SECTION-5
The Eigenfamilies and Simple Waves of Compressible Euler

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Math-280: A Mathematical Introduction to Shock Waves

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☐ The structure of simple waves for gas dynamics

(E)
$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + ((E+p)u)_x = 0$$

for smooth solutions these are equiv. to

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$(\rho s)_t + (\rho s u)_x = 0$$

(L)
$$V_t - u_x = 0$$
$$u_t + p_x = 0$$
$$\mathcal{E}_t + (pu)_x = 0$$

for smooth solutions these are equiv. to

$$V_t - u_x = 0$$
$$u_t + p_x = 0$$
$$S_t = 0$$

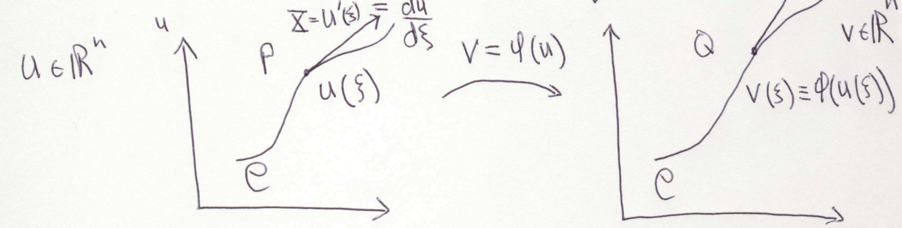
- First we see how eigenvalues and eigenvectors transform under nonlinear changes of the conserved quantities:

(CL) $$U_t + f(U)_x = 0$$

$U = g(V)$, $g: \mathbb{R}^n \to \mathbb{R}^n$ 1-1 regular

☐ Vector/Covectors & Tensors —

- Consider a curve in $\mathbb{R}^n$



Q: If the curve $\mathcal{C}$ has position $u(\xi)$ in u-coordinates and position $\varphi(u(\xi))$ in v-coordinates, how do the components of the tangent vectors $\frac{du^i}{d\xi}$ and $\frac{dv^\alpha}{d\xi}$ transform?

<u>Notation</u>: Let $e_i = \frac{\partial}{\partial u^i}$ denote the i-th basis vector in u-coordinates and $e_\alpha = \frac{\partial}{\partial v^\alpha}$ the i th coord basis vector in v-coordinates (to distinguish).

Then $X = \frac{du}{d\xi}$ has components $\frac{du^i}{d\xi}$, so

$$X = \sum_{i=1}^{n} \left(\frac{du^i}{d\xi}\right)\frac{\partial}{\partial x^i} = \underbrace{\frac{du^i}{d\xi}\frac{\partial}{\partial x^i}}_{\text{Einstein summation conv sum repeated up down indices}}$$

Also: the components of $\underline{V} = \frac{dv}{d\xi}$ are

$$\frac{dv^\alpha}{d\xi} = \frac{d}{d\xi}\varphi(u(\xi)) \Rightarrow \sum_{i=1}^{n} \frac{\partial \varphi^\alpha}{\partial u^i}\frac{du^i}{d\xi} = \frac{\partial v^\alpha}{\partial u^i}\frac{du^i}{d\xi}$$

nxn Jacobian of the transformation

since $v = \varphi(u)$

Sum repeated up-down indicies

Conclude: The representation of vector

$\underline{X} = \frac{du^i}{d\xi}\frac{\partial}{\partial u^i}$ in v-coordinates is $\underline{Y} = \frac{dv^\alpha}{d\xi}\frac{\partial}{\partial v^\alpha}$

where

$$\frac{dv^\alpha}{d\xi} = \frac{\partial v^\alpha}{\partial u^i}\frac{du^i}{d\xi} \quad (\text{sum } {}_i \leftrightarrow {}^i)$$

Defn: a vector field $\underline{X} = \underline{X}^i \frac{\partial}{\partial u^i}$ is a quantity ~~that~~ whose components transform by

$$\underline{X}^\alpha = \frac{\partial v^\alpha}{\partial u^i}\underline{X}^i$$

"let indicies keep track of coord systems $v: \alpha, \beta, \gamma \ldots$, $u: i, j, k \ldots$"

- The coordinate basis vectors $\frac{\partial}{\partial u^i}$ then transform by the inverse Jacobian

$$\frac{\partial}{\partial v^\alpha} = \frac{\partial u^i}{\partial v^\alpha}\frac{\partial}{\partial u^i}$$

