SECTION-6
Lax's Local Solution of the Riemann Problem for nxn Systems of Conservation Laws

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Math-280: A Mathematical Introduction to

Blake Temple, UC-Davis

Riemann Problem:

Defn: The Riemann Problem for (CL)

$$u_t + f(u)_x = 0 \qquad \text{(CL)}$$

is the ivp with i-data

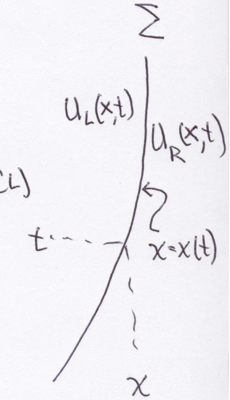
$$u(x,0) = \begin{cases} u_L & x \le 0 \\ u_R & x > 0 \end{cases} \equiv u_0(x) \qquad \text{(RP)}$$

for constant states $u_L, u_R \in \mathbb{R}^n$. For smooth solutions the eigenfamilies $(\lambda(u), R(u))$ of $df(u)$ determine the simple waves $u(\lambda(x,t))$ that propagate with values on $\mathcal{R}$ at speeds $\frac{dx}{dt} = \lambda$. Each such e-val of df is called a characteristic family, and we call the curves $\frac{dx}{dt} = \lambda$ the characteristics of a solution. We now consider the shock waves associated with (CL).

Shock Waves:

The main theorem about shock waves is the following:

Theorem: Let $u(x,t) \in \mathbb{R}^n$ be a function that consists of two smooth solutions $u_L(x,t)$, $u_R(x,t)$ of (CL) separated by a smooth timelike curve Σ described by $x = x(t)$.



That is, let

$$u(x,t) = \begin{cases} u_L(x,t) & x \le x(t) \\ u_R(x,t) & x \ge x(t) \end{cases}$$

st u_L & u_R are smooth, cont up to the boundary Σ. Then $u(x,t)$ is a weak soln of (CL) iff at each pt $(x_0, t_0) \in \Sigma$, the Rankine Hugoniot Jump Condts hold:

$$s[u] = [f] \qquad \text{(R-H)}$$

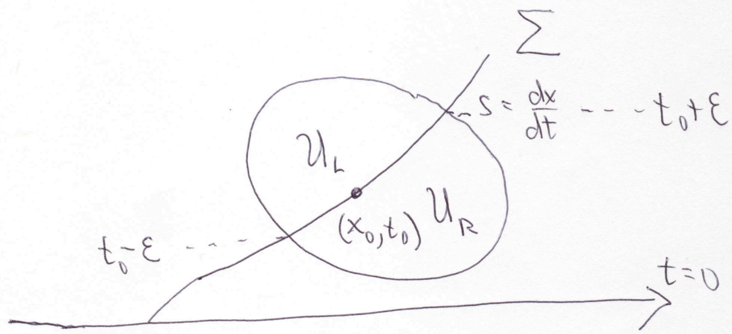
$[F]$ = "jump in F" $= F(u_R(x(t),t)) - F(u_L(x(t),t))$, $x,t \in \Sigma$

Proof: $u(x,t)$ is a weak soln of (CL) iff

$$\iint_{\substack{-\infty < x < +\infty \\ t \geq 0}} u\varphi_t + f(u)\varphi_x \, dx\,dt + \int_{-\infty}^{\infty} u(x,0)\varphi(x,0)\,dx = 0$$

$\forall$ test fn $\varphi(x,t) \in C_0^\infty(t \geq 0)$

. Now choose φ with support in a small nbhd $\mathcal{U} = \mathcal{U}_L \cup \mathcal{U}_R$ of a point $(x_0, t_0) \in \Sigma$, where $\Sigma = \{(x(t), t)\}$. Let $s \equiv s(t) = x'(t)$ be speed of shock Σ, & assume wlog $\mathcal{U} \cap \{t=0\} = \emptyset$.



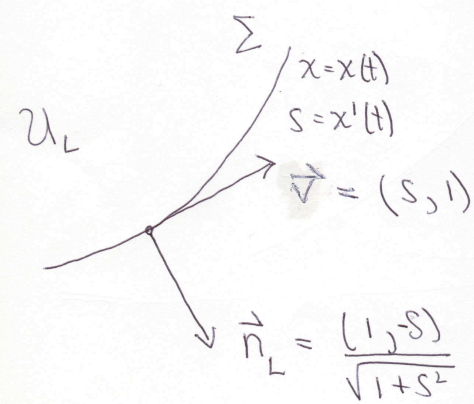
Then $u(x,t) = \begin{cases} u_L(x,t) & x \leq x(t) \\ u_R(x,t) & x \geq x(t) \end{cases}$ is a weak soln

$$\Rightarrow \iint_{\mathcal{U}} u\varphi_t + f(u)\varphi_x \, dx\,dt = 0$$

$$\Rightarrow \iint_{\mathcal{U}_L} u\varphi_t + f(u)\varphi_x \, dx\,dt + \iint_{\mathcal{U}_R} u\varphi_t + f(u)\varphi_x \, dx\,dt = 0 \qquad (*)$$

