SECTION-8
Traveling Waves for the p-system

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Math-280: A Mathematical Introduction to Shock Waves

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Traveling Wave Solutions for viscous p-system

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- System of cons laws: $u_t + f(u)_x = 0$ (CL)

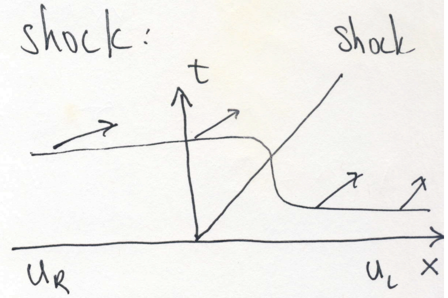
Shocks satisfy R-H jump condt: $s[u] = [f]$

- Viscous perturbation with artificial viscosity

$$u_t + f(u)_x = \underbrace{\varepsilon u_{xx}}_{\text{artificial viscosity}} \qquad \text{(VCL)}$$

- We look for a traveling wave soln of (VCL) that approximates the shock:

$u = u\left(\frac{x-st}{\varepsilon}\right)$ Picture



u = constant when

$x - st = \text{const} \iff x = st + \text{const} \iff \frac{dx}{dt} = s$ ✓

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- Plug $u\left(\frac{x-st}{\varepsilon}\right)$ into (VCL): $\xi = \frac{x-st}{\varepsilon}$; $' \equiv \frac{d}{d\xi}$

$$-\frac{su'}{\varepsilon} + df(u)\frac{1}{\varepsilon}u' = \frac{\varepsilon}{\varepsilon^2}u''$$

$$-su' + f(u(\xi))' = u''$$

integrate once

$$u' = -su + f + c$$

- Ask that: $\lim_{\xi \to -\infty} u' = 0$ & $\lim_{\xi \to -\infty} u = u_L$

$\Rightarrow$ $0 = -su_L + f_L + c \Rightarrow c = su_L - f_L$

$\Rightarrow$ $u' = -s(u - u_L) + f(u) - f_L$ (ODE)

<u>Conclude</u>: if $\lim_{\xi \to +\infty} u' = 0$ then $u_R = \lim_{\xi \to +\infty} u$ must satisfy

$$s[u] = [f]$$

<u>Conclude</u>: $\bar{U}_L, \bar{U}_R \in \mathrm{gf}(U_L, U_R)$ are rest pts of (ODE). Thus $\exists$ of traveling wave $\iff$ exist of connecting orbit of ODE

$\Rightarrow$ u_L, u_R satis RH jump condt's with $s = s(u_L, u_R) = \text{const}$ along whole profile!

③

- Consider the p-system:

$$V_t - u_x = 0 \qquad U = \begin{pmatrix} v \\ u \end{pmatrix}, \; f(U) = \begin{pmatrix} -u \\ p \end{pmatrix}$$

$$u_t + p(v)_x = 0$$

ODE is then:

$$U' = -s(U - U_L) + f(U) - f(U_L)$$

$$\Leftrightarrow \quad \begin{pmatrix} v \\ u \end{pmatrix}' = -s \begin{pmatrix} v - v_L \\ u - u_L \end{pmatrix} + \begin{pmatrix} -u + u_L \\ p - p_L \end{pmatrix}$$

$\Leftrightarrow$

$$v' = -s(v - v_L) + (-u + u_L) = \varphi(u,v)$$

$$u' = -s(u - u_L) + (p - p_L) = \psi(u,v)$$

④

Main Theorem: Let U_L, U_R be in Hugoniot locus

$$s[U] = [f].$$

Then the shock betw U_L & U_R has a viscous profile iff $U_R \in \mathcal{S}_i^-(U_L)$ $i = 1,2$.

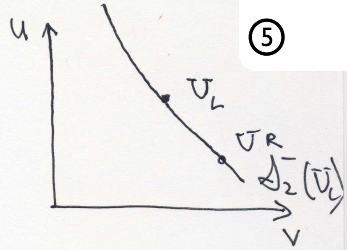
Note: The result requires $p' < 0$, $p'' > 0$

Note: This tells us that for p-system a shock satisfies the Lax entropy condt iff it is the limit as $\varepsilon \to 0$ of a viscous profile soln of $U_t + f(U)_x = \varepsilon U_{xx}$.