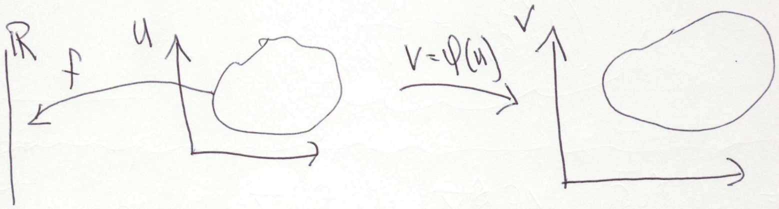
$$\frac{\partial u^i}{\partial v^\alpha} = \left(\frac{\partial v^\alpha}{\partial u^i}\right)^{-1}$$

Then eg:

$$\underline{X}^\alpha \frac{\partial}{\partial v^\alpha} = \underline{X}^i \underbrace{\frac{\partial v^\alpha}{\partial u^i} \cdot \frac{\partial u^j}{\partial v^\alpha}}_{\delta^j_i} \frac{\partial}{\partial u^j} = \underline{X}^i \frac{\partial}{\partial u^i} \quad \checkmark$$

Justify $\dfrac{\partial}{\partial v^\alpha} = \dfrac{\partial u^i}{\partial v^\alpha}\dfrac{\partial}{\partial u^i}$:

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $z = f(u)$ gives f in u-coordinates, then $z = f(\varphi^{-1}(v))$ gives f in v-coordinates



Then we want $\underbrace{\dfrac{\partial}{\partial v^\alpha} f}_{\substack{v\text{-rep of} \\ f = f(\varphi^{-1}(v))}} = \underbrace{\left(\dfrac{\partial u^i}{\partial v^\alpha}\dfrac{\partial}{\partial u^i}\right)}_{\substack{\text{rep of } \frac{\partial}{\partial v^\alpha} \text{ in} \\ u\text{-coords}}} \underbrace{f}_{\substack{u\text{-rep of } f \\ f = f(u)}}$

I.e., $\dfrac{\partial}{\partial v^\alpha} f = \dfrac{\partial}{\partial v^\alpha} f(\varphi^{-1}(v)) = \dfrac{\partial f}{\partial u^i}\dfrac{\partial u^i}{\partial v^\alpha} = \left(\dfrac{\partial u^i}{\partial v^\alpha}\dfrac{\partial}{\partial u^i}\right) f$ ✓

Defn: $X = X^i \dfrac{\partial}{\partial u^i}$ a vector field

X^i transforms contravariantly index up

$\left(X^i = X^\alpha \dfrac{\partial u^i}{\partial v^\alpha}\right)$ (1,0)-tensor

$\dfrac{\partial}{\partial u^i}$ transforms covariantly index down

$\left(\dfrac{\partial}{\partial u^i} = \dfrac{\partial v^\alpha}{\partial u^i}\dfrac{\partial}{\partial v^\alpha}\right)$ (0,1)-tensor

That's all we need

More generally: a (k,ℓ) tensor $T^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}$ transforms by

$$T^{i_1 \cdots i_k}_{j_1 \cdots j_\ell} \frac{\partial v^{\alpha_1}}{\partial u^{i_1}} \cdots \frac{\partial v^{\alpha_k}}{\partial u^{i_k}} \frac{\partial u^{j_1}}{\partial v^{\beta_1}} \cdots \frac{\partial u^{j_\ell}}{\partial v^{\beta_\ell}}$$

$$= T^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_\ell}$$

In particular, eigenvalues & eigenvectors are associated with (1,1)-tensors:

$$AR = \lambda R \iff A^i_j R^j_u = \lambda R^i_u$$

matrix multiplication $\sum A^i{}_j R^j = A \cdot R$
row column

u-components of vector $X = R^i_u \frac{\partial}{\partial u^i}$, etc

Under a change of coordinates $v = \varphi(u)$,
we get

$$\underbrace{\frac{\partial u^\nu}{\partial v^\beta} A^i_j \frac{\partial v^\alpha}{\partial u^i}}_{} \cdot \frac{\partial v^\beta}{\partial u^i} R^i_u \iff B^\alpha{}_\beta \cdot R^\beta_v$$

Summation Convention for $J^{-1}AJ = B$

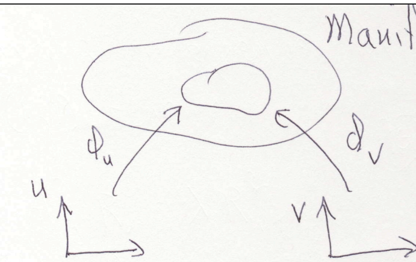
v-coord rep. of transformation

v-comp's of vector R

In General:

u, v represent different coord systems on a manifold, & the tensor transformation laws express the same tensor in different coordinates