Now u_L, u_R smooth in $\mathcal{U}_L, \mathcal{U}_R$ resp $\Rightarrow$

$$\iint_{\mathcal{U}_L} u\varphi_t + f\varphi_x \, dx\,dt \underset{\text{int by pts}}{=} -\iint_{\mathcal{U}_L} \underbrace{(u_t + f_x)}_{0}\varphi \, dx\,dt + \oint_{\Sigma} \varphi(f,u)\cdot\underbrace{(n_x,n_t)}_{\text{outer unit normal}} \underbrace{d\sigma}_{\text{arclength on }\Sigma}$$



So

$$\int_\Sigma (f,u)\cdot(n_x,n_t)\,d\sigma = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \varphi(f,u)\cdot\frac{(1,-s)}{\cancel{\sqrt{1+s^2}}}\,\cancel{\|\vec{v}\|}\,dt$$

$$= \int_{t_0-\varepsilon}^{t_0+\varepsilon} \varphi\{f-su\}\,dt$$

Similarly

$$\iint_{\mathcal{U}_R} u\varphi_t + f\varphi_x\,dx\,dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \varphi(-1)\{f-su\}\,dt$$

$\vec{n}_R = -\vec{n}_L$

and so $(*) \Rightarrow$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \varphi\{s(u_R-u_L)-(f_R-f_L)\}\,dt = 0 \quad \forall\varphi$$

So we must have $s[u]=[f]$ @ $t=t_0,\ x=x_0$

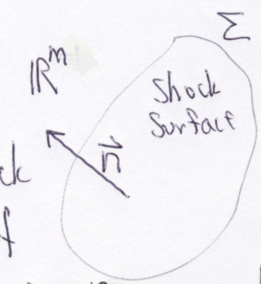
✓

Defn: The set of all $u\in\mathbb{R}^n$ that solve

$$s[u]=[f]$$

for $[u]=u-u_L$, $[f]=f(u)-f(u_L)$ is called the Hugoniot Locus of u_L.

Homework: Prove that $u_t+\operatorname{div}F=0$, $F=(F_1,\dots,F_m)$ is a system of cons laws in $x\in\mathbb{R}^m$, then a solution consisting of smooth soln's $u_L(x,t)$ & $u_R(x,t)$ on either side of a smooth shock surface Σ is a weak soln iff



$$s[u]=[F]\cdot\vec{n}$$ $\vec{n}\in\mathbb{R}^m$ is normal to Σ at each $t=\text{const}$

or $[u;\vec{F}]\cdot\vec{N}=0$ $\vec{N}\in\mathbb{R}^{m+1}$ normal to Σ in spacetime

General Riemann Problem:

(Lax 1957: with simplification by Conlon)

Assume: a system of conservation laws

$$u_t + f(u)_x = 0 \qquad \text{(CL)}$$

$$u \equiv u(x,t) = (u_1, \dots, u_n)$$

$$df(u) \equiv A \quad n \times n \text{ matrix field}$$

Assume(1) (CL) is strictly hyperbolic; i.e., A has n real & distinct evals at each u,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

and assume(2) that each characteristic family λ_i is either genuinely nonlinear or linearly degenerate

$$\nabla\lambda_i \cdot R_i \neq 0 \qquad R_i \text{ rt. e-vector of } \lambda_i \qquad \text{(GN)}$$

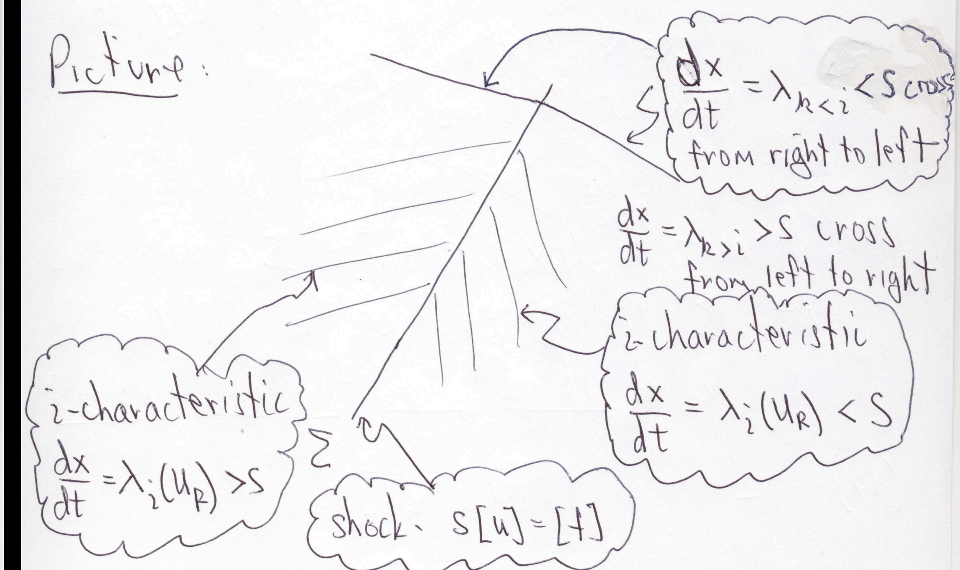
$$\nabla\lambda_i \cdot R_i \equiv 0 \qquad \text{(LD)}$$