☒ Assume $\bar U_R \in \mathcal{S}_2^-(\bar U_L)$

So $s[\bar U] = [f]$ and

$$u_R < u_L,\ v_R > v_L.$$



- Traveling wave equations:

$$v' = -s(v-v_L) + (-u+u_L) = \phi(u,v)$$
$$u' = -s(u-u_L) + (p-p_L) = \psi(u,v)$$

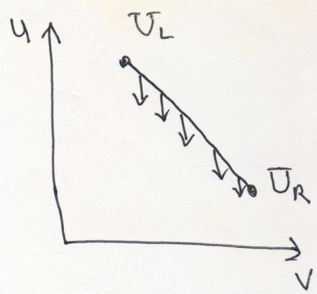
$s = s(\bar U_L, \bar U_R) > 0$ constant

- Two isoclines $\phi(u,v)=0$ & $\psi(u,v)=0$

☒ $\phi(u,v)=0 \iff \boxed{u-u_L = -s(v-v_L)}$

straight line of slope $-s<0$

- Claim: $u'<0$ on isocline $\overline{\bar U_L \bar U_R}$



Lemma 1 $u'<0$ on isocline $\overline{\bar U_L \bar U_R}$ ($u'=0$ @ $\bar U_L$, $\bar U_R$ rest pts)

Pf. Since $s[\bar U] = [f]$ we know

$$s = s(\bar U_L, \bar U_R) = \underbrace{\frac{p_R-p_L}{u_R-u_L}}_{s[u]=[-p]} = \underbrace{-\frac{p_R-p_L}{v_R-v_L}\frac{1}{s}}_{s[v]=-[u]}$$

thus

$$s^2 = -\frac{p_R-p_L}{u_R-u_L}$$

Now eqn for u' is

$$u' = -s(u-u_L) + p - p_L$$

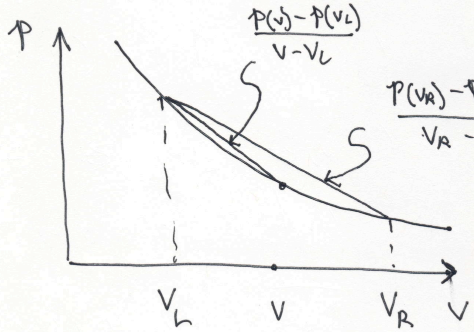
which along isocline $u-u_L = -s(v-v_L)$ gives

$$u' = s^2(v-v_L) + p - p_L = \left\{ s^2 + \frac{p-p_L}{v-v_L} \right\}(v-v_L)$$

$$\Rightarrow \quad u' = \left\{ -\frac{p_R-p_L}{v_R-v_L} + \frac{p-p_L}{v-v_L} \right\}(v-v_L)$$

⑦

Now Lemma1 follows from the convexity of $p(v)$: I.e., $p' < 0$, $p'' > 0$ $\Rightarrow$



$$\frac{p(v_R)-p(v_L)}{v_R-v_L} > \frac{p-p_L}{v-v_L}$$

"less negative"

$$\therefore \quad 0 > -\frac{p_R-p_L}{v_R-v_L} + \frac{p-p_L}{v-v_L} \qquad (*)$$

$$\Rightarrow \quad u' = \Big\{ \underbrace{-\frac{p_R-p_L}{v_R-v_L} + \frac{p-p_L}{v-v_L}}_{\text{neg}} \Big\} \underbrace{(v-v_L)}_{\text{pos}} < 0 \ \checkmark$$

⑧

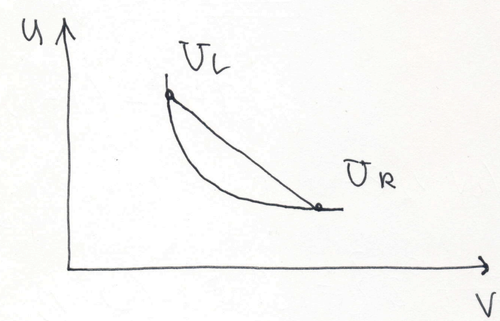
2nd isocline $\psi(u,v)=0 \Leftrightarrow \boxed{p-p_L = s(u-u_L)}$

$$\Leftrightarrow \quad u-u_L = \frac{p-p_L}{s}$$

$$\frac{du}{dv} = \frac{p'(v)}{s} < 0$$

$$\frac{d^2u}{dv^2} = \frac{p''(v)}{s} > 0$$

$\Rightarrow$ isocline is decreasing convex up & passes thru U_L, U_R



Lemma 2 : $V' < 0$ on the isocline $\Psi(u,v) = 0$ betw U_L & U_R.

