SECTION-4
Simple Waves, Genuine Nonlinearity
and
Hyperbolic Systems of Conservation Laws

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Math-280: A Mathematical Introduction
to
Shock Waves

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Simple Waves:

(CL) $u_t + f(u)_x = 0$ $\qquad f = (f_1, \dots, f_n)$, $u = (u_1, \dots, u_n) \equiv u(x,t)$

- Look for a smooth solution of (CL) of form

(S) $$u(\sigma(x,t))$$

where σ is a scalar function of x and t. Then u is constant where σ is constant. Note that for a given σ,

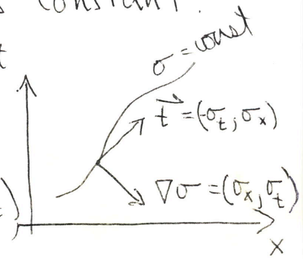
$$\sigma = \text{const}$$

defines the cure $\perp$ to $\nabla\sigma = (\sigma_x, \sigma_t)$ so $\sigma = \text{const}$ is tangent to

$$\vec{t} = (-\sigma_t, \sigma_x),$$

and so curves $\sigma = \text{const}$ move at speed

$$\frac{dx}{dt} = -\frac{\sigma_t}{\sigma_x}.$$



Plugging (S) into (CL) gives:

$$u'\sigma_t + df\, u'\sigma_x = 0$$

$$\left[df + \frac{\sigma_t}{\sigma_x} I\right] u' = 0 \qquad \text{(SW)}$$

$u'(\sigma)$ must be an evector while speed $\frac{dx}{dt} = -\frac{\sigma_t}{\sigma_x}$ along which $\sigma = \text{const}$ must be evalue

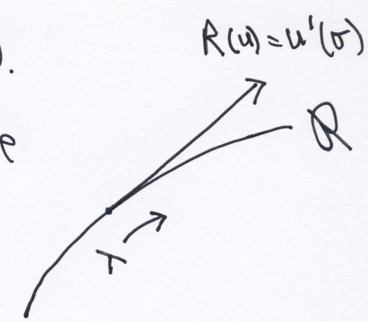
Thus: Let $(\lambda(u), R(u))$ be an e-pair,

$$[df - \lambda I]R = 0,$$

and let $u(\sigma)$ denote a parameterization of an integral curve $\mathcal{R}$ of e-vector field $R(u)$ satisfying

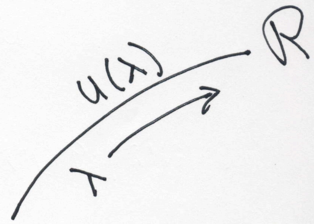
$$\frac{du}{d\sigma} = R(u(\sigma)).$$

Now assume λ is monotone along $\mathcal{R}$



• Since λ monotone, we can wlog assume $\sigma = \lambda$ and ~~set~~ rescale R so that

$$\frac{du(\lambda)}{d\lambda} = R(u(\lambda))$$



• Now define $\lambda(x,t)$ so that "λ is constant along lines of speed λ". Then with $\lambda = \sigma$ we have

$$\lambda = -\frac{\lambda_t}{\lambda_x}$$

So (SW) implies

$$\left[df + \frac{\lambda_t}{\lambda_x} I\right]u' = \left[df - \lambda I\right]R = 0$$

Theorem: If we make a state $u(\lambda)$ propagate as constant along lines of speed $\frac{dx}{dt} = \lambda$ in the xt-plane, then

$$u(\lambda(x,t)) \text{ solve (CL).}$$

Such a soln is called a λ-simple wave.