Definition (Lax Shock Condition)

We say that a shock wave is an admissible i-shock if the characteristic curves in the family of the shock impinge on the shock, & all other characteristic cross the shock. I.e. if

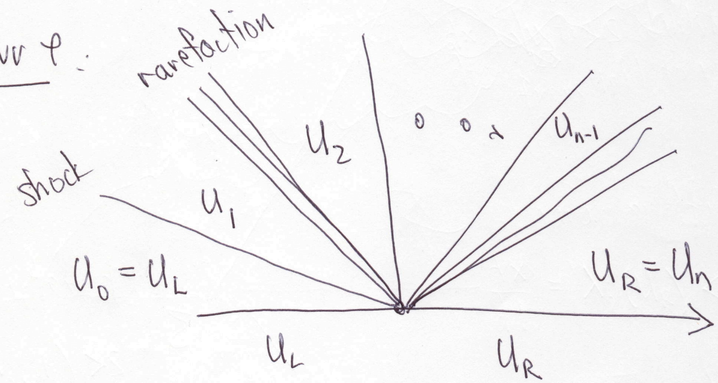
$$\lambda_i(u_R) < s < \lambda_i(u_L)\ ;\ s < \lambda_{i+1}(u_R)\ ;\ s > \lambda_{i-1}(u_L) \qquad \text{(LS)}$$

Picture:



Theorem: (Lax 1957) $\forall\ u_0 \in \mathbb{R}^n$ such that Assumptions (1) & (2) hold, $\exists$ nbhd $\mathcal{U} \ni u_0$ such that, if $u_L, u_R \in \mathcal{U}$, then there exists a unique soln of the R-P. $[u_L, u_R]$ in the class of i-centered simple waves (rarefaction waves) and i-admissible shock waves.

Picture:



Proof:

- Let $L_1, \dots, L_n$ and $R_1, \dots, R_n$ denote the left and right eigenvectors of $A = df$, normalized to unit length, so

$$L_i^t A = \lambda_i L_i^t \qquad \|L_i\| = 1$$

$$A R_i = \lambda_i R_i \qquad \|R_i\| = 1$$

and we have

Lemma (1) $\quad L_i \cdot R_k = 0 \quad \forall\ i = 1, \dots, n,\ i \neq k$

Assume wlog that $u_0 \in \mathcal{U}_0 \subseteq \mathbb{R}^n$ where $\mathcal{U}_0$ is so small that

$$\lambda_i(u) < \lambda_j(v) \quad \forall\ i < j,\ u, v \in \mathcal{U}_0$$

(11)

LEMMA (2): Assume only that (CL) is strictly hyperbolic. Then the soln of (R-H) $s[u]=[f]$ consists of n smooth curves (one associated with each eigenfamily of $A = df$) $\mathcal{S}_i(u_L)$ defined in a nbhd $\mathcal{U}_1$ for each $u_L \in \mathcal{U}_0 \subseteq \mathcal{U}_1$.

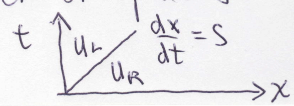
Proof: (View u, $f(u)$ as vertical vectors...)

$$f(u) - f(u_L) = \int_0^1 \frac{d}{d\sigma} f(u_L + \sigma(u-u_L))\, d\sigma$$

$$= \int_0^1 df(u_L + \sigma(u-u_L)) \cdot (u-u_L)\, d\sigma$$

$$\equiv G(u) \cdot (u-u_L)$$

Now the jump condt $s[u]=[f]$, $[u]=u-u_L$, $[f]=f(u)-f(u_L)$ determines the Hugoniot Locus of u_L — the set of all states u that can be connected to u_L by a shock of speed s



(12)

Thus

$$[f] = G(u)[u]$$

& (R-H) is equivalent to

$$[G(u) - s](u-u_L) = 0, \quad G(u) \text{ } n\times n \text{ matrix field}$$

$$G(u) = \int_0^1 df(u_L + \sigma(u-u_L))\, d\sigma$$

$$= \text{“Ave. of } df \text{ along line } \overline{u_L u}\text{”}$$

Now

$$\lim_{u\to u_L} G(u) = df(u_L)$$

so for u_L, u in a nbhd of u_0 $G(u)$ has real & distinct evals

$$\mu_1(u) < \mu_2(u) < \cdots < \mu_n(u).$$

Let $\ell_1(u), \ldots, \ell_n(u)$ denote the corresponding left e-vectors of $G(u)$, & $r_1(u), \ldots, r_n(u)$ the right e-vectors of $G(u)$, $\|\ell_i\| = 1 = \|r_i\|$.