Manifold



$$v = \varphi_v^{-1} \circ \varphi_u(u) \equiv \varphi(u)$$

- Metric $g_{ij}\, dx^i dx^j$ $\quad g_{ij} \sim (0,2)$-tensor
- 1-form $\omega_i\, dx^i$ $\quad \omega_i \sim (0,1)$-tensor
- $R^i_{jkl} \sim$ Riemann Curv. Tensor $\quad dx^i$ transforms contravariantly like comp's of vector

  transforms as a (1,3)-tensor
- Contraction: $R^\sigma_{j\sigma k} = R_{jk} \equiv$ Ricci Curvature
- Raise/Lower Indices with metric

  $R^i_{jkl} = g^{i\sigma} R_{\sigma jkl}$, $R_{ijkl} = g^{i\sigma} R^\sigma_{jkl}$, $g^{ij} = (g_{ij})^{-1}$

Ref: Adler, Bazin & Schiffer Intro to GR

Back to (U):

— (u) is equivalent to: $\boxed{(\mathcal{U})\ U_t + f(U)_x = 0}$

$U_t + df\, U_x = 0$.

Let $\lambda_1 \cdots \lambda_n$, $R_1 \cdots R_n$ be eigenvalues and corresponding e-vectors of df.

Thus

$$dg\, V_t + df\, dg\, V_x = 0$$

(V) $$V_t + dg^{-1} df\, dg\, V_x = 0.$$

Thus the eigenvalues of df are the e-values of $dg^{-1} df\, dg$, which would be dh if (V) could be written in conservation form. Let

$$A = df, \quad B = dg = \frac{\partial U^i}{\partial V^\alpha}$$

Then $AR = \lambda R$ implies

$$B^{-1}ABB^{-1}R = B^{-1}R.$$

V-comp's of e-vector R

$R^\alpha = B^{-1}R^i$ are the eigenvectors for (V), or

(R) $$B^i_\alpha R^\alpha = R^i \iff \boxed{\frac{\partial U^i}{\partial V^\alpha} R^\alpha = R^i}$$

"E-vectors transform like vectors under chnges of dep var's"

We can also check that genuine nonlinearity and linear degeneracy are coordinate independent notions. I.e.,

$$\nabla_U \lambda \cdot R = \frac{\partial \lambda}{\partial U^i} R^i = \frac{\partial \lambda}{\partial V} \underbrace{\frac{\partial V^\alpha}{\partial U^i} R^i}_{B^{-1}R} = \nabla \lambda_V \cdot R^\alpha$$

Note: (R) says that the eigenvectors of df transform like vectors under coordinate changes, so that if $U = g(V)$, then the integral curves satisfy $R_U = g(R_V)$.

- Let $U = \begin{bmatrix} v \\ u \\ E \end{bmatrix}$, $f(U) = \begin{bmatrix} -u \\ p \\ pu \end{bmatrix}$. We

obtain the e-vectors & e-values for the transformed system

$$V = \begin{bmatrix} v \\ u \\ s \end{bmatrix} = g^{-1}(U), \quad h(V) = \begin{bmatrix} -u \\ p \\ 0 \end{bmatrix}.$$

$$dh = \begin{bmatrix} 0 & -1 & 0 \\ p_v & 0 & p_s \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det|dh - \lambda I| = \det\begin{vmatrix} -\lambda & -1 & 0 \\ p_v & -\lambda & p_s \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 + p_v) = 0$$

$$(\lambda) \qquad \lambda_1 = -\sqrt{-p_v}, \quad \lambda_2 = 0, \quad \lambda_3 = +\sqrt{-p_v}.$$

For the p-system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0 \end{aligned}, \qquad \lambda_1 = -\sqrt{-p'}, \quad \lambda_2 = +\sqrt{-p'}$$

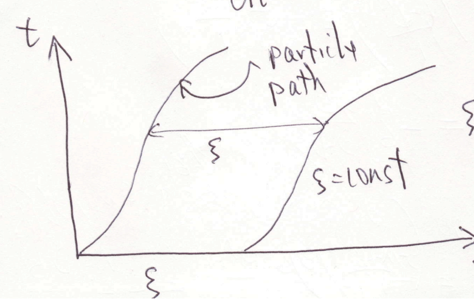
Conclude:

(1) So long as $p_v(v,s) < 0$, the system is strictly hyperbolic

(2) One wave speed is $\frac{dx}{dt} = \lambda_2 = 0$; ie., the wave moves with particle velocity. (Passive transport of temp gradients)

(3) Here x is standing in for the Lagrangian variable $\xi \longleftrightarrow x$. Letting ξ = Lagr, x = Eul then the speed of sound waves ($\frac{d\xi}{dt} = \lambda_1$ are back sound waves, $\frac{d\xi}{dt} = \lambda_3$ are forward sound waves) give (roughly) the speed of waves relative to the moving fluid: (we will see ∃ factor ρ missing...)

$$\frac{d\xi}{dt} = \pm\sqrt{-p_v}$$



$$\xi = \int_{x(\xi,0)}^{x(\xi,t)} \rho(z,t)\,dz$$

x = Eulerian coord

Compute e-vectors in Lagrangian coords V:

- For the eigenvectors, let $R = (1, b, c)$:

$$dh - \lambda I = \begin{bmatrix} -\lambda & -1 & 0 \\ p_v & -\lambda & p_s \\ 0 & 0 & -\lambda \end{bmatrix} \quad (dh - \lambda I)R^t = 0 \iff \begin{array}{l} -\lambda - b = 0 \\ p_v - b\lambda + c\,p_s = 0 \\ -c\lambda = 0 \end{array}$$

- For $\lambda = \lambda_1 = -\sqrt{-p_v}$, $b = \sqrt{-p_v}$, $c = 0$, so

$$R_1 = \begin{bmatrix} 1 \\ \sqrt{-p_v} \\ 0 \end{bmatrix}$$

- For $\lambda = \lambda_3 = +\sqrt{-p_v}$, $b = -\sqrt{-p_v}$, $c = 0$, so

$$R_3 = \begin{bmatrix} 1 \\ -\sqrt{-p_v} \\ 0 \end{bmatrix}$$

- For $\lambda = \lambda_2 = 0$, $b = 0$, $c = -\dfrac{p_v}{p_s}$

$$R_2 = \begin{bmatrix} 1 \\ 0 \\ -p_v/p_s \end{bmatrix} \parallel \begin{bmatrix} -p_s \\ 0 \\ p_v \end{bmatrix}.$$

- We check for genuine nonlinearity:

- $$\nabla\lambda_1 \cdot R_1 = \nabla_V\left(-\sqrt{-p_v}\right) \cdot \left(1, \sqrt{-p_v}, 0\right)$$
$$= -\frac{\partial}{\partial v}\left(-\sqrt{-p_v}\right) + \cancelto{0}{\frac{\partial}{\partial u}\sqrt{-p_v}} = \frac{-p_{vv}}{2\sqrt{-p_v}} \neq 0$$

so long as $p_{vv} > 0$, $p_v < 0$

$\Rightarrow$ λ_1 is <u>genuinely nonlinear</u>

- $\nabla\lambda_0 \cdot R_2 = 0 \Rightarrow \lambda_2$ <u>linearly degenerate</u>

- $$\nabla\lambda_3 \cdot R_3 = \nabla_V\left(\sqrt{-p_v}\right) \cdot \left(1, -\sqrt{-p_v}, 0\right) =$$
$$= \frac{\partial}{\partial v}\left(\sqrt{-p_v}\right) = \frac{p_{vv}}{2\sqrt{-p_v}} \neq 0$$

$\Rightarrow$ λ_3 is <u>genuinely nonlinear</u>

- We determine the eigenvectors R for (L):

Let $g(V) = U$, so

$$g\begin{bmatrix} v \\ u \\ s \end{bmatrix} = \begin{bmatrix} v \\ u \\ \mathcal{E} \end{bmatrix},$$

where $\mathcal{E} = e + \frac{1}{2}u^2 = e(v,s) + \frac{1}{2}u^2$.