Proof: Assume $u(\lambda(x,t))$ solves (CL). Then

$$u_t + f(u(\lambda(x,t)))_x = 0$$

$$\iff u'(\lambda)\lambda_t + df\, u'(\lambda)\lambda_x = 0$$

$$\iff u'(\lambda)\lambda_t + \lambda u'(\lambda)\lambda_x = 0$$

$$\iff u'(\lambda)\left[\lambda_t + \lambda \lambda_x\right] = 0$$

<u>Conclude</u>: If λ solves the Burgers equation $\lambda_t + \lambda\lambda_x = 0$, which says "$\lambda$ is constant along lines of speed λ" then $u(\lambda(x,t))$ solves (CL), & "u is constant value $u(\lambda)$ along lines of speed λ" ✓

Recall we have a blowup result for Burgers:

<u>Thm</u>: Solutions of $\begin{cases} u_t + uu_x = 0 \\ u(x,0) = u_0(x) \end{cases}$

will blow up in the derivative (form a shock-wave) before time

$$T = \frac{1}{\max(-u_0'(x))}$$

<u>Cor</u>: a simple wave $u(\lambda(x,t))$ must blow up in deriv before time $T = \frac{1}{\max(-\lambda_x(x,0))}$

Pf. $\lambda(x,t)$ solves $\begin{cases} \lambda_t + \lambda\lambda_x = 0 \\ \lambda(x,0) \equiv \lambda_0(x) \end{cases}$ ✓

<u>Note</u>: we can get a blowup time for simple waves in terms of i-data $u_0(x)$:

Given $u(\lambda)$, $u'(\lambda) = R(u(\lambda))$, let

$$\Lambda \equiv \underset{u(\lambda)\in R}{\mathrm{Max}} \|u'(\lambda)\| \quad \left(\Lambda \to \infty \text{ as } \lambda \text{ becomes const along } R\right)$$

so Λ is a measure of the "strength" of nonlinearity

Then

$$u_0(x) = u(\lambda(x,0))$$

$$u_0'(x) = u'(\lambda)\cdot\lambda_x(x,0)$$

$$\frac{u_0'(x)\cdot r_0(x)}{\|u'(\lambda)\|} = \lambda_x(x,0)$$

where $r_0(x) = \dfrac{u'(\lambda(x,0))}{\|u'(\lambda)\|}$ = unit eigenvector at $t=0$

Then
$$M = \mathrm{Max}\{-\lambda_x(x,0) : \lambda_x < 0\}$$
$$= \mathrm{Max}\left\{-\frac{u_0'(x)\cdot r_0(x)}{\|u'(\lambda)\|} : u_0'\cdot r_0 < 0\right\}$$
$$\geq \mathrm{Max}\left\{-\frac{u_0'(x)\cdot r_0(x)}{\Lambda} : u_0'\cdot r_0 < 0\right\}$$

So blowup occurs before time

$$T = \frac{1}{\mathrm{Max}\{-\lambda_x(x,0)\}} \leq \frac{1}{\mathrm{Max}\left\{-\frac{u_0'(x)\cdot r_0(x)}{\Lambda}\right\}}$$

Maxed over $\lambda_x < 0$, $u_0'(x)\cdot r_0 < 0$

Defn ①: An eigenfamily (λ, R) of df is called genuine nonlinear (named after Lax 1957) if $\boxed{\nabla\lambda \cdot R \neq 0}$
(we say λ is G.N.)

Note: This says λ is monotone along integral curves $\mathcal{R}$ of $R(u)$, so $\mathcal{R}$ can be parameterized by λ & we can construct simple wave solutions

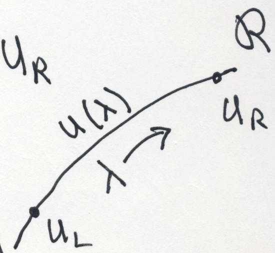
$$u(\lambda(x,t))$$
$$\lambda_t + \lambda\lambda_x = 0$$

Note: If λ is G.N. and $u(\lambda)$ is a param. of $\mathcal{R}$ by λ, then $\lambda = \lambda(u(\lambda))$ so

$$(\lambda) \qquad 1 = \nabla\lambda \cdot u'(\lambda) = \nabla\lambda \cdot R$$

Example: Let (λ, R) be a smooth G.N. family for (CL). Let u_L and u_R be two points on the same integral curve $\mathcal{R}$ of $R(u)$, and assume

