 \cap J_2}} |\gamma_i| \left(|\varepsilon_j| - |\alpha_j| - |\beta_j| \right) \leq \frac{1}{n} G_0 L(J_1) D$$

If sign $\varepsilon_j \neq$ sign α_j or sign B_j, then $\exists$ two cases: if $0 <$ sign ε_j (ε_j rarefaction), then ε_j approaches <u>fewer</u> γ_i than α_i or B_i if α_i or B_i is a shock. $\therefore$ in this case

$$\text{LHS} \leq \sum_{\substack{\text{App } J_2 \\ \gamma_i \in J_1 \cap J_2}} |\gamma_i| \left(|\varepsilon_j| - |\alpha_j| - |B_j| \right) \leq \tfrac{1}{n} G_0 L(J_1) D$$

(I.e., ignore the neg terms $-|\gamma_i||\alpha_j|$ where γ_i is an i-rarefaction wave).

If sign $\varepsilon_j < 0$ (ε_j shock) then we argue as follows: by $(*)$,

$$0 \leq |\varepsilon_j| = -\varepsilon_j \leq -\alpha_j - B_j + G_0 D \tfrac{1}{n}$$

But if either α_j or B_j is positive, then we can ignore it; ie, if both $\alpha_j \geq 0$, $B_j \geq 0$ then $|\varepsilon_j| \leq G_0 D \frac{1}{n}$ & we are done. If one is ≥ 0, say $\beta_j \geq 0$, then

$$\text{(e)} \qquad -\varepsilon_j + \alpha_j \leq -B_j + G_0 D \tfrac{1}{n} \leq G_0 D \tfrac{1}{n}.$$

Thus, if $|\alpha_j| < |\varepsilon_j|$, LHS(e) is positive, and we have

$$|\varepsilon_j| - |\alpha_j| \leq G_0 D \tfrac{1}{n}.$$

If $|\alpha_j| > |\varepsilon_j|$, then $|\varepsilon_j| - |\alpha_j| \leq 0$ and we are done ie., in either case,

$$\text{LHS}(*) \leq \sum_{\gamma_i \in J_1 \cap J_2} |\gamma_i| \left(|\varepsilon_j| - |\alpha_j| \right) \leq \tfrac{1}{n} G_0 L(J_2) D.$$

& CLAIM is PROVED ✓

<u>Thus</u>: $Q(J_2) - Q(J_1) \leq -D + G_0 \sum_n \frac{1}{n} L(J_1) D$

$$\leq \left(-1 + G_0 L(J_1)\right) D.$$

In particular, if $G_0 L(J_1) \leq 1$, then

$$Q(J_2) - Q(J_1) \leq 0.$$

Now Consider

$$\Delta F = F(J_2) - F(J_1) = L(J_2) - L(J_1) + G\left(Q(J_2) - Q(J_1)\right)$$

where G is yet to be chosen.

$$\Delta F \leq G_0 D + G\left(-1 + G_0 L(J_1)\right) D$$

$$= \left\{ G_0 - G + G G_0 L(J_1) \right\} D$$

Or

$$\Delta F \leq G \left\{ -1 + \frac{G_0}{G} + G_0 L(J_1) \right\} D$$

Idea: we attempt to show that if

$$L(J_0) \leq \varepsilon$$

then $L(J) \leq K\varepsilon$ for all J, by showing that $\Delta F \leq 0 \;\forall\, J_1, J_2$. So set $\boxed{G \geq 2G_0}$ so that

$$\Delta F \leq G\left\{ -\tfrac{1}{2} + G_0 L(J_1) \right\} D \leq 0$$

if $G_0 L(J_1) < \frac{1}{2}$, or

$$L(J_1) < \frac{1}{2G_0}.$$

Thus we need

$$L(J_1) \leq K\varepsilon \leq \frac{1}{2G_0}$$

to insure $\Delta F \leq 0$.