Pf. $u' = \Psi(u,v) = 0 = -s(u-u_L) + (p-p_L)$

$$\Rightarrow \qquad p - p_L = s\,(u-u_L)$$

Thus

$$v' = -\underbrace{s(v-v_L)}_{\text{neg}} + \underset{\text{pos}}{(-u+u_L)} = -s(v-v_L) - \frac{p-p_L}{s}$$

$$= \left\{ -s^2 - \frac{p-p_L}{v-v_L} \right\} \frac{v-v_L}{s}$$

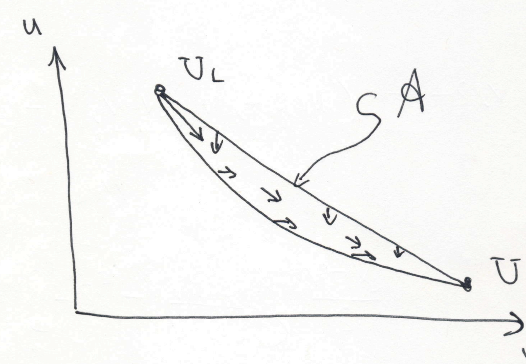
$$= \underbrace{\left\{ \frac{p_R-p_L}{v_R-v_L} - \frac{p-p_L}{v-v_L} \right\}}_{>0 \text{ by } (*)} \frac{v-v_L}{s} > 0 \quad \checkmark$$

Picture: autonomous system of ODE's determines states on traveling wave:

$$v' = -s(v-v_L) + (-u+u_L) = \varphi(u,v)$$

$$u' = -s(u-u_L) + (p-p_L) = \Psi(u,v)$$

$$s = s(U_L, U_R) = \text{const} > 0, \quad U = (v,u)$$



Rest pts U_L, U_R & region betw isoclines in an invariant region $\mathcal{A}$. We prove $\exists !$ orbit connecting U_L to U_R

(Ie. $U_R \in \mathcal{S}_2^-(U_L)$ is only pt that meets Hug. locus $s[U] = [f]$ with speed s because s monotone along $\mathcal{S}(U_L)$)

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- Check: $v'>0$, $u'<0$ in $\underbrace{}^{\text{interior of}}$ region A:

Pf. The isoclines $\phi(u,v)=0$ & $\psi(u,v)=0$ are where v' & u' change sign. Since A is bounded by isoclines, v' & u' have const sign in int$A$. But $u'<0$ on $\phi(u,v)=0$ & $v'>0$ on $\psi(u,v)=0$ $\Rightarrow$ these signs are maintained thruout.

- Thus – the existence of a unique orbit connecting U_L to U_R follows so long as U_L is a saddle pt whose unstable direction points into A. I.e. then the unstable orbit @ A starts into A, & since int$A$ has no rest pts, Poincare-Bendixson $\Rightarrow$ orbit must end at U_R. (u & v are monotone along orbit)

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Proof: that U_L is a saddle, & unstable direction points into A:

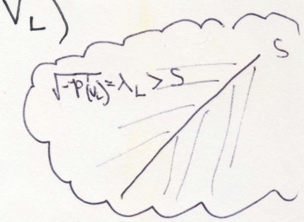
$$\begin{pmatrix} v \\ u \end{pmatrix}' = \begin{pmatrix} -S(v-v_L)+(-u+u_L) \\ -S(u-u_L)+(p-p_L) \end{pmatrix} = F(v,u)$$

$$\left.\frac{\partial F}{\partial U}\right|_{U=U_L} = \begin{bmatrix} -S & -1 \\ p'(v_L) & -S \end{bmatrix} = dF$$

$$0 = |dF-\lambda I| = (-S-\lambda)^2 + p'(v_L)$$

$$(S+\lambda)^2 = -p'(v_L)$$

$$\lambda = -S \pm \sqrt{-p'(v_L)}$$



Lax Shock condt

$\Rightarrow$ one pos / one neg eval $\Rightarrow$ saddle ✓

Claim: unstable direction pts into $\mathcal{A}$:

• $\lambda_+ = -s + \sqrt{-p'(v_L)}$

$$\begin{bmatrix} -\sqrt{-p'_L} & -1 \\ p'_L & -\sqrt{-p'_L} \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = 0$$

