THE EXISTENCE AND LIMIT BEHAVIOR OF THE ONE-DIMENSIONAL SHOCK LAYER.*

By David Gilbarg.

Introduction. We consider steady, one-dimensional flows of a viscous, heat-conducting fluid which approach finite limit values at $x = +\infty$ and $x = -\infty$. Such flows display the character of a shock wave (for small viscosity, μ, and heat conductivity, λ), in that they differ sensibly from their end states at $x = \pm\infty$ only in a small interval of rapid transition. In analogy with the classical boundary layer, and also to distinguish these flows from the shock waves which belong properly to the theory of ideal fluids, we follow Weyl [1] in naming such a flow a *shock layer*. The one-dimensional shock layer is in certain respects the prototype of all shock phenomena and has therefore been studied widely, with particular emphasis on the problem of thickness of the shock front [2, 3, 5, 6, 8]. However, basic problems concerning these flows, such as those of existence, and limit behavior for small λ, μ, remain open. Their solution, which we consider here, is a step towards placing on a sound basis the relation between the theories of real and ideal fluids.

The general problem of existence of the shock layer for a fluid with given λ, μ, and with the preassigned end states, has been studied inconclusively by Rayleigh [4] and Weyl [1]. Until now, the existence of the shock layer seems to have been definitely proved only for an exceptional set of ideal gases for which a postulated relation between λ, μ, and the specific heat at constant pressure,[1] permits explicit integration of the equations of motion; (Becker [2], also [5, 6]). We succeed here in obtaining an essentially complete solution of the existence problem by proving the existence and uniqueness of the shock layer for the general class of fluids considered by Weyl, with λ, μ arbitrary functions of the state, and for arbitrary end states satisfying the shock relations (Theorem 1). This result, therefore, establishes for general fluids an exact correspondence between the steady one-dimensional shock waves and the shock layers.

* Received June 3, 1950; revised November 13, 1950.

Prepared under Navy Contract N6onr-180, Task Order V, with Indiana University.

[1] Namely, Prandtl number $c_p\mu/\lambda = 3/4$.

The problem of the limit behavior of the shock layers for small λ and μ is that of providing a rigorous proof for the statement, generally accepted on physical grounds, that any shock wave in an ideal fluid is the limit of the corresponding shock layers, and, conversely, that the shock layers (with the same end states) approach a shock wave in the limit as $\lambda, \mu \rightarrow 0$.[2] Mathematically, this problem belongs to the class of perturbation problems for differential equations in which the vanishing of certain parameters, (λ, μ in this case), reduces the order of the equations. A complete solution to the problem is given in Theorems 2 and 5 for the same general fluids as in the existence theorem. The limit behavior makes apparent the unequal dependence of the flows on viscosity and heat conductivity in the fact that as $\lambda \rightarrow 0$ (with μ fixed), the shock layers converge to a continuous thermally non-conducting shock layer (Theorem 3), whereas for $\mu \rightarrow 0$ (with λ fixed), they converge to a (generally) *discontinuous* non-viscous shock layer (Theorem 4).

We remark that the methods used here provide easily estimated bounds on the thickness of the shock front, although not the precise determinations sought by previous writers [2, 3, 5, 6] on this subject.

1. Existence and uniqueness of the shock layer. The equations of motion for a steady one-dimensional flow of a viscous heat conducting fluid are: [3]

$$\rho u = \text{constant} = c_1; \tag{1}$$

$$\rho u^2 + p - \mu u_x = \text{constant} = c_2 \tag{2}$$

$$\rho u(\tfrac{1}{2}u^2 + e + p/\rho) - \mu u u_x - \lambda \theta_x = \text{const.} = c_3. \tag{3}$$

These equations express respectively the conservation conditions for mass, momentum, and energy. The coefficient of viscosity, μ (which combines the two viscosity coefficients appearing in the general Navier-Stokes equations), and the heat conductivity, λ, are in general functions of the thermodynamic state, and will here be considered in this generality. The other quantities appearing in the equations are the density ρ, pressure p, velocity u, internal energy e per unit mass, and temperature θ. We assume that the variables θ and $\tau = 1/\rho =$ specific volume, fix uniquely the thermodynamic state of the fluid, and that λ, μ, p, and e are sufficiently smooth (e. g. twice differentiable), functions of these variables.

---

[2] This limit behavior is apparent in the explicit solution obtained by Becker [2]; however, in this case, λ and μ are connected by the relation, $c_p\mu/\lambda = 3/4$, and do not approach zero independently. For discussion of the problem of limit behavior, see, for example, [7] pp. 135-138, also [8], pp. 218-222.

[3] See, for example, [7] p. 134; a complete derivation is contained in [9].

Eliminating u from equations (1)-(3), we obtain,

$$\text{(E)}\quad \begin{aligned} \lambda(d\theta/dx) &= b[e(\tau,\theta) - \tfrac{1}{2}b^2(\tau - a)^2 - c] \equiv L(\tau,\theta) \\ \mu(d\tau/dx) &= 1/b[p(\tau,\theta) + b^2(\tau - a)] \equiv M(\tau,\theta), \end{aligned}$$

where,

$$a = c_2/c_1^2, \qquad b = c_1, \qquad c = c_3/c_1 - c_2^2/2c_1^2,$$

and $e = e(\tau,\theta)$, $p = p(\tau,\theta)$ are (given) equations of state of the fluid. With certain minor modifications, this is the form of the equations of motion used by both Weyl [1] and Becker [2].

We enumerate now a set of conditions on the functions $L(\tau,\theta)$, $M(\tau,\theta)$ which will be sufficient to prove the desired existence and limit theorems. In Section **3**, it will be verified that the corresponding functions for the general fluids of Weyl, and, in particular, for polytropic gases, satisfy these conditions. We assume:

The domain of definition is a set of points $Z = (\tau,\theta)$, hereafter called the Z-plane, forming a simply connected region in the quadrant $\tau > 0$, $\theta > 0$. In this region, we have:

(A) $L_\theta > 0$, $M_\theta > 0$.