(1) $$K\varepsilon \leq \frac{1}{2G_0}.$$

Now assume for induction that $L(J) \leq K\varepsilon \ \forall\ J \leq J_1$. Then

$$\begin{aligned} L(J_2) \leq F(J_2) \leq F(J_0) &\leq L(J_0) + G\,Q(J_0) \\ &\leq L(J_0) + G\,L(J_0)^2 \\ &\leq \varepsilon + G\varepsilon^2 \end{aligned}$$

Thus, $L(J_2) \leq K\varepsilon$ if

$$\varepsilon(1 + G\varepsilon) \leq K\varepsilon,$$

or,

$$(1 + G\varepsilon) \leq K.$$

Thus choose

$$\boxed{K = 1 + G\varepsilon} \quad \text{for} \quad \boxed{G = 2G_0},$$

and choose $\varepsilon \ll 1$ so that

$$K\varepsilon \leq \frac{1}{2G_0} = \frac{1}{G}$$

$$\Leftrightarrow \quad \boxed{(1 + G\varepsilon)\varepsilon \leq \frac{1}{G}}$$

We show by induction that if $L(J_0) < \varepsilon$, then $L(J) \leq K\varepsilon \ \forall\ J$. I.e., assume for induction that $L(J) \leq K\varepsilon \ \forall\ J \leq J_1$.

Then (11)

$$L(J_2) \le F(J_2) \le F(J_0) \le L(J_0) + G\,L(J_0)^2$$

$G = 2G_0$

$K \le 1 + G\varepsilon$

$\Rightarrow \Delta F \le 0$

$$\le (1 + G\varepsilon)\varepsilon$$

$$\le K\varepsilon \checkmark$$

We have: if

$$\varepsilon_i = \alpha_i + B_i \pm \tfrac{1}{n} G_0 D$$

defines G_0; then, if

$$L(J_0) \le \varepsilon,$$

where $(1 + 2G_0\varepsilon)\varepsilon \le \frac{1}{2G_0}$,

(12)

then

$$L(J_j) \le K\varepsilon \quad \text{all } j \ge 0$$

where $K = 1 + 2G_0\varepsilon$.

This gives the total variation estimate

Actually: we also need that the soln stays within a nbhd $\mathcal{U}$ where R.P.'s are defined and estimate (*) holds. Thus, take $u_0(-\infty) = \bar{u} \in \mathcal{U}$ so that $L(J_j) \le K\varepsilon \Rightarrow u_{\Delta x}(\cdot, t_j) \subseteq \mathcal{U} \;\forall\, t_j$.

This is a further restriction on ε ✓

Glimm's Method: (3rd Order Extension by Young-Te)

- R.P. soln $\alpha \equiv (\alpha_1, \dots, \alpha_n)$ $\beta \equiv (\beta_1, \dots, \beta_n)$

  $\gamma \equiv (\gamma_1, \dots, \gamma_n)$ any sequence of waves

- R^i = unit right eigenvector at $\bar{u}$.

Improved Interaction Estimate:

$$\sum \varepsilon_i R^i = \sum (\alpha_i + \beta_i) R^i + \sum_{j>k} \alpha_j \beta_k [R^j, R^k] + D(\alpha,\beta)O(S(\alpha,\beta)) \qquad (Y)$$

$$D(\alpha,\beta) = \sum_{App} |\alpha||\beta|$$

$$S(\alpha,\beta) = \max(|\alpha|, |\beta|)$$

Here: wave strengths must be measured in a coordinate system that is as close as possible to a coord. syst. of Riemann Invariants

Here: solutions are restricted to a nbhd of $\tilde{u} \ni \mathcal{U}$ in which G.N., L.D., S.H. and Riemann problems can be solved via Lax. (λ_i, r^i) are eigenpairs, and a coordinate system of "almost" Riemann invariants is chosen to satisfy $\nabla W_i \cdot r^i = 1$, together with the condition that

r^i points toward increasing λ_i

$$W_i(\tilde{u}) = 0$$

$$W_i(u) = 0 \quad \forall\, u \in \mathcal{R}_k(\tilde{u}),\ k \neq i.$$

In general we can choose r^i as unit eigenvectors, and we set

$$R^i \equiv r^i(\tilde{u}).$$

Wave strength γ_i

Note: since (Y) is a vector statement, each component must read:

$$\mathcal{E}_i = \alpha_i + \beta_i + \sum_{j>k} \alpha_j \beta_k [R^j, R^k]_i$$
$$+ D(\alpha,\beta)\, O(S(\alpha,\beta))$$