$\iff -\sqrt{-p'_L} - b = 0 \iff \boxed{b = -\sqrt{-p'_L}}$

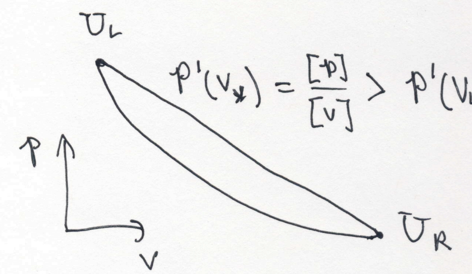
$$\Rightarrow R_+ = \begin{array}{cc} \begin{bmatrix} 1 \\ -\sqrt{-p'_L} \end{bmatrix} & \begin{array}{l} \leftarrow v \\ \leftarrow u \end{array} \end{array} \sim \frac{du}{dv} = -\sqrt{-p'_L} \text{ in direction } R_+$$

• $0 = \phi(u,v) = -s(v-v_L) + (-u+u_L) \sim \boxed{\frac{du}{dv} = -s}$

• $0 = \psi(u,v) = -s(u-u_L) + (p-p_L) \sim \boxed{\frac{du}{dv} = \frac{p'_L}{s}}$

• Now $s = \sqrt{-\frac{p_L - p_R}{v_L - v_R}} = \sqrt{-p'(v_*)}$

& $p' < 0$ $p'' > 0$ $\Rightarrow$



$p'(v_*) = \frac{[p]}{[v]} > p'(v_L)$

$\therefore$ $-p'(v_*) < -p'(v_L)$

$\sqrt{-p'(v_*)} < \sqrt{-p'(v_L)}$

$-\sqrt{-p'(v_*)} > -\sqrt{-p'(v_L)}$

$$\Rightarrow -\sqrt{-p'(v_*)} > -\sqrt{-p'(v_L)} > -\underbrace{\frac{\sqrt{-p'(v_L)}}{\sqrt{-p'(v_*)}}}_{>1} \sqrt{-p'(v_L)}$$

$\frac{du}{dv}$ along $\phi = 0$ @ U_L

$\frac{du}{dv}$ along R_1 at U_L

$\frac{du}{dv}$ along $\psi = 0$ @ U_L

Picture

U_L

$$\frac{du}{dv} = -\sqrt{-p'(v_*)} \qquad \phi = 0$$

$$\frac{du}{dv} = -\sqrt{-p'(v_L)} \qquad R_1$$

$$\frac{du}{dv} = \frac{p'(v_L)}{\sqrt{-p'(v_*)}} = -\frac{\sqrt{-p'(v_L)}}{\sqrt{-p'(v_*)}}\sqrt{-p'(v_L)}$$

$$\psi = 0$$

Proves that R_1 pts into A at U_L.

Theorem: if $U_R \in S_2^-(U_L)$ then $\exists$ a unique traveling wave soln of the viscous p-system (CL) $U_t + f(U)_x = \varepsilon U_{xx}$ connecting U_L to U_R & propagating at speed s.

Sim. result for $U_R \in S_1^-(U_L)$.

Note ① we required $p' < 0$, $p'' > 0$

② The argument fails for $U_R \in S_{1,2}^+(U_L)$

Conclude: For the p-system the Lax entropy condt for shocks is equivalent to the condition that shocks be limits of traveling wave soln's of

$$U_t + f(U_x) = \varepsilon U_{xx}$$

in limit $\varepsilon \to 0$.

$\boxed{}$

☒ Existence of traveling wave soln's for Navier-Stokes (Gilbarg, Am Jour Math Vol 73, No 2. pp 256-274 (1951))

From (MA), (Mo), (En) pg 7-13 Section 2 NS

(MA) $\rho_t + \mathrm{div}(\rho u) = 0$

(Mo) $(\rho u)_t + \mathrm{div}(\rho u \otimes u - \sigma) = 0$

(En) $E_t + \mathrm{div}\left[(E - \sigma) u\right] = 0 \Leftrightarrow$

with NS stress tensor

(NS) $\sigma = -pI + \tilde{\sigma}$, $\tilde{\sigma} = (\lambda\,\mathrm{div}\,u) I + 2\mu D$

$D \equiv$ "symm part of vel grad" $= \frac{1}{2}\left(u^i_{x_j} + u^j_{x_i}\right)$

Assuming (NS), (Mo) & (En) $\Leftrightarrow$

(Mo) $(\rho u)_t + \mathrm{div}(\rho u \otimes u + p) = (\lambda + \mu)\nabla\,\mathrm{div}\,u + \mu \Delta T$