(R-H) $\iff$ $[G(u)-s](u-u_L)=0$ (R-H)

- By cont, in a nbhd of u_L, $G(u)$ has real & distinct e-vals $\mu_1 < \cdots < \mu_n$ & e-vectors $\{r_1, \dots, r_n\}$. Thus to solve

$$[G(u)-s](u-u_L)=0$$

Need $s=\mu_i$, $u-u_L=\varepsilon r_i$ some i. $(\|r_i\|=1)$

- We show $\forall i$ this defines a smooth curve $u_i(\varepsilon)$

I.e., $u-u_L=\varepsilon r_i(u)$

has a unique soln in nbhd of $\varepsilon=0, u=u_L$ by IFT:

$$0=u_L-u+\varepsilon r_i(u)=F(u,\varepsilon) \qquad (F)$$

$$\left.\frac{\partial F}{\partial u}\right|_{\substack{u=u_L\\ \varepsilon=0}} = -I \Rightarrow \text{can solve for } u=u(\varepsilon)$$

uniquely in a nbhd of $\varepsilon=0$ ✓

<u>Indeed</u>: The IFT says: if $z=f(x;y)$ $z\in\mathbb{R}^n, x\in\mathbb{R}^n, y\in\mathbb{R}^m$, and we have

$$0=f(x_0;y_0) \qquad (Z)$$

$$\left|\frac{\partial f}{\partial x}(x_0;y_0)\right|\neq 0$$

then we can solve (Z) in a nbhd by

$$x=g(y), \quad y\in\Omega\ni y_0$$

so that

$$0=f(g(y);y).$$

Thus for (F) we can take

$$0=u_L-u+\varepsilon r_i(u)=F(u,\varepsilon,u_L) \qquad (F')$$

$$\frac{\partial F}{\partial u}(u_L,0,u_L)=-I$$

$\Rightarrow$ $u=u(\varepsilon;u_L)$ solves $u_L-u+\varepsilon r_i(u)=0$ in a nbhd of $\varepsilon=0, u=u_L$. This $\Rightarrow$ shock curves depend <u>smoothly</u> on u_L as well ✓

Theorem: In some nbhd of u_L, the solution of the R-H jump conditions

$$s[u]=[f]$$

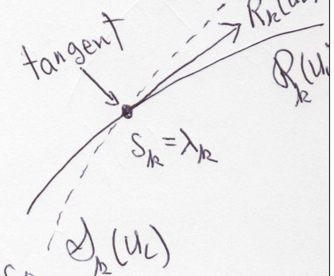
consists of n smooth curves $\mathcal{S}_1(u_L),\dots,\mathcal{S}_n(u_L)$. States u on the curve $\mathcal{S}_i(u_L)$ satisfy

$$u-u_L=\varepsilon\, r_i(u), \quad s=\mu_i(u) \qquad (*)$$

where $(\mu_i(u), r_i(u))$ is an eigenpair of $G(u,u_L)$, $G(u_L,u_L)=df(u_L)$. I.e., (by IFT) (*) can be solved in a nbhd $|\varepsilon|<\bar{\varepsilon}$ by smooth curve $u_i(\varepsilon)$ which define $\mathcal{S}_i(u_L)$, $u_i(0)=u_L$. In this case the shock speed is given by

$$s=s_i(\varepsilon)=\mu_i(u_i(\varepsilon)) \qquad (**)$$

Lemma 3: The i-th Shock Curve $\mathcal{S}_i(u_L)$ is tangent to $\mathcal{R}_i(u_L)$ at $u=u_L$, and the i-shock speed tends to $\lambda_i(u_L)$ as $\varepsilon\to 0$.



Pf. From (F′) we have

$$0=u_L-u+\varepsilon\, r_k(u)$$

is solved by $u=u_k(\varepsilon;u_L)$, so

$$0=u_L-u_k(\varepsilon;u_L)+\varepsilon\, r_k(u_k(\varepsilon,u_L)).$$

Diff wrt ε & set $\varepsilon=0$:

$$0=-\frac{\partial}{\partial\varepsilon}u_k(0;u_L)+r_k(\underbrace{u_k(0,u_L)}_{u_L})$$

or

$$\dot{u}_k(0)\equiv\frac{\partial}{\partial\varepsilon}u_k(0;u_L)=r_k(u_L)=c\,R_k(u_L)$$

Also

$$s_i(\varepsilon)=\mu_i(u_i(\varepsilon))\xrightarrow{\varepsilon\to 0}\mu_i(u_L)=\lambda_i(u_L)$$

Now let $u_i(\varepsilon) \equiv u_i(\varepsilon, u_L)$ denote the i-th shock curves and $V_i(\varepsilon) \equiv V_i(\varepsilon, u_L)$ the i-th rarefaction curves, $\mathcal{S}_i(u_L)$ & $\mathcal{R}_i(u_L)$, respectively, and assume ε is the <u>arclength</u> parameterization of these curves, $u_i(0) = V_i(0) = u_L$, $\dot{u}_i(0) = \dot{V}_i(0) = R_i(u_L)$

The following was first proved by Lax:

Theorem: at $\varepsilon = 0$ we have:

(1) $s_i(0) = \lambda_i(0)$ $\quad [s_i \equiv \text{shock speed} \equiv s_i(u_i(\varepsilon))]$

(2) $\dot{s}_i(0) = \frac{1}{2}\dot{\lambda}_i(0)$ $\quad [\lambda_i(\varepsilon) \equiv \lambda_i(V(\varepsilon))]$

(3) $\dot{u}_i(0) = \dot{V}_i(0) = R_i(u_L)$

(4) $\ddot{u}_i(0) = \ddot{V}_i(0) = (R_i \cdot \nabla) R_i \big|_{u=u_L}$

Pf. We have already proven (1) & (3). Now for this proof let $u(\varepsilon) \equiv u_i(\varepsilon)$ and $V(\varepsilon) \equiv V_i(\varepsilon)$ (suppress i). To obtain (2), (4) we diff the (R-H) jump conditions & the e-value relations:

$\boxed{\text{For } \mathcal{S}(u_L):}$ $\qquad s[u] = [f] \qquad A = df$

Diff Once: $\qquad \dot{s}[u] + s\dot{u} = A\dot{u} \qquad (1)$

Diff Twice: $\qquad \ddot{s}[u] + 2\dot{s}\dot{u} + s\ddot{u} = \dot{A}\dot{u} + A\ddot{u} \qquad (2)$

Now (1) at $\varepsilon = 0$ gives: $\qquad sR = \lambda R \qquad (3)$

(2) at $\varepsilon = 0$ gives $\qquad 2\dot{s}R + \lambda\ddot{u} = \dot{A}R + A\ddot{u} \qquad (4)$

$\boxed{\text{For } \mathcal{R}(u_L):}$ $\qquad (A - \lambda)\dot{V} = 0$

Diff Once: $\qquad (\dot{A} - \dot{\lambda})\dot{V} + (A - \lambda)\ddot{V} = 0 \qquad (5)$

(5) at $\varepsilon = 0$ gives $\qquad \dot{A}R = \dot{\lambda}R - (A - \lambda)\ddot{V} \qquad (6)$

Subst (6) into (4):

$$2\dot{s}R + \lambda\ddot{u} = \dot{\lambda}R - (A - \lambda)\ddot{V} + A\ddot{u}$$

or

$$(2\dot{s} - \dot{\lambda})R = (A - \lambda)(\ddot{u} - \ddot{V}) \qquad (7)$$

Now Consider: $A-\lambda : \mathbb{R}^n \to \text{Span}\{R_1 \cdots \hat{R}_i \cdots R_n\}$

i.e., R is not in Range of $A-\lambda$. Thus by (7)

$$2\dot{s} - \dot{\lambda} = 0$$

Thus by (7) again, $\ddot{u} - \ddot{v} = cR$ some const c.

But since $u(\varepsilon), v(\varepsilon)$ are arclength parameterized and $\dot{u}(0) = \dot{v}(0) = R$, we must have

$$(\ddot{u} - \ddot{v}) \perp R$$

Therefore $\ddot{u} = \ddot{v}$ verifying (4).

HW. Show that for $u_R \in \mathcal{S}(u_L)$ near u_L,

$$s = \frac{\lambda(u_L) + \lambda(u_R)}{2} + O(\varepsilon^2)$$

where s = shock speed for shock jumping u_L to u_R

Lemmas 2, 3 require only that (CL) be strictly hyperbolic in some nbhd.

Assume now that (CL) is genuinely nonlinear (GN) in the kth char. field λ_k, & set

$$\lambda \equiv \lambda_k, \quad R \equiv R_k, \quad |R| \equiv 1$$

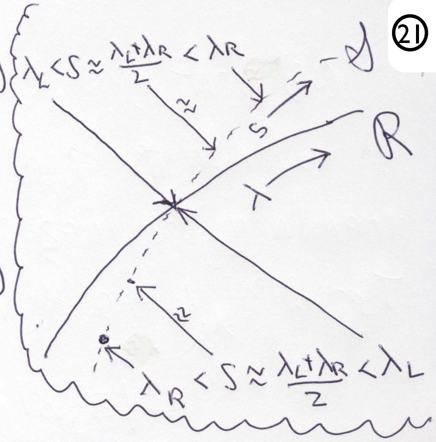
& assume (normalize R so that)

$$\nabla\lambda \cdot R > 0 \qquad \text{(GN)}$$

Let $u(\varepsilon)$ denote the arclength parameterization of $\mathcal{S}_k(u_L) \equiv \mathcal{S}$, &

$$u(0) = u_L, \quad \dot{u}(0) = R.$$

Lemma 4: $u_R = u(\varepsilon) \in \mathcal{S}$ corresponds to an admissible shock wave (for u_R suff close to u_L) iff $\varepsilon < 0$.