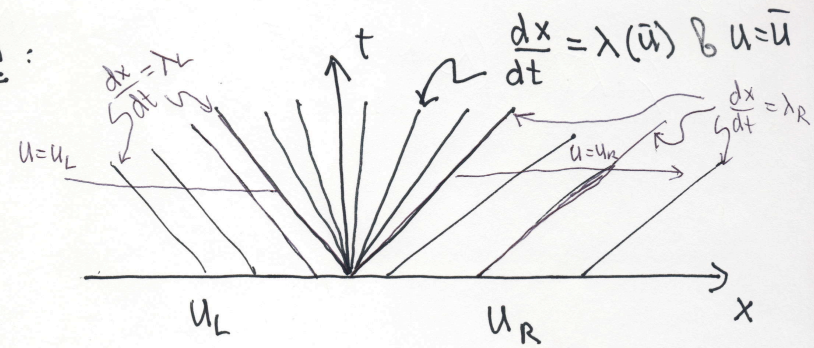
$$\lambda_L = \lambda(u_L) < \lambda(u_R) = u_R$$



Then we can solve the Riemann Problem $\Leftrightarrow$ (CL) w. i-data

$$\text{(RP)} \quad u(x,0) = u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x \geq 0. \end{cases}$$

Soln:

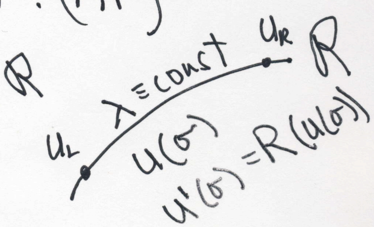


- That is: Let $u(x,t)$ take values $u(\lambda)$ along lines of speed λ. Since λ increases from left to right, the prescription is consistent, and the wave takes u_L to u_R.
- This is called a rarefaction wave or a centered simple wave
- Note: this succeeds because λ increases from u_L to u_R along $\mathcal{R}$.

$\Rightarrow$ No such wave exists if u_R is on the left & u_L on the right:

Defn: We say (λ, R) is linearly degenerate if $\nabla\lambda \cdot R \equiv 0 \iff \lambda$ is constant along the integral curves of $\mathcal{R}$. Then we can construct simple waves by letting $u \in \mathcal{R}$ propagate with speed $\frac{dx}{dt} = \lambda \equiv \text{const}$. (FIP)

Example: For any $u_L, u_R \in \mathcal{R}$ we can solve $\approx$ Riemann Problem consisting of (CL) together with i-data



$$u_0(x) = \begin{cases} u_L & x < -\varepsilon \\ u_R & x \geq \varepsilon \end{cases}$$

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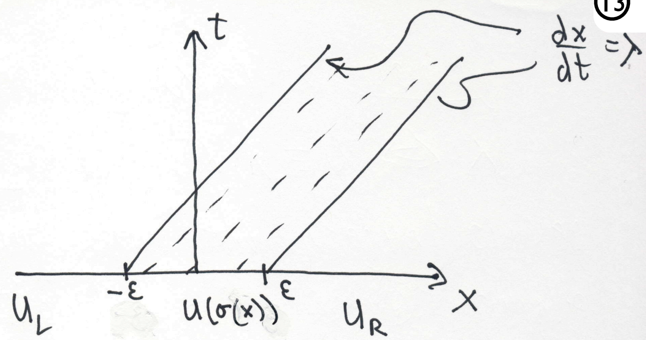
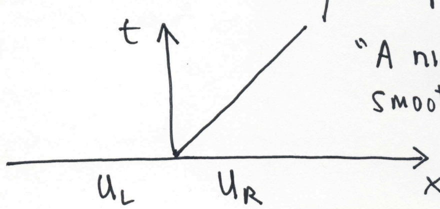
Soln:



Here $\sigma(x)$ can be any smooth fn such that $\sigma(-\varepsilon) = u_L$ & $\sigma(\varepsilon) = u_R \Rightarrow$ smooth transition from u_L to u_R.