(En) $E_t + \mathrm{div}((E+p)u) = \mathrm{div}\,\tilde{\sigma} u + k \Delta T$

Restrict to $x \in \mathbb{R}$:

$\boxed{\text{1-D}}$ (MA) $\rho_t + (\rho u)_x = 0$

(Mo) $(\rho u)_t + (\rho u^2 + p)_x = (\lambda + 2\mu) u_{xx}$

(En) $E_t + ((E+p)u)_x = (\lambda + 2\mu)(u u_x)_x + k T_{xx}$

Set $\lambda + 2\mu \leftrightarrow \mu$ & $k \leftrightarrow \lambda$ & $T \leftrightarrow \theta$

$\Leftrightarrow$

(NS)

(MA) $\rho_t + (\rho u)_x = 0$

(Mo) $(\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}$

(En) $E_t + [(E+p)u]_x = \mu (u u_x)_x + \lambda \theta_{xx}$

* Look for traveling waves ≡ soln's that depend on $x-st$ some speed $s \equiv$ const. WLOG do a Galilean Trans. so $s=0$ $\Rightarrow$ $\rho=\rho(x)$, $u=u(x)$, $E=E(x)$ plug in (NS)

(MA) $\rho u = \text{const} = c_1$

(Mo) $\rho u^2 + p - \mu u_x = \text{const} = c_2$

(En) $\rho u\left(\frac{1}{2}u^2 + e + p/\rho\right) - \mu u u_x - \lambda \theta_x = \text{const} = c_3$

Use (MA) to solve for $u = c_1 \tau$, plug into (Mo) & (En) & use cancelation to get

$$\text{(En)} \quad \lambda \frac{d\theta}{dx} = b\left[e - \frac{1}{2}b^2(\tau-a)^2 - c\right] \equiv L(\tau,\theta)$$

$$\text{(Mo)} \quad \mu \frac{d\tau}{dx} = \frac{1}{b}\left[p + b^2(\tau-a)\right] \equiv M(\tau,\theta) \qquad \text{(E)}$$

$b = c_1$, $a = \frac{c_2}{c_1^2}$, $c = \frac{c_3}{c_1} - c_2^2/2c_1^2$; $p = p(\tau,\theta)$, $e = e(\tau,\theta)$

Now assume $Z_0 = (\tau_0, \theta_0)$ & $Z_1 = (\tau_1, \theta_1)$ are the left & right states of a Lax shock-wave, so $s[U] = [f(U)]$ & WLOG $s=0$ $\Rightarrow [f(U)] = 0$. In 1952, Gilbarg identified the following cond'ts, valid at Lax shocks for Euler w. polytropic eqn of state, sufficient to imply $\exists!$ of shock profiles

Assume:

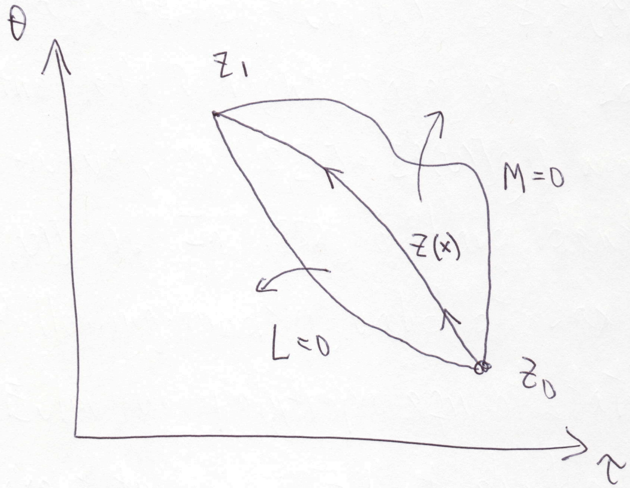
(A) $L_\theta, M_\theta > 0$

(B) $\exists$ two curves L & M on which $L(\tau,\theta)=0$ & $M(\tau,\theta)=0$ resp., which intersect at Z_0, Z_1 st these are the only solns $L=0 \cap M=0$

(C) $L_\tau > 0$ on L $\quad \tau_1 \le \tau \le \tau_0$

(D) $L_\tau / L_\theta < M_\tau / M_\theta$ at Z_0; $L_\tau / L_\theta > M_\tau / M_\theta$ at Z_1

Thm: $\exists!$ shock profile soln (E), $Z(x) \xrightarrow[x \to \mp\infty]{} Z_{0,1}$.



Pf. Condition ⇒ same argument as p-system goes thru . . . (see paper for details)

Homework: Fill in details