# SECTION-1
# The Polytropic Equation of State
# and
# The Speed of Sound

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## Math-280: A Mathematical Introduction
## to
## Shock Waves

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## Kinetic Theory of Gases: (Classical)

1. $p = \frac{2}{3} n \langle \frac{1}{2} m v^2 \rangle$ for ideal gas in equilibrium

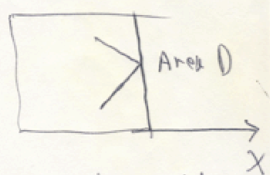
$p$: pressure; $n$: $\frac{\#\text{ particles}}{\text{vol}}$; $\langle \frac{1}{2} m v^2 \rangle$: average KE of CM of particle

"Proof" Assume gas consists of collections of particles moving at different velocities

$V_A = (V_A^x, V_A^y, V_A^z) \equiv (V_A^1, V_A^2, V_A^3) =$ vel of A-part.

$m_A \equiv$ mass of A-part

$N_A \equiv \frac{\#\text{ A-particles}}{\text{vol}}$




- Consider pressure due to A-particles on piston at $x = \text{const}$, area $D$
- Each particle delivers $2 m_A V_A^x$ of momentum after collision with $D$.
- In time $dt$, particles within $dx = V_A^x dt$ will hit $D$
- $N_A D dx = N_A D V_A^x dt =$ # that hit $D$ in time $dt$
- $(N_A D V_A^x dt) 2 m_A V_A^x =$ total mom. delivered to $D$ in time $dt$



- $F = \frac{d(\text{mom})}{dt} = 2 N_A D\, m_A (V_A^x)^2 =$ force on $D$

$p_A = \frac{F}{D} = 4 N_A \left(\frac{1}{2} m_A (V_A^x)^2\right)$ [only applies if $V_A^x$ is toward piston]

- Thus $p = \sum_A p_A = \sum_A 4 N_A \left(\frac{1}{2} m_A (V_A^x)^2\right)$

$$= 2n \langle \tfrac{1}{2} m (V^x)^2 \rangle$$

where

$$\langle \tfrac{1}{2} m (V^x)^2 \rangle = \frac{\sum_A N_A \left(\frac{1}{2} m_A (V_A^x)^2\right)}{n}$$

= Average KE of C.M. motion of molecule.

$n =$ # density = total # of particles per vol.

<u>Note</u>: we lost factor 2 because half of all molecules move <u>away</u> from $D$.

- In equilibrium, $\langle \frac{1}{2}m(v^x)^2\rangle = \langle \frac{1}{2}m(v^y)^2\rangle$

$$= \langle \tfrac{1}{2}m(v^z)^2\rangle$$

since no preferred direction

$$= \tfrac{1}{3}\langle \tfrac{1}{2}mv^2\rangle$$

$$\Rightarrow \boxed{p = \frac{2}{3} n \langle \tfrac{1}{2}m v^2\rangle}$$

2 The temperature is proportional to the ave KE. of the particles

$$\langle \tfrac{1}{2}mv^2\rangle = \tfrac{3}{2}kT \quad \text{defines } k$$

$k \equiv$ Boltzmann's Const relates Temp to ave KE. of a particle.

$\Rightarrow$ Ideal Gas Law: $p = nkT$

or, macroscopically, if $\exists$ $N$ particles in vol $V$,

$$\boxed{pV = NkT} \qquad \text{Boyle's Law}$$



- If we let $\tilde{N}$ = # of moles, then

$$Nk = \tilde{N}R_0$$

& it is written

$$pV = \tilde{N}R_0T$$

- Energy is stored in the vibrations of complicated molecules. Each degree of freedom stores same amt energy (on average), of which KE of center of mass contains 3 degrees of freedom $\Rightarrow \frac{1}{3}\left(\frac{3}{2}kT\right) = \frac{1}{2}kT$ is the KE. stored in each degree of freedom.

[I.e. as you raise the temp, the collisions distribute the KE. equally among all degrees of freedom: but only the KE. in the C-M motion of molecule will contribute to the pressure thru impact with the wall, because the internal vibrations are constrained to have equal & opp momenta

Thus, the internal energy stored in the motions and vibrations of an r-atom molecule at temp T is <u>on average</u> (including KE!)