& since $\sum_{j>k} |\alpha_j \beta_k [R^j, R^k]| \leq D(\alpha,\beta)$

(Y) implies Glimm's estimate

(G) $$\mathcal{E}_i = \alpha_i + \beta_i + O(1) D \cdot \frac{1}{n}$$

Cor: (Glimm) if there exists a coordinate system of Riemann Invariants, then we can choose $[R^i, R^j] = 0$ & (Y) gives

(G2) $$\mathcal{E}_i = \alpha_i + \beta_i + O(1) \underbrace{SD}_{\text{3rd order}} \quad \text{(improved estimate)}$$

We show how to prove Glimm's Thm, how to improve it using (G2), and how Young gets this improvement when ∄ a coord system of R.I. by taking advantage of estimate (Y) & a cancellation in quadratic terms that improves the method to 3rd order.

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Thm (Glimm): [Assume $\mathcal{U} \ni \tilde{u}$ is chosen so all assumptions about R.P.'s hold, & following inequalities hold in $\mathcal{U}$:]

(1) $|\varepsilon_i - (\alpha_i + \beta_i)| \leq \frac{K}{n} D(\alpha, \beta)$

(2) $V(J) = \sum_J |\gamma_i|$

(3) $Q(J) = \sum_{App(J)} |\gamma_i||\gamma_j|$

(4) $G(J) = V(J) + c\,Q(J)$

Choose: $c = 3K$

(L1) $Q(J_+) - Q(J_-) \leq D(\Delta)(K V(J_-) - 1)$

choose: $v = \frac{1}{c} = \frac{1}{3K} < 1$

Then if $V(J_0) \leq v$, then $V(J) \leq 2v$ $\forall J$ (so long as $V(J) \leq 2v$ guarantees $\mathcal{U}_\Delta \subseteq \mathcal{U}$)

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Proof: $V(J_0) \leq v \leq 2v$. Thus by induction it suffices to show that if $V(J) \leq 2v$ $\forall J \leq J_-$, then $V(J_+) \leq 2v$ also.

But for any successors $\tilde{J}_+, \tilde{J}_-$ with $\tilde{J}_- \leq J_-$, we have

$$G(\tilde{J}_+) - G(\tilde{J}_-) \leq KD + cD(KV(\tilde{J}_-) - 1)$$

$c = 3K$

$V(\tilde{J}_-) \leq 2v = \frac{2}{3K}$

$$\leq D\{K + cKV(\tilde{J}_-) - c\}$$

$$\leq D\{K + 3K^2 \cdot \frac{2}{3K} - 3K\}$$

$$\leq 0 \checkmark$$

$\therefore$ $V(J_+) \leq G(J_+) \leq G(J_0) \leq V(J_0) + cV(J_0)^2$

$$\leq v + cv^2 = v\left(1 + \frac{3K}{3K^2}\right) \leq 2v$$

$\checkmark$

☒ Improved Existence Theorem:

Let $\delta = \frac{3}{4}\eta \exp\{-\frac{L}{3K^2}\}$, (some const L to be given)

where η is chosen so that the ball of radius η around $\tilde{u}$ lies entirely within $\mathcal{U}$.

<u>Thm</u>: $\exists$ positive constants δ and ν such that if the initial data u_0 is chosen so that

$$S(J_0) \leq \delta, \quad \bar{V}(J_0) \leq \nu$$

then u_Δ can be defined $\forall$ time, and on any I-curve J we have

$$S(J) \leq P(J) \leq \eta, \quad \bar{V}(J) \leq 2\nu$$

<u>Here</u>: $S(J)$ is a measure of the

$$\sup |u_\Delta - u_0(-\infty)|$$

given by

$$(*) \quad S(J) = \sup_{p,\gamma} \left| \sum_{i=1}^{M} \gamma_i \right|,$$

where $\gamma = (\gamma_1, \dots, \gamma_M)$ is any sequence of <u>consecutive</u> <u>p-waves</u> along J.