Pf: We have
$$S(\varepsilon) = \frac{\lambda_R+\lambda_L}{2} + O(1)\,\varepsilon^2$$

But
$$\dot\lambda(0) = \frac{d}{d\lambda}\lambda(u(\varepsilon))\Big|_{\varepsilon=0} = \nabla\lambda\cdot\dot u\Big|_{\varepsilon=0} = \nabla\lambda\cdot R > 0$$

Thus for $u \approx u_L$, $u \in \mathcal{S}$, $\lambda(\varepsilon) = \lambda_R < \lambda_L$; for $\varepsilon < 0$,
$$S(\varepsilon) = \frac{\lambda(\varepsilon)+\lambda_L}{2} + O(1)\,\varepsilon^2$$
$$\Rightarrow \quad \lambda_R = \lambda(\varepsilon) < S(\varepsilon) < \lambda_L$$

For $\varepsilon > 0$, the reverse inequality holds.

Moreover, $\varepsilon < 0$ & $\varepsilon << 1 \Rightarrow \lambda_k(u_L) < S(\varepsilon) < \lambda_{k+1}(u_L)$ because $S(\varepsilon) \approx \lambda_k$ for $\varepsilon << 1$.

DEFN: If the ith characteristic family is (GN), let

$R_i^+(u_L) \equiv$ portion of $R_i(u_L)$ such that
$$\lambda_i > \lambda_i(u_L)$$

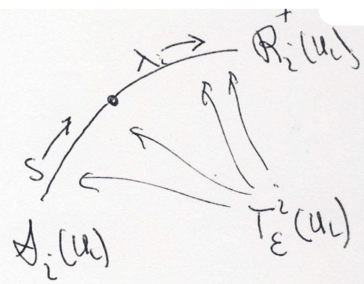
$\mathcal{S}_i^-(u_L) \equiv$ portion of $\mathcal{S}_i(u_L)$ such that
$$S_i < \lambda_i(u_L).$$

Let $\mathcal{T}_i(u_L) \equiv \mathcal{S}_i^-(u_L) \cup R_i^+(u_L)$.

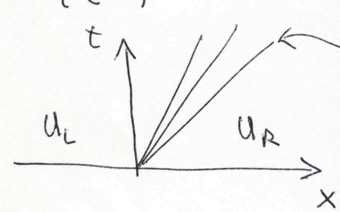
By our theorem, $\exists$ a C^2-parametrization of $\mathcal{T}_i(u_L)$ in a nbhd of u_L, and the 3rd derivative has at most a jump discontinuity:

Let $T^i_\varepsilon(u_L) \equiv T^i(u_L;\varepsilon)$ be the arclength parametrization of $\mathcal{T}_i(u_L)$ in a nbhd of u_L,
$$|\varepsilon| < \varepsilon_0,$$
$T^i_0(u_L) = u_L$, $T^i_\varepsilon(u_L) \subseteq R_i^+(u_L)$ $\varepsilon > 0$, $T^i_\varepsilon(u_L) \subseteq \mathcal{S}_i^-(u_L)$ $\varepsilon < 0$

Note: $\forall\, u_R = T^i_\varepsilon(u_L)$ we can solve the Riemann Problem with a wave of speed $\simeq \lambda_i(u_L)$ i.e.



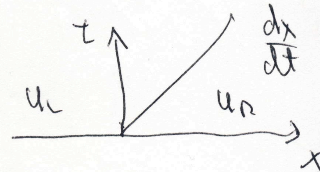
$\varepsilon > 0$



$\frac{dx}{dt} = \lambda_i(u)$

$u \in R_i^+(u_L)$ between u_L and u_R

$\varepsilon < 0$



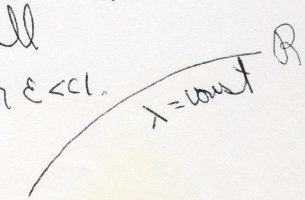
$\frac{dx}{dt} = s$ where $s[u] = [f]$.

Moreover, the shocks are admissible.

DEFN: we call the above solution i-waves "i-shock and i-rarefaction waves"

Assume: (LD) $\nabla\lambda \cdot R \equiv 0$ in the k-th char. field. Then

LEMMA 5: The shock curve S coincides with the integral curve R, which is also the level curve $\lambda = \text{const}$. In this case, $s(\varepsilon) = \lambda(\varepsilon) = \lambda(u_L)$ is constant on S, and all shocks are contact discontinuities for $\varepsilon \ll 1$.



Proof: We show that every point on $R(u_L)$ satisfies the (R-H) with $s = \lambda$. First, let $u(\varepsilon)$ denote a smooth parameterization of R, $u(0) = u_L$. Then

$$\dot f = \frac{d}{d\varepsilon} f(u(\varepsilon)) = df \cdot \dot u = \lambda \dot u \quad \text{since } \dot u = R.$$

Thus $f(u(\varepsilon)) = \lambda u(\varepsilon) + \text{const}$

or $f_R - f_L = \int_0^\varepsilon \dot f\, d\varepsilon = \int_0^\varepsilon \lambda \dot u\, d\varepsilon = \lambda[u]$

$$\Rightarrow \lambda[u(\varepsilon) - u_L] = [f(u(\varepsilon)) - f(u_L)]$$

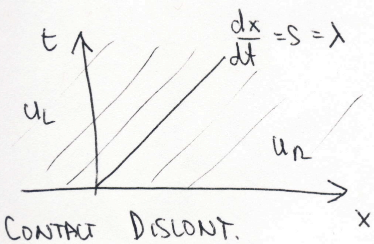
so R must be the portion of Hugoniot locus corresp. to $\lambda = \lambda_i$