(B) There are two curves, L and M, on which $L(\tau,\theta) = 0$ and $M(\tau,\theta) = 0$, respectively, and which intersect in two points, $Z_0 = (\tau_0,\theta_0)$, $Z_1 = (\tau_1,\theta_1)$, $(\tau_0 > \tau_1)$; (τ_0,θ_0) and (τ_1,θ_1) are the only simultaneous solutions of $L(\tau,\theta) = 0$ and $M(\tau,\theta) = 0$.

(C) $L_\tau > 0$ on L for $\tau_1 \leq \tau \leq \tau_0$.

(D) $L_\tau/L_\theta < M_\tau/M_\theta$ at Z_0; $L_\tau/L_\theta > M_\tau/M_\theta$ at Z_1.

It is an immediate consequence of (A) that L and M are representable as single valued functions of τ, namely, $\theta = l(\tau)$, $\theta = m(\tau)$, respectively. By virtue of (C), $l(\tau)$ is monotonically decreasing in the interval $\tau_1 \leq \tau \leq \tau_0$, and thus $\theta_1 > \theta_0$. From (B) and (D) it follows $m(\tau) > l(\tau)$ for $\tau_1 < \tau < \tau_0$.

The closed curve formed by the arcs $L^* \subset L$ and $M^* \subset M$ which join Z_0 and Z_1 bounds a simply connected region R of the Z-plane (Fig. 1). We conclude from (A) that everywhere in R, $L(\tau,\theta) > 0$ and $M(\tau,\theta) < 0$.

The preceding geometric facts contain all the information that is needed about the (τ,θ) phase plane for the proofs which follow. More general conditions could be obtained, but would be of no interest for the ensuing applications.

Consider the points $Z_0(\tau_0,\theta_0)$ and $Z_1(\tau_1,\theta_1)$. These satisfy the shock conditions,

$$p_0 + b^2\tau_0 = p_1 + b^2\tau_1 = b^2 a \tag{4}$$

$$e_0 - b^2/2\,(\tau_0 - a)^2 = e_1 - b^2/2\,(\tau_1 - a)^2 = c. \tag{5}$$

Z_0 and Z_1 then represent possible initial and final states, respectively, of a normal shock wave of an ideal fluid having the same equations of state as the given fluid. A solution, $S(x) = (\tau(x), \theta(x))$, $(-\infty < x < +\infty)$, of equations (E), (for given $\lambda(\tau, \theta)$, $\mu(\tau, \theta)$), will be called a *shock layer* if

$$\lim_{x \to -\infty} S(x) = Z_0, \qquad \lim_{x \to -\infty} S(x) = Z_1.$$

The corresponding *shock layer curve* in the Z-plane is the integral curve

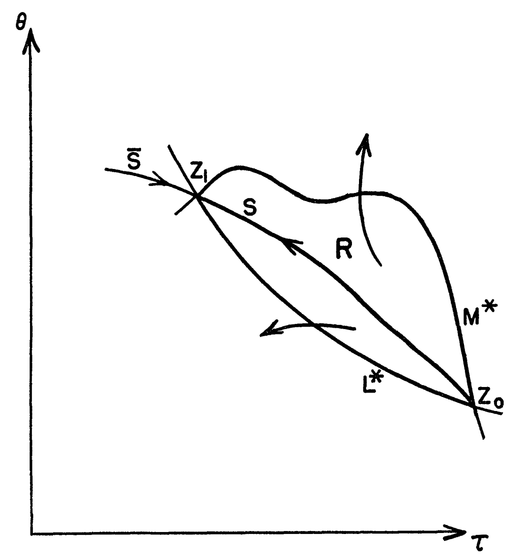

FIG. 1.

represented by the set of "equivalent" shock layers, $S(x + h)$, (h = constant). A shock layer will be called *parametrized* if a particular representative of this class is designated, and, unless so noted, equivalent shock layers will be considered identical.

We observe that if an integral of (E) has the limits Z'_0, Z'_1 at $x = -\infty$, $+\infty$, respectively, then Z'_0 Z'_1 must satisfy the shock conditions (4), (5). It is necessary, therefore, to restrict the definition of shock layer, as we have done, to end states defining a normal shock wave.

We consider first the question of existence and uniqueness of the shock layer for arbitrary positive $\lambda = \lambda(\tau, \theta)$, $\mu = \mu(\tau, \theta)$.

By condition (B), Z_0 and Z_1 are the exclusive singular points of the system (E).[4] We prove that Z_0 is an unstable node and Z_1 a saddle point of (E).[5] For, the characteristic equation of the system (E) at Z_0 and Z_1 is

$$(6) \qquad 0 = \begin{vmatrix} L_\theta/\lambda - \kappa & L_\tau/\lambda \\ M_\theta/\mu & M_\tau/\mu - \kappa \end{vmatrix} = \kappa^2 - (M_\tau/\mu + L_\theta/\lambda)\kappa + (M_\theta L_\theta/\lambda\mu)(M_\tau/M_\theta - L_\tau/L_\theta),$$

where the values λ, μ, L_τ, M_θ, etc., are to be taken at Z_0 and Z_1. The discriminant of this equation is

$$[(M_\tau/\mu - L_\theta/\lambda)^2 + 4(M_\theta L_\tau/\lambda\mu)]_{Z=Z_0,Z_1};$$

by conditions (A) and (C), M_θ, $L_\tau > 0$, so that the discriminant is positive at both Z_0 and Z_1, and therefore the roots of (6) are real. The constant terms of (6), is, by conditions (A) and (D), positive at Z_0, negative at Z_1. The latter point is therefore a saddle, the former a node, and, since $M_\tau, L_\theta > 0$ (conditions (A), (C), (D)), an unstable node.