Thus

$$\begin{bmatrix} v \\ u \\ \mathcal{E} \end{bmatrix} = \begin{bmatrix} v \\ u \\ e(v,s) + \frac{1}{2}u^2 \end{bmatrix} = g\begin{bmatrix} v \\ u \\ s \end{bmatrix},$$

and

$$dg = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_v & u & e_s \end{bmatrix} = \left[\frac{\partial U^\alpha}{\partial V^i}\right] = B.$$

$$R_U^\alpha = \frac{\partial U^\alpha}{\partial V^i} R_V^i \qquad R_U = B R_V$$

(a) We now use the evals/evectors of the Lagrangian formulation of Compressible Euler to derive the evals/evectors for the Eulerian Equations; i.e., let

$$V_\tau + dh(V)V_\xi = 0 \qquad (L)$$

$$V = \begin{bmatrix} v \\ u \\ s \end{bmatrix}, \quad h(V) = \begin{bmatrix} -u \\ p \\ 0 \end{bmatrix}, \quad dh = \begin{bmatrix} 0 & -1 & 0 \\ p_v & 0 & p_s \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we have (λ_i, R_i) for dh. The idea then is to use the Lagrangian mapping

$$(\xi, \tau) \longmapsto (x, t)$$

to obtain the Eulerian system in variables V,

$$V_t + K(V) \cdot V_x = 0,$$

get the eigen-pairs for $K(V)$, then transform back to Eulerian variables $W = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$ using vector relations $R_W^i = \frac{\partial W^i}{\partial V^\alpha} R_V^\alpha$, $\lambda_W = \lambda_V$.

• The Lagrangian map $(\xi,\tau) \longrightarrow (x,t)$ is a transformation of independent variables satisfying

$$t = \tau, \quad \xi = \int_{x_0(t)}^{x(\xi,t)} \rho(z,t)\,dz, \text{ with}$$

$$\frac{\partial x}{\partial \tau} = u, \quad \frac{\partial x}{\partial \xi} = \frac{1}{\rho}$$

$$\frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial t}{\partial \xi} = 0,$$

so that

$$\frac{\partial}{\partial \tau} = \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = u\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial x}.$$

Using these in (L) gives the Eulerian System

$$uV_x + V_t + dh \cdot \frac{1}{\rho} V_x = 0$$

or

$$V_t + \left[uI + \frac{1}{\rho}dh\right]V_x = 0,$$

so the Eulerian equations in variables V must be

$$V_t + K(V)V_x = 0$$

with

$$K(V) = \left[uI + \frac{1}{\rho}dh\right].$$

Thus if (λ_i, R_i) is an eigenpair for dh (already computed), so that $dh \cdot R_i = \lambda_i R_i$, then

$$K(V)R_i = \left[uI + \frac{1}{\rho}dh\right]R_i = \left[u + \frac{1}{\rho}\lambda_i\right]R_i$$

(19)

Since $\lambda_1 = -\sqrt{-P_V(V,S)}$, $\lambda_2 = 0$, $\lambda_3 = +\sqrt{-P_V(V,S)}$

it follows that the Eulerian evals are

$$\lambda_1^E = u - \frac{1}{\rho}\sqrt{-P_V},$$

$$\lambda_2^E = u,$$

$$\lambda_3^E = u + \frac{1}{\rho}\sqrt{-P_V},$$

(independent of chnges in dep variables)

and the Eulerian e-vectors are <u>the same</u>

(subject to vector chnges of dep. variables)

$$R_1^E = \begin{pmatrix} 1 \\ \sqrt{-P_V} \\ 0 \end{pmatrix}, \quad R_2^E = \begin{pmatrix} -P_S \\ 0 \\ P_V \end{pmatrix}, \quad R_3^E = \begin{pmatrix} 1 \\ -\sqrt{-P_V} \\ 0 \end{pmatrix}$$

given in variables $V = \begin{pmatrix} V \\ u \\ S \end{pmatrix}$ in terms of

$P = P(V,S)$, V = spec volume, S = spec. entropy.

(20)

• <u>Note</u>: $\frac{1}{\rho}\sqrt{-P_V}$ is the <u>sound speed</u>, the speed of waves relative to an observor moving with the particles $\frac{dx}{dt} = u$.

<u>Note</u>: $P_V(V,S) = \frac{\partial P}{\partial V} = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial V} \underset{\rho = \frac{1}{V}}{\uparrow}{=} -\frac{1}{V^2}\frac{\partial P}{\partial \rho}$

$$= -\rho^2 \frac{\partial P}{\partial \rho}$$

Thus

$$\sigma = \frac{1}{\rho}\sqrt{-P_V} = \sqrt{P_\rho(\rho,S)} = \text{sound speed},$$

and so

$$\boxed{\begin{aligned} \lambda_{1,3}^E &= u \pm \sqrt{P_\rho} = u \pm \sigma \\ \lambda_2^E &= u \end{aligned}}$$