- Conclude: in limit $\varepsilon \to 0$ we get nice convergence to the soln of the R.P. which is a contact discontinuity of speed $\frac{dx}{dt} = \lambda$



"A nice limit of smooth soln's"

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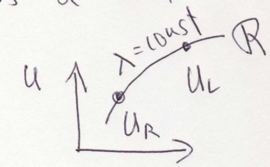
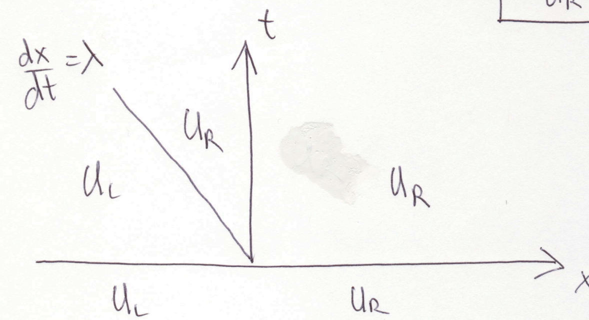
Thm: If (λ, R) is a linearly degenerate eigenfamily, then $\forall\, u_L, u_R \in \mathcal{R}$ we can solve the R.P. with a contact discontinuity of speed $\frac{dx}{dt} = \lambda$. This will be a weak soln because it is a nice limit of smooth soln's





Defn ③: We say (CL) is <u>strictly hyperbolic</u> if at each $u \in \mathbb{R}^n$ we have

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

I.e., df has n real & distinct evals at every pt.

Cor: $df(u)$ has a <u>basis</u> of e-vectors at every u (FIP)

Note: This says $\exists$ n-distinct families of waves moving (locally) at distinct speeds.

Conclude: A system of conservation laws $u_t + f(u)_x = 0$ which is strictly hyperbolic, and such that each characteristic (eigen) family is either G.N. or L.D. is a class of equations that is a natural generalization of C.Euler (Lax ~57)

Defn: a (constant) $n\times n$ matrix A is <u>strictly hyperbolic</u> if it has n real & distinct evals —

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

Such matrices share similar properties to symmetric matrices

Thm ① $\exists$ a basis of e-vectors (FIP)

Thm ② If $R_1 \cdots R_n$ is a basis of e-vectors, Let $M_k = \mathrm{Span}_{i \neq k}\{R_i\}$, and set $L_k = M_k^{\perp}$

Then L_k is a left evector & $L_k^t A = \lambda_k L_k^t$

Cor: $\exists$ basis of left eigenvectors $\{L_1, \ldots, L_n\}$

P.f. of Thm (2) Let $L, R \in \mathbb{R}^n$ & consider the product $L^t A R$ which we can write

$$\langle L^t A, R \rangle = \langle L^t, AR \rangle$$

$$\langle x^t, y \rangle = \underset{\text{matrix}}{x^t \cdot y} = \underset{\text{dot}}{x \cdot y}$$

(If A symmetric, then $\langle AL, R \rangle = \langle L, AR \rangle$)

<u>Claim</u>: if $L_k \in M_k^\perp$ then $L_k^t \cdot A \in M_k^\perp$

I.e. $L_k \in M_k^\perp$ iff $L_k \cdot R_j = 0 \ \forall j \neq k$. But $L_k \in M_k^\perp \Rightarrow$

$\langle L_k^t A, R_j \rangle = \langle L_k^t, AR_j \rangle = \langle L_k^t, \lambda_j R_j \rangle = 0$

if $j \neq k$ so $L_k^t A \in M_k^\perp$ as well.

Thus $A : L_k^t \in M_k^\perp$ to $L_k^t A \in M_k^\perp$.