It follows from well-known general considerations that there are exactly two integral curves of (E) which approach the saddle Z_1 as $x \to +\infty$, and exactly two which approach it as $x \to -\infty$, these pairs corresponding to the negative and positive roots, respectively, of the characteristic equation (6). The two members of each pair have the same slope at Z_1, but approach it from opposite directions. The slopes are given by

$$-L_\tau/(L_\theta - \kappa\lambda) = -(M_\tau - \kappa\mu)/M_\theta,$$

and, in particular, for $\kappa < 0$ at Z_1, this is negative. Hence, by considering the sign of $L(\tau,\theta)/M(\tau,\theta)$ in the neighborhood of Z_1, it is seen that one of the solutions converging to Z_1 as $x \to +\infty$ approaches it from the region R, in which the ratio is negative. Let this solution be designated by $S(x)$. We establish that $S(x)$ is a shock layer. For, consider the integral curves of (E) which pass through the points of M^* and L^*. On M^*, $M(\tau,\theta) = 0$, $L(\tau,\theta) > 0$, so that all integral curves have vertical tangent vectors, and are directed outwards from R for increasing x. Similarly, on L^*, all solutions have zero slope, with $L(\tau,\theta) = 0$, $M(\tau,\theta) < 0$, and are directed outwards from R, since the slope of L^* is negative (by conditions (A) and (C)). Thus, for *decreasing* x, all integral curves of (E) passing through L^* and M^*

---

[4] This makes clear that the shock transition from Z_0 to Z_1 cannot (strictly) occur in a finite x-interval, since the singular points of (E) can be approached only for $x = \pm\infty$.

[5] This is proved by Weyl in the (τ, p) plane ([1], Theorem 6).

are directed *into* R. In R itself, all integral curves have negative slope and are traversed for decreasing x in the direction of increasing τ and decreasing θ. Let us now follow $S(x)$ for decreasing x. By virtue of the preceding, $S(x)$ cannot intersect L^* or M^* between Z_0 and Z_1. Since there are no other singular points of (E), and since $S(x)$ is monotonic while in R, and cannot terminate inside R, it must approach the node Z_0 as $x \to -\infty$. This proves the existence of a shock layer $S(x) = (\tau(x), \theta(x))$.

To prove the uniqueness of the corresponding shock layer curve S, we show that no other integral curve of (E) can join Z_0 and Z_1. For Z_0, being an unstable node, which can be approached by an integral curve only as $x \to -\infty$, the only solution other than S which could join Z_0 and Z_1 would have to be the second integral curve, call it $\bar{S}$, which enters Z_1 as $x \to +\infty$. This is seen to be impossible as follows. If $\bar{S}$ is also a shock layer curve, then the arcs S and $\bar{S}$ form a closed curve bounding a simply connected region G in the Z-plane. One of the two integral curves of (E) which approach Z_1 as $x \to -\infty$ enters this region. It cannot terminate in G, since Z_0 and Z_1 are the exclusive singular points of (E); nor can it approach asymptotically a limit cycle in G, for this in turn would have to contain in its interior a singular point of (E).[6] The only possibility remaining[6] is that the curve intersect S or $\bar{S}$, thereby contradicting the uniqueness of the integral curves of (E). This contradiction completes the uniqueness proof.

Thus, under assumptions (A)-(D), we can state the following existence and uniqueness theorem.

THEOREM 1. *Let $Z_0 = (\tau_0, \theta_0)$ and $Z_1 = (\tau_1, \theta_1)$ be states of a fluid which satisfy the shock conditions, (4), (5); then, for any viscosity $\mu = \mu(\tau, \theta)$, and heat conductivity $\lambda = \lambda(\tau, \theta)$, there exists a unique shock layer joining Z_0 to Z_1.*

2. Limit behavior of the shock layer as $\lambda \to 0$, $\mu \to 0$. For given $\lambda(Z)$, $\mu(Z)$, let the shock layers of the preceding section be designated by $S(x; \lambda, \mu) = (\tau(x; \lambda, \mu), \theta(x; \lambda, \mu))$, and the associated layer curve by $S(\lambda, \mu)$. We assume now concerning $\lambda(Z)$ and $\mu(Z)$ that these coefficients depend on parameters (independent of τ, θ) in such a way that λ and μ can independently be made arbitrarily small in R. In the following, when bounds are placed on λ, μ, or λ/μ, these bounds are to be understood as holding throughout the region R.

[6] Kamke, *Differentialgleichungen Reeller Funktionen*, Chelsea, New York (reprinted 1947), pp. 216, 220-222.

It is our chief object in this section to establish that, as λ and μ approach zero (in whatever manner), the corresponding shock layers approximate a shock wave. In a sense, this "justifies" the usual shock wave theory by showing it to be the limit case of the corresponding theory of real fluids. In the details, it is shown that the limits as $\lambda, \mu \to 0$ exist in the sense of function limits. This requires the individual shock layers to be so parametrized—since they are fixed only to within an x-translation—that the appropriate limits exist under this parametrization. We consider first the double limit, under the same assumptions as in Theorem 1.

THEOREM 2. *Let $Z_0 = (\tau_0, \theta_0)$ and $Z_1 = (\tau_1, \theta_1)$ be initial and final states of the shock wave, $Z = Z_0$, $-\infty < x < \xi$, $Z = Z_1$, $\xi < x < \infty$. Then, the corresponding family of shock layers, $S(x; \lambda, \mu)$, if suitably parametrized, approaches this shock wave as $\lambda \to 0$ and $\mu \to 0$ independently; that is,*

$$\lim_{\substack{\lambda \to 0 \\ \mu \to 0}} S(x; \lambda, \mu) = Z_0, \quad -\infty < x < \xi;$$
$$= Z_1, \quad \xi < x < \infty,$$

the convergence being uniform in every closed interval not containing $x = \xi$.

Proof.[7] For $\epsilon > 0$, let $R(\epsilon)$ be the subregion of R outside the circles of radius ϵ about Z_0, Z_1. Consider the equations (E) for any integral curve in R in the form,

$$\text{(E)} \qquad d\theta/dx = L(\tau, \theta)/\lambda, \qquad d\tau/dx = M(\tau, \theta)/\mu,$$

and take the difference,

$$d(\theta - \tau)/dx = L(\tau, \theta)/\lambda + |M(\tau, \theta)|/\mu > 0$$

Define $\eta = \text{Max}(\lambda, \mu)$ in R, and let $C(\epsilon)$ be a constant for which

$$L(\tau, \theta) + |M(\tau, \theta)| \geq C(\epsilon) > 0, \qquad (\tau, \theta) \in R(\epsilon);$$

such a $C(\epsilon)$ exists, since $L(\tau, \theta) > 0$ on M^*, and $|M(\tau, \theta)| > 0$ on L^* (except at Z_0, Z_1), and are also positive in $R(\epsilon)$. Then, we have for any integral curve in $R(\epsilon)$,

$$(7) \qquad d(\theta - \tau)/dx > C(\epsilon)/\eta > 0$$

Now consider any shock layer $S(x; \lambda, \mu)$, and let (τ_M, θ_M) and (τ_m, θ_m) designate the value of $S(x; \lambda, \mu)$ where it intersects the ϵ circles about Z_1