<u>Moreover</u>, for $u_R = u(\epsilon) \approx u_L$, $\lambda = \lambda_i$,

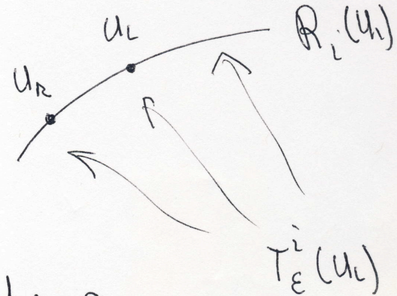
$$\lambda_i = \lambda_i(u_R) = S = \lambda_i(u_L) = \lambda_i$$

$$\lambda_{i-1}(u_L) \le S \le \lambda_{i+1}(u_L)$$

$$\lambda_{i-1}(u_R) \le S \le \lambda_{i+1}(u_R)$$



CONTACT DISCONT.



Let $T^i_\varepsilon(u_L)$ $|\varepsilon| < \varepsilon_0$ be a smooth param. of $R_i(u_L)$ in a nbhd of u_L. Then $\cancel{\forall}$ u_R the Riemann problem has a soln given by a contact discont. $\forall\, u_R = T^i_\varepsilon(u_L)$.

$\boxed{\text{ASIDE}}$

<u>THM</u> (TE): An integral curve $R(u_L)$ of (λ, R) lies in the Hugoniot locus of u_L iff $R(u_L)$ is a level curve of λ or else $R(u_L)$ is a straight line in u-space:

<u>RESULT</u>: Can classify the 2x2 systems of cons. laws with coinciding shock and rarefaction curves:

<u>CLASS I</u>	<u>CLASS II</u>	<u>Class III</u>
2 contact fields	one line one contact	two line fields

CLASS II $\quad u_t + (u\varphi(u,v))_x = 0 \quad$ Polymer Eqn's

$v_t + (v\varphi(u,v))_x = 0 \quad$ Nonlinear Elastic

CLASS III: $\quad u_t + \left(\dfrac{u}{1+u+v}\right)_x = 0 \quad$ Multicomponent chromatography

$v_t + \left(\dfrac{kv}{1+u+v}\right)_x = 0$

[BOOK: Aris, R. & Amundson, N. Math methods in Chem. Engineering prentice-Hall]

NASC on flux fn's were given for system to lie in I, II, III.

B) Local Soln of RP:

(CL) $\quad u_t + f(u)_x = 0$

Theorem: (Lax 1957) Assume (CL) is strictly hyperbolic and GN or LD in each characteristic field in some nbhd $\mathcal{U}_0 \ni u_0$. Then $\exists\ \delta$ st if $u_L, u_R \in \mathcal{U}_\delta = \{u : |u-u_0| < \delta\}$, then $\exists!$ soln of RP in the class of i-simple waves, call it $[u_L, u_R](x,t)$. Moreover, $\exists\ \delta' = O(\delta)$ st all intermediate states in $[u_L, u_R](x,t)$ lie in $\mathcal{U}_{\delta'}$; i.e., $[u_L, u_R](x,t) \in \mathcal{U}_{\delta'}$ $\forall\ x \in \mathbb{R}, t \geq 0$.

Proof. We have defined the i-wave curve

$$\mathcal{Y}_i(u_L) = \mathcal{R}_i^+(u_L) \cup \mathcal{S}_i^-(u_L)$$

Let $u_R = T^i_{\varepsilon_i}(u_L)$ denote the state ε_i-arc length units from u_L along $\mathcal{Y}_i(u_L)$, $\varepsilon_i > 0$ along $\mathcal{R}_i^+$ & $\varepsilon_i < 0$ along $\mathcal{S}_i^-$. Then for

$$\underset{\sim}{\varepsilon} \equiv (\varepsilon_1, \dots, \varepsilon_n)$$

we can define

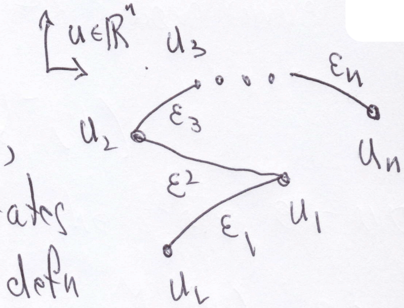
$$u_R = T(\underset{\sim}{\varepsilon}, u_R)$$

by

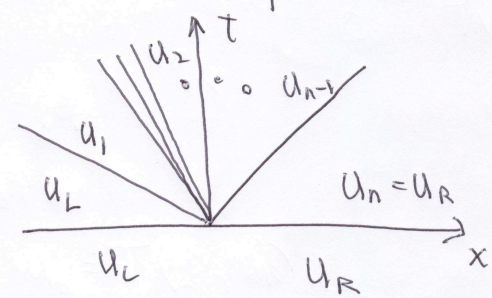
$$T(\underset{\sim}{\varepsilon}, u_R) = T^n_{\varepsilon_n} \circ T^{n-1}_{\varepsilon_{n-1}} \circ \cdots \circ T^1_{\varepsilon_1} u_L$$