Since $M_k^\perp$ is 1-dimensional, let $L_k \in M_k^\perp$ be fixed, so that $\forall k$

$$L_k^t A = \mu_k L_k^t$$

for some μ_k. We show $\mu_k = \lambda_k$. I.e.,

$$\mu_k \langle L_k^t, R_k \rangle = \langle \mu_k L_k^t, R_k \rangle = \langle L_k^t A, R_k \rangle$$
$$= \langle L_k^t, AR_k \rangle = \langle L_k^t, \lambda_k R_k \rangle$$
$$= \lambda_k \langle L_k^t, R_k \rangle$$

Since $\langle L_k^t, R_k \rangle \neq 0$ ($\{R_j\}$ is a basis & $L_k \perp M_k$) it follows that $\mu_k = \lambda_k$ and so

$$L_k^t A = \lambda_k L_k^t \quad \text{for } L_k \in M_k^\perp \quad ✓$$

(19)

<u>Note</u>: $L_k = M_k^\perp$ must be the only left e-vector with e-val λ_k because $\{L_k\}$ forms a basis of e-vectors with diff e-vals.
In particular, note that if $L_k^t A = \lambda_k L_k^t$, then

$$\lambda_k L_k^t \cdot R_j = L_k^t A R_j = L_k^t \lambda_j R_j$$

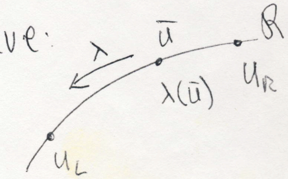
so $\lambda_k \neq \lambda_j \Rightarrow L_k \perp R_j \Rightarrow L_k = M_k^\perp$ ✓

(20)

skip

<u>Note</u>: The rarefaction wave soln is ~~of~~ only Lipschitz continuous along lines $\frac{dx}{dt} = \lambda_L$ and $\frac{dx}{dt} = \lambda_R$ through the origin. We could smooth it out by smoothing the discont.

<u>Example</u> (2) Let (λ, R) be a smooth, genuinely nonlinear family for (CL). Let u_L and u_R be points on the same integral curve of R, such that $\lambda_L = \lambda(u_L) > \lambda(u_R) = \lambda_R$.
Then the following is a simple wave:



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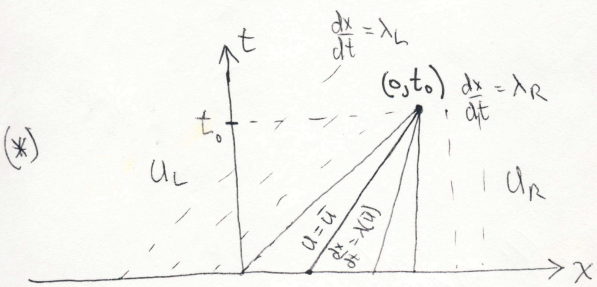

<u>Note</u>: This is a simple wave because states $\bar{u}$ on R propogate with speed $\lambda(\bar{u})$. If we let $u(\lambda)$ denote the parameterization of R

by λ, then for $t<t_0$, (*) is a smooth solution of the initial value problem (CI) together with

HW

$$\text{(ID)} \qquad u_0(x) = \begin{cases} u_L & x \leq -\lambda_L t_0, \\ u\left(-\frac{x}{t_0}\right). & -\lambda_L t_0 \leq x \leq -\lambda_R t_0, \\ u_R & x \geq -\lambda_R t_0. \end{cases}$$

I.e., $x = \bar{\lambda} t + b$ defines the line of speed $\bar{\lambda}$ along which the solution value is $\bar{u}$. Since this line goes through $(0, t_0)$, we have

$$0 = \bar{\lambda} t_0 + b$$

$$b = -\bar{\lambda} t_0 .$$

Thus $x = \bar{\lambda} t - \bar{\lambda} t_0$ is the line on which $u = \bar{u}$, and so at $t=0$, $x = -\bar{\lambda} t_0$ is the point where $u_0(x) = \bar{u} = u(\bar{\lambda}) = u\left(-\frac{x}{t_0}\right)$ ✓

Now differentiating the relation

$$u_0(x) = u\left(-\frac{x}{t_0}\right),$$

we obtain

$$u_0'(x) = u'\left(-\frac{x}{t_0}\right)\left(-\frac{1}{t_0}\right),$$

or

$$(t_0) \qquad t_0 = \frac{\left\| u'\left(-\frac{x}{t_0}\right)\right\|}{\| u_0'(x) \|}.$$