(3) $$U = N\,3r\,\tfrac{1}{2}kT$$ ⟺ "Internal energy is prop to temp"

$$\frac{2}{3r}U = NkT$$

Says only 3 of the 3r degrees of freedom that contribute to U also contribute to pressure

so the ideal gas law reads

Ideal gas law $$\boxed{pV = \frac{2}{3r}U = (\gamma-1)U}$$ (macroscopically)

$$\boxed{\begin{array}{l}\gamma - 1 = \frac{2}{3r} \Rightarrow \gamma = 1 + \frac{2}{3r} \\ \gamma \to 1 \text{ as } r \gg 1 \\ \Rightarrow 1 < \gamma \le \frac{5}{3}\end{array}}$$

For a monatomic gas,

$$pV = \frac{2}{3}U \Rightarrow \gamma = 5/3$$

If we divide the total mass M, we get local law

$$\boxed{pv = \frac{2}{3r}e} \Rightarrow \boxed{(\gamma-1)e = \underbrace{\frac{N}{M}k}_{R}T = RT}$$

$v$ = specific volume $= \frac{1}{\rho} = \frac{\text{vol}}{\text{mass}}$

$e$ ≡ specific internal energy $= \frac{\text{energy}}{\text{mass}}$



• <u>Note</u>: Air at stand temp press ≈ ideal gas
Diatomic molecules $(N_2, O_2)$

$$\Rightarrow r = 2 \Rightarrow \gamma - 1 = \frac{2}{6} = \frac{1}{3} \Rightarrow \gamma = \frac{4}{3}$$

$\gamma \approx 1.33$

(Actually, modelled with $\gamma = 1.4$)

This would be classical theory - in fact, air ≈ $N_2$ $O_2$ & QM ⇒ only rot' degrees of freedom ⇒ 3+2 = 5 degrees not 6 ⇒ $\gamma - 1 = \frac{2}{5}$ ⇒ $\gamma \approx 1.4$

• The <u>equation of state</u> in statistical mechanics is

$$p = f(\rho, T)$$

In an ideal gas, ∄ local interactions

Major problem of stat mech's — find the eqn of state as a function of local interactions (eg inverse square force, etc). Except very <u>simple cases</u>, f can only be approximated

— Craig Tracy

⑦

- Equation of state for idea gas:

Assume: $pv = RT = (\gamma-1)e$ $\quad \gamma - 1 = \frac{2}{3r}$ $\quad (*)$

so that $e = \frac{R}{\gamma-1}T$. (Internal energy $\propto$ temperature)

Defn ① $C_v \equiv$ spec. ht at const volume

$\equiv$ "heat required to raise unit mass 1 degree at constant volume"

Since $dv = 0$, all energy of heating goes into $e$, so

$$C_v = \frac{de}{dT} = \frac{R}{\gamma-1}$$

- Thermodynamics: $de = Tds - pdv$ (2nd Law Thermo)

(This introduces entropy as a state variable)

2nd Law says: $\frac{\partial e(s,v)}{\partial s} = T \quad \frac{\partial e(s,v)}{\partial v} = -p$

The existence of an (integrable) state variable s follows from $\nexists$ perpetual motion machines, but we show that 2nd law assuming $(*)$ can be integrated!

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To integrate 2nd law, introduce free energy:

$$\psi = e - sT$$

$$d\psi = de - sdT - Tds$$

But: 2nd Law $\Rightarrow$ $Tds = de + pdv$

$$d\psi = \cancel{de} - sdT - \cancel{de} - pdv$$

$$d\psi = -sdT - pdv$$

$$\therefore \boxed{\frac{\partial \psi(T,v)}{\partial T} = -s, \quad \frac{\partial \psi(T,v)}{\partial v} = -p}$$

By $(*)$, $p(T,v) = \frac{RT}{v}$ $\Rightarrow$ can integrate

$$\frac{\partial \psi(T,v)}{\partial v} = -\frac{RT}{v}$$

$$\boxed{\psi(T,v) = -RT\ln v + g(T)}$$

Some fn $g(T)$.