---

[7] The author is indebted to E. Hopf for this proof, which strengthens and simplifies the author's original proof.

and Z_0 respectively, with $S(x_0;\lambda,\mu)=(\tau_m,\theta_m)$ and $S(x_1;\lambda,\mu)=(\tau_M,\theta_M)$. We have, for $x_0\leq x\leq x_1$,

$$\tau_0>\tau_m\geq\tau(x;\lambda,\mu)\geq\tau_M>\tau_1;$$
$$\theta_0<\theta_m\leq\theta(x;\lambda,\mu)\leq\theta_M<\theta_1.$$

From inequality (7) for $S(x;\lambda,\mu)$, it follows,

$$(8)\qquad x_1-x_0\leq\eta/C(\epsilon)[\theta_M-\tau_M-(\theta_m-\tau_m)]<\eta/C(\epsilon)[\theta_1-\theta_0+\tau_0-\tau_1],$$

Thus, given ϵ and δ, and $S(x;\lambda,\mu)$ for which

$$\eta=\mathrm{Max}(\lambda,\mu)\leq\delta C(\epsilon)/(\theta_1-\theta_0+\tau_0-\tau_1),$$

the simultaneous inequalities,

$$(9)\qquad |\tau(x;\lambda,\mu)-\tau_{0,1}|>\epsilon,\qquad |\theta(x;\lambda,\mu)-\theta_{0,1}|>\epsilon,$$

can hold only when the values of $S(x;\lambda,\mu)$ lie in $R(\epsilon)$, and therefore in an interval of length less that $x_1-x_0<\delta$. In other words, by making heat conductivity and viscosity sufficiently small, the width of the shock front can be made arbitrarily small. To complete the proof of the theorem, one need only observe that the solutions $S(x;\lambda,\mu)$ can be so parametrized that the intervals (x_0,x_1) contain $x=\xi$. The convergence of the $S(x;\lambda,\mu)$ to the discontinuous shock is then uniform outside of every open interval containing $x=\xi$.

We note that with arbitrary parametrization, either the double limit is a shock wave, as above, or does not exist in an entire interval, or is a constant, Z_0 or Z_1. Hence, among all parametrizations, the only non-trivial limits of shock layers, as $\lambda,\mu\rightarrow 0$ independently, are the shock waves.

We note also that (8) gives a (probably very poor) upper bound on the width of the shock transition in the sense of inequality (9). Similarly, lower bounds as well can be obtained from reverse inequalities of the type (7). It is possible that, by taking other linear combinations of the two equations (E), and thereby obtaining inequalities analogous to (7), more satisfactory bounds can be achieved.

We turn now to the single limit problems, $\lim_{\lambda\rightarrow 0}S(x;\lambda,\mu)$ and $\lim_{\mu\rightarrow 0}S(x;\lambda,\mu)$, for fixed $\mu(Z)$ and $\lambda(Z)$, respectively. First we prove,

LEMMA 1. *Let G be any open neighborhood of the closed arc L^*. Then for sufficiently small values of λ/μ, all shock layer curves $S(\lambda,\mu)$ lie entirely in G.*

Proof. Let $\bar{L}$ be an arc of bounded negative slope with endpoints $\neq Z_0$, Z_1 on M^*, and lying so close to L^* that the subregion of R contained between L^* and $\bar{L}$ contains only points of G. Such a curve $\bar{L}$ can always be found by virtue of the monotonicity of L^* between Z_0 and Z_1. Let D be the subregion of R bounded by $\bar{L}$ and M^*; D contains the complement of G in R. On $\bar{L}$, $L(\tau, \theta) \geq k_1 > 0$, $|M(\tau, \theta)| \leq k_2$, and $|\text{slope } \bar{L}| \leq N$, where k_1, k_2, and N are suitable positive constants. Let now $\mu/\lambda > (k_2/k_1)N$; then since the slopes of the corresponding integral curves of (E) satisfy on $\bar{L}$ the inequality,

$$-d\theta/d\tau = -(\mu/\lambda)(L(\tau,\theta)/M(\tau,\theta)) \geq (\mu/\lambda)(k_1/k_2) > N,$$

we have that these integral curves must be directed into D. Thus, any integral curve of (E) for which $\mu/\lambda > (k_2/k_1)N$ and which contains a point of D cannot intersect $\bar{L}$ beyond this point (i.e. for increasing x), and hence cannot pass through Z_1. As a consequence, if $S(\lambda, \mu)$ is a shock layer curve for which $\lambda/\mu < k_1/k_2N$, $S(\lambda, \mu)$ must lie entirely in the region G (indeed, in the smaller region between L^* and $\bar{L}$).

Similar considerations apply in case of small μ/λ. An interesting difference arises from the possibility that M^* is not monotonic. If the arc is monotonic, then exactly the same argument as above applies, with but evident verbal changes. If, however, M^* is not monotonic between Z_0 and Z_1, consider then the arc $\bar{M}^*$ defined by the monotonic function in $[\tau_1, \tau_0]$,

$$\theta = \bar{m}(\tau) = \operatorname*{Min}_{\tau_1 \leq t \leq \tau} m(t); \qquad \tau_1 \leq \tau \leq \tau_0.$$

This arc (Fig. 2) joining Z_1 and Z_0[8] encloses between it and L^* a subregion of R in which, we assert, all shock layer curves must be contained. For, if $\theta = \sigma(\tau)$ is the equation of any shock layer, we have, for every τ in $[\tau_1, \tau_0]$, and some t such that $\tau_1 \leq t \leq \tau$,

$$\sigma(\tau) \leq \sigma(t) \leq m(t) = \bar{m}(\tau),$$

with the inequality, $\sigma(\tau) < \bar{m}(\tau)$, holding provided $\tau \neq \tau_0, \tau_1$. Hence, all shock layer curves lie below $\bar{M}^*$ in R. (The same proof shows that $\bar{M}^*$ is the maximum among all monotonic arcs dominated by M^*.) Replacing M^* by $\bar{M}^*$, we find that the same argument as in the case that M^* is monotonic proves that, for sufficiently small μ/λ, the shock layer curves $S(\lambda, \mu)$ lie in any preassigned neighborhood of $\bar{M}^*$. Thus we can state,

---

[8] That $Z_0 \varepsilon \bar{M}^*$ is clear; for, if not, there is a $\bar{\tau}$, $\tau_1 < \bar{\tau} < \tau_0$, for which $l(\bar{\tau}) < m(\bar{\tau}) < m(\tau_0) = l(\tau_0)$, contradicting the monotonic decreasing character of $l(\tau)$.