"... move ε_1 units from u_L along $\mathcal{Y}_1(u_L)$ to get u_1,
ε_2 units from u_1 along $\mathcal{Y}_2(u_1)$ to get u_2, ...,
ε_n " " u_{n-1} " $\mathcal{Y}_n(u_{n-1})$ " " u_n"

Picture



• Claim: if $u_R = T(\underset{\sim}{\varepsilon}, u_L)$, then the intermediate states and wave curves in the defn of T determine a unique soln of R.P. $[u_L, u_R]$



$$u_1 = T^1_{\varepsilon_1} u_L, \quad u_2 = T^2_{\varepsilon_2} T^1_{\varepsilon_1} u_L = T^2_{\varepsilon_2} u_1,$$

$$u_{i+1} = T^{i+1}_{\varepsilon_{i+1}} u_i.$$

- Thus to prove the Thm, it suffices to prove that for u_L, u_R suff close to u_0, $\exists\ \underset{\sim}{\varepsilon} \in \mathbb{R}^n$ st $u_R = T(\underset{\sim}{\varepsilon}; u_L)$.

<u>Inverse</u> <u>Fn</u> <u>Thm</u>: If $\left|\frac{\partial T}{\partial \underset{\sim}{\varepsilon}}\right|_{\substack{\underset{\sim}{\varepsilon}=0 \\ u_L = u_0}} \neq 0$, (IFT)

$u_L = T(0, u_L)$

then $\exists$ nbhd $\mathcal{U}_{\delta_1} \ni u_0$, $\mathcal{U}_{\delta_2} \ni u_0$ st if

$u_L \in \mathcal{U}_{\delta_1}$, $u_R \in \mathcal{U}_{\delta_2}$ then we can solve $u_R = T(\underset{\sim}{\varepsilon}; u_L)$ uniquely for $\underset{\sim}{\varepsilon}$; that is

$$\underset{\sim}{\varepsilon} = T^{-1}(u_R; u_L), \quad u_R, u_L \in \mathcal{U}_\delta$$

$$\delta = \min\{\delta_1, \delta_2\}$$

- To verify (IFT), our C^2-contact betw R_i^+ & S_i^- implies

$$T^k_{\varepsilon_k} u = u + \varepsilon_k R_k(u) + O(1)\varepsilon_k^2$$

Thus:

$$u_1 = T^1_{\varepsilon_1} u_L = u_L + \varepsilon_1 R_1(u_L) + O(1)\varepsilon_1^2$$

$$u_2 = T^2_{\varepsilon_2} T^1_{\varepsilon_1} u_L = T^2_{\varepsilon_2} u_1 = u_1 + \varepsilon_2 \underbrace{R_2(u_1)}_{\substack{= R_2(u_L) + O(1)|\underset{\sim}{\varepsilon}| \\ \text{by cont}}} + O(1)\varepsilon_2^2$$

$$= u_L + \varepsilon_1 R_1(u_L) + \varepsilon_2 R_2(u_L) + O(1)|\underset{\sim}{\varepsilon}|^2$$

$\vdots$

$$u_R = T(\underset{\sim}{\varepsilon}, u_L) = T^n_{\varepsilon_n} \circ\circ\circ T^1_{\varepsilon_1} u_L = u_L + \varepsilon_1 R_1(u_L) + \cdots + \varepsilon_n R_n(u_L) + O(1)|\underset{\sim}{\varepsilon}|^2$$

(Uses... $R_i(u_{i-1}) = R_i(u_L) + O(1)|\underset{\sim}{\varepsilon}|$)

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Thus: $\left.\dfrac{\partial T}{\partial \varepsilon_i}\right|_{\varepsilon=0} = R_i(u_L)$

$$\Rightarrow \left.\frac{\partial T}{\partial \underset{\sim}{\varepsilon}}\right|_{\varepsilon=0} = \begin{bmatrix} -R_1(u_L)- \\ \vdots \\ -R_n(u_L)- \end{bmatrix}$$

nonzero det because R_i indept at u_L by strict hyp.

Conclude: $\left|\dfrac{\partial T}{\partial \underset{\sim}{\varepsilon}}\right|_{\substack{\varepsilon=0 \\ u_L=u_0}} \neq 0$

So by IFT $\underset{\sim}{\varepsilon} = T^{-1}(u_R, u_L)$ & done ✓

That all states in $[u_L, u_R](x,t)$ lie within some $\mathcal{U}_{\delta'} \supseteq \mathcal{U}_\delta$ follows because the indept. of e-vectors $\Rightarrow |u_i - u_L| \leq Const\,|u_R - u_L|$

(FIP)

HW: Prove Lax's Soln of RP is a weak soln of (CL) (Hint: RP is PW smooth with shock boundaries)

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