Thus

$$s = -\frac{\partial \psi}{\partial T} = -\left(-R \ln v + g'(T)\right) = R \ln v - g'(T)$$

Moreover,

$$C_v T = e = \psi + sT = -RT \cancel{\ln v} + g(T) + RT \cancel{\ln v} - T g'(T)$$

$$C_v T = g(T) - T g'(T)$$

diff: $C_v = \cancel{g'(T)} - \cancel{g'(T)} - T g''(T)$

$$g''(T) = -\frac{C_v}{T}$$

$$g'(T) = -C_v \ln T + \text{Const.}$$

$\Rightarrow$ $s = R \ln v + C_v \ln T + \cancel{\text{Const.}} \to 0$ (Only changes in entropy can be measured)

$$= C_v\{(\gamma - 1)\ln v + \ln T\}$$

$$\boxed{s = C_v \ln(v^{\gamma-1} T)}$$ equation of state



<u>But</u>: $e = C_v T$

and $s = C_v \ln(v^{\gamma-1} T) \Rightarrow T = v^{1-\gamma} \exp(s/C_v)$

$$\Rightarrow \boxed{e = C_v \frac{1}{v^{\gamma-1}} \exp\left(\frac{s}{C_v}\right) \equiv e(s,v)}$$

2nd law: $de = T ds - p dv \Rightarrow$

$$\boxed{p = -\frac{\partial e}{\partial v}(s,v) = C_v(\gamma - 1) \frac{1}{v^{\gamma}} \exp(s/C_v)}$$

"Equation of state for polytropic or $\gamma$-law gas"

- Meaning of $\gamma$:

<u>Defn</u>: $C_p \equiv$ "specific heat at constant pressure"
$\equiv$ Energy required to raise unit mass 1 deg at const pressure

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<u>Claim</u>: $\gamma = \frac{C_p}{C_v}$ (For this let $E \equiv \frac{\text{total energy}}{\text{mass}}$ in vol $\bar{V}$)

Pf. $dE = de + P\,dV = C_v\,dT + P\,dV$

$\uparrow$ change in tot energy; $\uparrow$ $\Delta$ internal energy; $\uparrow$ work done by expansion

Now: $PV = RT \Rightarrow V = \frac{RT}{p}$

Thus: at constant pressure, $\frac{dV}{dT} = \frac{R}{p}$

But

$$C_p = \left.\frac{dE}{dT}\right|_{P=\text{const}} = C_v + p\left.\frac{dV}{dT}\right|_{p=\text{const}} = C_v + R$$

$$C_p = C_v + R = C_v + (\gamma-1)C_v = \gamma C_v$$

$$\boxed{\gamma = \frac{C_p}{C_v}}$$

Note: $C_v$, $C_p$ easy to measure.

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We now show that the compressible Euler equations reduce to the linear theory of sound in the limit of weak signals

- Said differently: we use the polytropic equation of state together with CEE's to derive the speed of sound – A major open problem in the time of Newton, first resolved by Euler $\approx 1750$
- Compressible Euler Equations:

(MA) $\rho_t + \mathrm{div}(\rho u) = 0$

(MO) $(\rho u)_t + \mathrm{div}(\rho u \otimes u + PI) = 0$

(En) $E_t + \mathrm{div}((E+P)u) = 0$

($\mathcal{S}$) $\mathcal{S}_t + \mathrm{div}(\mathcal{S}u) = 0$

- 1st note: $\mathcal{S} = \rho s = \frac{\text{entropy}}{\text{vol}}$ = entropy density,

$s$ = specific entropy = $\frac{\text{entropy}}{\text{mass}}$

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Thus (S) $\Rightarrow$

$$0 = (\rho S)_t + \operatorname{div}(\rho S u) = \rho_t S + \rho S_t + S \operatorname{div}(\rho u) + \rho \nabla S \cdot u$$

$$= S(\rho_t + \operatorname{div} \rho u) + \rho (S_t + \nabla S \cdot u)$$

$$\Rightarrow \boxed{S_t + \nabla S \cdot u = 0}$$

- Now the fluid particles follow trajectories $x(t)$ satisfying

$$\frac{dx}{dt} = u(x(t), t)$$

Thus $\frac{d}{dt} S(x(t), t) = \nabla S \cdot \dot{x}(t) + S_t$

$$= S_t + \nabla S \cdot u = 0$$

Conclude: $S \equiv$ const along particle paths $\Rightarrow$

<u>Thm</u>: If $S \equiv$ constant at $t = 0$, then $S \equiv$ const for all time in solutions of (MA), (MO), (En) & (S).