LEMMA 2. *Let $\bar{M}^*$ be the arc defined by the function,*

$$\theta = \bar{m}(\tau) = \operatorname*{Min}_{\tau_1 \leq t \leq \tau} m(t), \qquad (\tau_1 \leq \tau \leq \tau_0);$$

and let G be any open neighborhood of $\bar{M}^*$. Then all shock layer curves $S(\lambda, \mu)$, for which μ/λ is sufficiently small, lie entirely in G.

Consider now the reduced systems,

$$\text{(10a)} \quad \begin{aligned} \mu(d\tau/dx) &= M(\tau, \theta) \\ 0 &= L(\tau, \theta) \end{aligned}; \qquad \text{(10b)} \quad \begin{aligned} \lambda(d\theta/dx) &= L(\tau, \theta) \\ 0 &= M(\tau, \theta). \end{aligned}$$

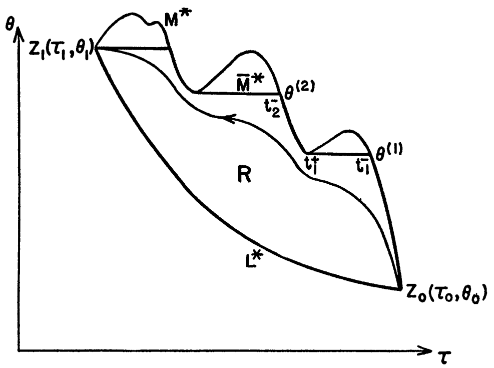

FIG. 2.

The former, (10a), are the equations of a viscous, thermally non-conducting flow, and (10b) of a non-viscous, thermally conducting flow. Let us call any solution of the reduced system (10a), for τ in $[\tau_1, \tau_0]$ and for all x, a *thermally non-conducting shock layer*; (the constant solutions, $(\tau(x), \theta(x)) = Z_0, Z_1$, are thus included in this definition, chiefly for convenience). Then we have,

THEOREM 3. *Let $\bar{S}(x;\bar{\mu})$ be a thermally non-conducting shock layer with viscosity $\bar{\mu}$. Then, the family of shock layers, $S(x;\lambda,\bar{\mu})$, if suitably parametrized, approaches $\bar{S}(x;\bar{\mu})$ as $\lambda \to 0$; that is,*

$$\lim_{\lambda \to 0} S(x;\lambda,\bar{\mu}) = \bar{S}(x;\bar{\mu}),$$

the convergence being uniform in x in every finite interval.

Proof. Let $\zeta = \bar{S}(\bar{x};\bar{\mu}) = (\bar{\tau}(\bar{x};\bar{\mu}), \bar{\theta}(\bar{x};\bar{\mu}))$ be a point on L^*, and x_λ, values such that $x_\lambda \to \bar{x}$ as $\lambda \to 0$. Choose the parametrization of the shock layers, $S(x;\lambda,\bar{\mu}) = (\tau(x;\lambda,\bar{\mu}), \theta(x;\lambda,\bar{\mu}))$, so that $\lim_{\lambda \to 0} S(x_\lambda;\lambda,\bar{\mu}) = \zeta$; this is possible by virtue of Lemma 1. Also from this lemma, we have for every shock layer,

$$\theta = l(\tau) + \epsilon(\tau, \lambda), \tag{11}$$

where $\lim_{\lambda \to 0} \epsilon(\tau, \lambda) = 0$ uniformly in $\tau_1 \leq \tau \leq \tau_0$, and therefore in $-\infty < x < +\infty$. Thus,

$$\bar{\mu} d\tau(x; \lambda, \bar{\mu})/dx = M(\tau, l(\tau) + \epsilon(\tau, \lambda)).$$

Because of the choice of the $\tau(x; \lambda, \bar{\mu})$, we have from this equation, that $\lim_{\lambda \to 0} \tau(x; \lambda, \bar{\mu})$ exists uniformly in every finite x-interval and satisfies

$$\bar{\mu} d\tau/dx = M(\tau, l(\tau)),$$

that is, satisfies the reduced system (10a). Since there is a unique solution of the reduced system passing through the point ζ for $x = \bar{x}$, we conclude $\lim_{\lambda \to 0} \tau(x; \lambda, \bar{\mu}) = \bar{\tau}(x; \bar{\mu})$, and from (11) $\lim_{\lambda \to 0} \theta(x; \lambda, \bar{\mu}) = \bar{\theta}(x; \bar{\mu})$, these limits, by the above, being uniform in every finite interval.

We observe from the above proof that if the shock layers are so parametrized that $\lim_{\lambda \to 0} S(x; \lambda, \bar{\mu})$ exists for a single value of x, then it exists for all x and defines a thermally non-conducting shock layer.

For λ fixed and $\mu \to 0$ the limit solution of the reduced system is no longer continuous in general, as the following Theorem 4 shows. This points up a basic difference in the effects of viscosity and heat conduction on the structure of the shock layers.

To simplify considerations, it will be assumed in the following that M^* has only a finite number of minima, so that the arc $\bar{M}^*$ contains at most a finite number of intervals on which θ is constant. We note that Z_0 cannot lie in such an interval, while Z_1 may or may not, in general. In particular, for an ideal gas, M^* is a parabola; if it is not monotonic, $\bar{M}^*$ consists of the segment of $\theta = \theta_1$ intercepted by M^* plus the arc of M^* joining the segment to Z_0.