(Eg., take $u_0(-\infty) = \tilde{u}$, & (*) measures the supnorm distance from $u_{\Delta x}$ to $\tilde{u}$)

To prove the theorem we define a <u>new potential</u> $P(J)$ for $S(J)$ that satisfies

(A) $$S(J) \le P(J) \le S(J)\left(1 + K\bar{V}(J)\right),$$

and

$$P(J_+) - P(J_-) \le L \underbrace{D\, P(J_-)}_{\text{3rd order error}} \; ;$$

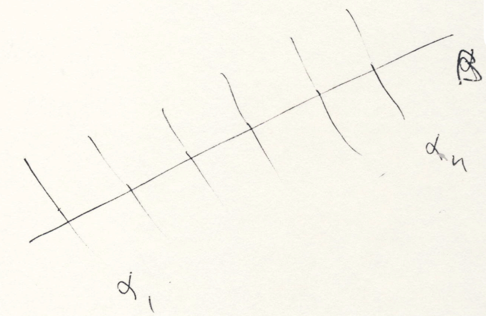
$\Leftrightarrow$

(B) $$P(J_+) \le P(J_-)(1 + LD)$$

Some constant L (that appears in the theorem).

The theorem <u>follows from</u> (A), (B). The construction of P is very technical.

<u>The idea</u>:



If a j-wave β crosses a string of consecutive 1-waves $\alpha_1 \cdots \alpha_n$, then wave on order of $\alpha_i \beta\, [R_1, R_j]$ are generated. But if supnorm $|u_\Delta - \tilde{u}| << 1$, then the waves $\alpha_1 \cdots \alpha_n$ must alternate in sign $\Rightarrow$ the reflected waves must alternate in sign.

$\Rightarrow$ even tho $\Sigma |\alpha_i \beta|$ is quadratic,

$|\Sigma\, \alpha_i \beta\, [R_i R_j]|$ is cubic. $\Rightarrow$ "cannot generate large supnorm"

Idea:

- $$P(J) = \sup_{\tau \in A(\gamma)} S(i^{\tau}(\gamma))$$

where $i^{\tau}(\gamma)$ denotes the sequence of waves obtained from γ by interchanging waves and including the quadratic interaction errors given by (Y), according to permutation τ.

- Only certain admissible "interaction maps" i^{τ} are allowed (ie. approaching waves can un-approach, but un-approaching waves cannot approach.)
- It turns out that P satisfies (A) & (B). [Pf see Young - beautiful !]

Proof: For the induction, suppose

$$S(J_0) \leq \delta,$$

and

$$V(J_0) \leq \nu = \frac{1}{3K}.$$

By previous argument we have

$$G(J_+) - G(J_-) \leq 0 \quad \forall\, J_-\, J_+,$$

and

$$V(J) \leq 2\nu \quad \forall\, J.$$

Consider now

$$P(J_+) \leq P(J_-)(1 + LD)$$

and so moving thru a sequence of successors backwards from $P(J_-)$ we obtain

$$P(J_-) \le P(J_0) \prod_{\Delta} (1 + L D_\Delta) \quad (\Delta \overset{\text{below}}{<} J_-)$$

$$\le P(J_0) \exp\{L \sum D_\Delta\}$$

where the sum is over all diamonds Δ preceding J_-. But

$$\sum_{\Delta < J_-} D_\Delta \le 3\,(Q(J_0) - Q(J_-))$$

$$\le 3\,Q(J_0) \le 3\,\bar{V}(J_0)^2 \le \frac{1}{3K^2}$$

because

$$Q(\tilde{J}_+) - Q(\tilde{J}_-) \le D_\Delta (K \bar{V}(\tilde{J}_-) - 1)$$

$$\le D_\Delta \left(K \tfrac{2}{3K} - 1\right) \le -\tfrac{1}{3} D_\Delta .$$

Thus

$$P(J_-) \le P(J_0) \exp\left(L \frac{1}{3K^2}\right)$$

But:

$$P(J_0) \le \underbrace{S(J_0)(1 + K\bar{V}(J_0)) \le \tfrac{4}{3}\,\delta}_{\text{assumption ✓}} \le \eta \exp\left\{-\frac{L}{3K^2}\right\}$$

So

$$S(J_+) \le P(J_+) \le P(J_0) \exp\left(\frac{L}{3K^2}\right)$$

$$\le \tfrac{4}{3}\,\delta \exp\left(\frac{L}{3K^2}\right) \le \eta$$