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This justifies statement that Compressible Euler is <u>reversible</u> on smooth sol'n's.

② The sound speed: (linearize equations) (Use $P = P(\rho, S)$, $S \equiv$ const)

$$\rho = \rho_0 + \varepsilon \tilde{\rho}(x,t) \qquad S \equiv S_0$$

$$u = \varepsilon \tilde{u}(x,t) \qquad \text{(sol'n near } u = 0)$$

$$P(\rho, S_0) = P(\rho_0 + \varepsilon\tilde{\rho}, S_0) = P(\rho_0) + \frac{\partial}{\partial \rho} P(\rho_0, S_0)\varepsilon\tilde{\rho} + O(\varepsilon^2)$$

Plug into (MA), (MO):

(MA) $\quad (\rho_0 + \varepsilon\tilde{\rho})_t + \operatorname{div}\left((\rho_0 + \varepsilon\tilde{\rho})(\varepsilon\tilde{u})\right) = 0$

(1) $\quad \varepsilon\tilde{\rho}_t + \varepsilon\rho_0 \operatorname{div}\tilde{u} = 0 \quad$ (neglecting $O(\varepsilon^2)$)

(MO) $\quad 0 = \left[(\rho_0 + \varepsilon\tilde{\rho})(\varepsilon\tilde{u})\right]_t + \operatorname{div}\left[(\rho_0 + \varepsilon\tilde{\rho})\varepsilon\tilde{u} \otimes \varepsilon\tilde{u} + P(\rho_0) I + \frac{\partial}{\partial\rho} P(\rho_0, S_0)\varepsilon\tilde{\rho} I + O(\varepsilon^2)\right]$

(2) $\quad \varepsilon \rho_0 \tilde{u}_t + \varepsilon \frac{\partial}{\partial\rho} P(\rho_0, S_0) \operatorname{div}(\tilde{\rho} I) = 0 \quad$ (neglecting $O(\varepsilon^2)$)

diff (1): $\quad \tilde{\rho}_{tt} + \rho_0 \operatorname{div}\tilde{u}_t = 0 \qquad \tilde{u}_t = -\frac{1}{\rho_0}\frac{\partial}{\partial\rho} P(\rho_0, S_0)\operatorname{div}(\tilde{\rho} I)$

$$= -\frac{1}{\rho_0}\frac{\partial}{\partial\rho} P(\rho_0, S_0)\nabla\tilde{\rho}$$

$$\boxed{\tilde{\rho}_{tt} - \frac{\partial}{\partial\rho} P(\rho_0, S_0)\Delta\tilde{\rho} = 0}$$

Conclude: The sound speed for compressible Euler is:

$$\sigma = \sqrt{\frac{\partial p}{\partial \rho}(\rho, s)}$$

This holds true in non-linear problem as well:

For $\gamma$-law gas,

$$p(v,s) = C_v(\gamma-1)\frac{1}{v^\gamma}\exp(s/C_v)$$

$$\Rightarrow \quad p(\rho,s) = C_v(\gamma-1)\rho^\gamma \exp(s/C_v)$$

$$\Rightarrow \quad \boxed{\sigma^2 = \frac{\partial p}{\partial \rho}(\rho,s) = C_v\gamma(\gamma-1)\rho^{\gamma-1}\exp(s/C_v)}$$

Note: $e = C_v \rho^{\gamma-1}\exp(s/C_v)$

$$\Rightarrow \quad \sigma^2 = \gamma(\gamma-1)e = \gamma(\gamma-1)C_v T$$

$\Rightarrow$ sound speed $\propto \sqrt{T}$ ✓