Let the function $\theta = \bar{m}(\tau)$, $(\tau_1 \leq \tau \leq \tau_0)$, be constant on the intervals, $[t_i^+, t_i^-]$, $(i = 1, \cdots, n)$, which are ordered so that $t_i^- > t_i^+ > t_{i+1}^-$, (Fig. 2). We enlarge the notion of solution of the reduced system (10b) by admitting certain *discontinuous* solutions, namely, functions $\bar{\tau}(x)$, $\bar{\theta}(x)$ satisfying (10b) for $\bar{\tau}$ in $[\tau_1, \tau_0]$, except at points $x = x_i$, $i = 1, \cdots, n$, where $\bar{\tau}(x_i - 0) = t_i^-$, $\bar{\tau}(x_i + 0) = t_i^+$; these solutions are uniquely determined in $-\infty < x < +\infty$ up to an x-translation. If $n = 0$, that is, if $m(t)$ is strictly monotonic, then the solutions are, of course, continuous. Any solution of (10b) for τ in

$[\tau_1, \tau_0]$ and for all x, whether continuous, or discontinuous in the above sense, we term a *non-viscous shock layer.* We now prove,

THEOREM 4. *Let $\bar{S}(x;\bar{\lambda})$ be a non-viscous shock layer with heat conductivity $\bar{\lambda}$. Then, the family of shock layers, $S(x;\bar{\lambda},\mu)$, if suitably parametrized, approaches $\bar{S}(x;\bar{\lambda})$ as $\mu \to 0$; that is,*

$$\lim_{\mu\to 0} S(x;\bar{\lambda},\mu) = \bar{S}(x;\bar{\lambda}),$$

the convergence being uniform in x in any closed interval not containing a point of discontinuity of $\bar{S}(x;\bar{\lambda})$.

Proof. If M^* is strictly monotonic, the theorem is proved exactly as Theorem 3. If, however, this is not the case, we proceed as follows. Let $\bar{S}(x;\bar{\lambda}) \equiv \bar{S}(x) = (\bar{\tau}(x), \bar{\theta}(x))$; (for notational convenience in the following, we omit reference to $\bar{\lambda}$, since it will be the common value of the heat conductivity for all the shock layers under consideration). Let $\zeta = \bar{\tau}(\bar{x}), \bar{\theta}(\bar{x}))$ be a point on M^*, $(\bar{x} < \bar{x}_1)$, where $\bar{x}_1$ is the first point of discontinuity of $\bar{\tau}(x)$. For any set x_μ, such that $x_\mu \to \bar{x}$ as $\mu \to 0$, let the shock layers $S(x;\mu) = (\tau(x;\mu), \theta(x;\mu))$ be so parametrized that $\lim_{\mu\to 0} S(x_\mu;\mu) = \zeta$. We have, by Lemma 2,

$$\tau = m^{-1}(\theta) + \epsilon(\theta,\mu), \tag{12}$$

where $m^{-1}(\theta)$ is the inverse of $m(\tau)$ for $t_1^- \leq \tau \leq \tau_0$, $(\theta_0 \leq \theta \leq \theta^{(1)} = m(t_1^-))$, and $\lim_{\mu\to 0} \epsilon(\theta,\mu) = 0$ uniformly in every closed interval of $\theta_0 \leq \theta < \theta^{(1)}$. For $\theta(x;\mu)$ we therefore have,

$$\bar{\lambda} d\theta(x;\mu)/dx = L[m^{-1}(\theta) + \epsilon(\theta,\mu), \theta]. \tag{13}$$

Since $\theta(x_\mu, \mu) \to \theta(\bar{x})$ and $x_\mu \to \bar{x}$ as $\mu \to 0$, and $\bar{\tau} = m^{-1}(\bar{\theta})$ for $\theta_0 \leq \theta < \theta^{(1)}$, if follows that $\lim_{\mu\to 0} \theta(x;\mu) = \bar{\theta}(x)$, $-\infty < x < \bar{x}_1$, and from (12), $\lim_{\mu\to 0} \tau(x;\mu) = \bar{\tau}(x)$, the convergence in both cases being uniform in every closed interval to the left of $x = \bar{x}_1$. To prove convergence for $x > \bar{x}_1$, consider first the case that $\tau_1 \neq t_1^+$. By the preceding there are values $x_\mu^{(1)} < \bar{x}_1$, such that $x_\mu^{(1)} \to \bar{x}_1$, $\tau(x_\mu^{(1)}, \mu) \to t_1^-$, and $\theta(x_\mu^{(1)}, \mu \to \theta^{(1)}$, as $\mu \to 0$. With the same parametrization of the $S(x;\mu)$ as chosen above, let ξ_μ be values for which $\theta(\xi_\mu;\mu) = \theta^{(1)}$; then $\tau(\xi_\mu;\mu) \to t_1^+$ as $\mu \to 0$ (Lemma 2). We show now that $|\xi_\mu - x_\mu^{(1)}| \to 0$ with μ. For, $L(\tau,\theta)$ is bounded away from zero in the neighborhood of the segment $\theta = \theta^{(1)}$ of $\bar{M}^*$; say $L(\tau,\theta) > k > 0$ in such a neighborhood; and, if μ is sufficiently small, the arcs of $S(x;\mu)$ for

$x_\mu^{(1)} \leq x \leq \xi_\mu$ lie in this neighborhood (Lemma 2). For these μ, we have from the first member of (E),

$$(14) \quad |\xi_\mu - x_\mu^{(1)}| \leq (\lambda/k)[\theta(\xi_\mu, \mu) - \theta(x_\mu^{(1)}, \mu)] = (\lambda/k)[\theta^{(1)} - \theta(x_\mu^{(1)}, \mu)],$$

which approaches zero as $\mu \to 0$. Hence $\xi_\mu \to \bar{x}_1$. We consider equation (12) and (13) again, where now $m^{-1}(\theta)$ is the inverse of $\theta = m(\tau)$ for $t_2^- \leq \tau \leq t_1^+$, $(\theta^{(1)} \leq \theta \leq \theta^{(2)} = m(t_2^-))$, and $\lim_{\mu \to 0} \epsilon(\theta, \mu) = 0$ uniformly in every closed interval of $\theta^{(1)} \leq \theta < \theta^{(2)}$. As before, since $\theta(\xi_\mu, \mu) \to \theta^{(1)} = \bar{\theta}(x_1)$ as $\xi_\mu \to \bar{x}_1$, and $\bar{\tau} = m^{-1}(\bar{\theta})$ for $\theta^{(1)} \leq \bar{\theta} \leq \theta^{(2)}$, and also since the solution of

$$\lambda d\theta/dx = L(m^{-1}(\theta), \theta)$$

is unique for $\theta(\bar{x}_1) = \theta^{(1)}$, and $\theta^{(1)} \leq \theta \leq \theta^{(2)}$, it follows that $\lim_{\mu \to 0} \theta(x; \mu) = \bar{\theta}(x)$ and $\lim_{\mu \to 0} \tau(x; \mu) = \bar{\tau}(x)$ for $\bar{x} \leq x < \bar{x}_2$, the convergence being uniform in the closed intervals of $(\bar{x}_1, \bar{x}_2)$. One proceeds in this way until the arc of $\bar{M}^*$ containing Z_1 is reached. If Z_1 is not contained in one of the intervals of $\bar{M}^*$, then the above process shows the uniform convergence of $S(x; \mu)$ to $(\bar{\tau}(x), \bar{\theta}(x))$ in every closed subinterval of the half line $\bar{x}_n < x < \infty$. If, however, Z_1 belongs to one of the intervals of $\bar{M}^*$, namely $\theta = \theta_1$, then, by definition of the discontinuous non-viscous shock layer, $\tau_1 = t_n^+$, and $(\bar{\tau}(x), \bar{\theta}(x)) = (\tau_1, \theta_1) = Z_1$ for all x in $\bar{x}_n < x < \infty$. In this case, we prove the convergence, $S(x; \mu) \to Z_1$ for $\bar{x}_n < x < \infty$, as follows. For any small $\delta > 0$, let $R(\delta)$ be the subregion of R outside the circle of radius δ about Z_1. Let $k(\delta)$ be such that $L(\tau, \theta) \geq k(\delta) > 0$ in a neighborhood of $\theta = \theta_1$ in $R(\delta)$. If now the values $x_\mu^{(n)} < \bar{x}_n$ are such that $x_\mu^{(n)} \to \bar{x}_n$, $\tau(x_\mu^{(n)}, \mu) \to t_n^-$, and $\theta(x_\mu^{(n)}, \mu) \to \theta^{(n)} = \theta_1$, as $\mu \to 0$, then, for any arguments $\xi_\mu > x_\mu^{(n)}$ such that $S(\xi_\mu; \mu) \in R(\delta)$, it follows from the first of equations (E),

$$|\xi_\mu - x_\mu^{(n)}| \leqq (\lambda/k(\delta))[\theta_1 - \theta(x_\mu^{(n)}; \mu] \to 0, \text{ as } \mu \to 0;$$

in other words, those $x > \bar{x}_n$ for which $\tau(x; \mu) - \tau_1 > \delta$ lie in an interval about $\bar{x}_n$ which grows arbitrarily small as $\mu \to 0$. Since δ is arbitrary, this proves the uniform convergence, $\tau(x; \mu) \to \tau_1$, $\theta(x; \mu) \to \theta_1$, in any closed half line of $\bar{x}_n < x < \infty$, and completes the proof of the theorem.

One observes from the above proof that if the shock layers are so parametrized that $\lim_{\mu \to 0} S(x; \lambda, \mu)$ exists for a single value of x, then it exists for all x and defines a non-viscous shock layer.

To complete discussion of the limit behavior of the shock layers, it remains to establish the existence of the iterated limits as $\lambda, \mu \to 0$, and to show their equality with the double limit (and therefore with a shock wave). We require an appropriate parametrization of the shock layers, with respect to which the double limit and the two iterated limits exist simultaneously. For this, take a circle of sufficiently small radius ϵ about Z_0 (it suffices $\epsilon < \tau_0 - t_1^-$); this intersects each shock layer curve, $S(\lambda, \mu)$, at exactly one point, which we designate by $\zeta_{\lambda\mu}$; and let ζ_λ, ζ_μ represent the point of intersection with M^* and L^*, respectively. We note that $\lim_{\lambda \to 0} \zeta_{\lambda\mu} = \zeta_\mu$ (Lemma 1), and $\lim_{\mu \to 0} \zeta_{\lambda\mu} = \zeta_\lambda$ (Lemma 2). For fixed ξ, now let

$$x_{\lambda\mu} = \xi - \tfrac{1}{2}\delta_{\lambda\mu}, \qquad \bar{x}_\mu = \xi - \tfrac{1}{2}\delta_\mu, \qquad \bar{x}_\lambda = \xi - \tfrac{1}{2}\delta_\lambda,$$

where,

$$\delta_{\lambda\mu} = (\eta/C(\epsilon))[\theta_1 - \theta_1 + \tau_0 - \tau_1], \qquad \eta = \mathrm{Max}(\lambda, \mu),$$

$$\delta_\mu = (\mathrm{Max}\, \mu/C(\epsilon))[\theta_1 - \theta_0 + \tau_0 - \tau_1],$$

$$\delta_\lambda = (\mathrm{Max}\, \lambda/C(\epsilon))[\theta_1 - \theta_0 + \tau_0 - \tau_1],$$

$C(\epsilon)$ being the same bound as in inequality (7) of Theorem 2. We make the assignment $S(x_{\lambda\mu}; \lambda, \mu) = \zeta_{\lambda\mu}$; this will prove to be the desired parametrization. From the proof of Theorem 2, we see that, since ξ is contained in all the intervals $(x_{\lambda\mu}, x_{\lambda\mu} + \delta_{\lambda\mu})$ as $\eta \to 0$, it must follow,

$$\lim_{\substack{\lambda \to 0 \\ \mu \to 0}} S(x; \lambda, \mu) = Z_0, \quad -\infty < x < \xi$$
$$= Z_1, \quad \xi < x < +\infty.$$

Also, since for fixed μ,

$$\lim_{\lambda \to 0} \delta_{\lambda\mu} = (\mathrm{Max}\, \mu/C(\epsilon))[\theta_1 - \theta_0 + \tau_0 - \tau_1] = \delta_\mu,$$

we have $\lim_{\lambda \to 0} x_{\lambda\mu} = \xi - \tfrac{1}{2}\delta_\mu = \bar{x}_\mu$, $\lim_{\lambda \to 0} S(x_{\lambda\mu}; \lambda, \mu) = \zeta_\mu$. Therefore as in the proof of Theorem 3, $\lim_{\lambda \to 0} S(x; \lambda, \mu) = (\bar{\tau}(x; \mu), \theta(x; \mu))$ where $(\bar{\tau}(x; \mu), \bar{\theta}(x; \mu))$ is the solution of the reduced system (10a) which satisfies $(\bar{\tau}(\bar{x}_\mu; \mu), \bar{\theta}(\bar{x}_\mu; \mu)) = \zeta_\mu$. This, combined with the existence of the double limit, establishes the existence of the iterated limit, $\lim_{\mu \to 0} (\lim_{\lambda \to 0} S(x; \lambda, \mu))$, and its equality with the double limit. Identical considerations apply to $\lim_{\lambda \to 0} (\lim_{\mu \to 0} S(x; \lambda, \mu)$. Consequently, we may state, THEOREM 5. *Let $Z_0 = (\tau_0, \theta_0)$, $Z_1 = (\tau_1, \theta_1)$ be initial and final states of the shock wave, $Z = Z_0$, $-\infty < x < \xi$, $Z = Z_1$, $\xi < x < +\infty$. Then the shock layers $S(x; \lambda, \mu)$ can be parametrized so that,*

$$\lim_{\substack{\lambda\to 0\\ \mu\to 0}} S(x;\lambda,\mu) = \lim_{\lambda\to 0}(\lim_{\mu\to 0} S(x;\lambda,\mu))$$
$$= \lim_{\mu\to 0}(\lim_{\lambda\to 0} S(x;\lambda,\mu)) = \begin{matrix} Z_0, & -\infty < x < \xi, \\ Z_1, & \xi < x < +\infty. \end{matrix}$$

It is clear, conversely, that the iterated limit, if it exists (for some parametrization), must be either a constant or a shock wave.

In the case of polytropic gases with λ, μ taken independent of τ and θ, the convergence of the iterated limits to a shock wave is visible from explicit formulas for the thermally non-conducting shock layers, $\lim_{\lambda\to 0} S(x;\lambda,\mu)$, and for the non-viscous shock layers, $\lim_{\mu\to 0} S(x;\lambda,\mu)$, (presupposing, however, the existence of these single limits; see Becker [2], for example).

The advantages of the (τ, θ) plane as phase plane should be noted. In these variables, the right members of equations (E), and therefore the curves L and M, are independent of λ and μ. This fact makes the proof of Theorem 1 quite simple, and also permits study, with respect to a fixed geometry, of the dependence of the shock layers on λ and μ. In contrast, the topologies of other phase planes, such as the (τ, p) plane used by Weyl in [1], are such as to obscure a satisfactory proof of the existence and limit theorems.

3. Proof of Conditions (A) — (D). It remains to show that conditions (A)-(D) of Section **1** are satisfied for the fluids under consideration. We examine first the polytropic gases because of their particular importance and simplicity. For polytropic gases, $e(\tau,\theta) = \alpha\theta$, $p(\tau,\theta) = \beta\theta/\tau$, where $\alpha, \beta > 0$ are constants of the medium. Thus

$$L(\tau,\theta) = b[\alpha\theta - b^2/2(\tau - a)^2 - c], \quad M(\tau,\theta) = 1/b[\beta\theta/\tau + b^2(\tau - a)].$$

The Z-plane is the quadrant $\tau > 0$, $\theta > 0$, and the constants a, b, c are assumed to be such that the parabolas L: $L(\tau,\theta) = 0$, and M: $M(\tau,\theta) = 0$, intersect in two points, $Z_0 = (\tau_0, \theta_0)$, $Z_1 = (\tau_1, \theta_1)$, $\tau_0 > \tau_1$. Conditions (A)-(D) are now easily seen to hold. (A) $M_\theta, L_\theta > 0$, since $\alpha, \beta, b > 0$; (B) the parabolas L, M have only one component in the Z-plane, and cannot intersect in more than the two assumed points, Z_0, Z_1; (C) $L_\tau = -b^3/2(\tau - a) > 0$ for $\tau < a$, hence for $\tau_1 \leq \tau \leq \tau_0 < a$; (D) follows from the fact that M lies above L between their points of intersection.

We turn now to the general class of fluids defined by Weyl. These are characterized by the following conditions:

I. $d\tau/dp\big)_{S=\text{const.}} < 0, \quad S = \text{entropy};$

Ia. $S_p(\tau, p) > 0, \ \theta_p(\tau, p) > 0;$

II. $d^2\tau/dp^2\big)_{S=\text{const.}} > 0;$

III. in the continuous process of adiabatic compression one can raise pressure arbitrarily high;

IV. the thermodynamic state Z is uniquely specified by pressure p and specific volume τ, and the points (τ, p) representing the possible states Z in a (τ, p) diagram form a convex region.

To these conditions must of course be added the basic thermodynamic relation,

V. $de = \theta dS - p d\tau.$

This class of fluids evidently includes the polytropic gases, and, as Weyl has shown, the theory of shock waves generalizes in entirety to these fluids.

For fluids subject to the above conditions, Weyl proves a number of important results, among which we need the following.

(a) The Hugoniot contour,

$$H(Z, Z_0) = e(\tau, p) - e(\tau_0, p_0) - \tfrac{1}{2}(p + p_0)(\tau_0 - \tau) = 0$$

is a simple curve on which $s = (p - p_0)/(\tau_0 - \tau)$ grows strictly monotonically from $m_0 = - dp/d\tau\big)_{S=S_0}$ at Z_0 to $+\infty$ as Z moves on the upper branch, $p > p_0$, $\tau < \tau_0$.

(b) If $H(Z_1, Z_0) = 0, Z_1$ lying on the upper branch of $H(Z, Z_0) = 0$, then the straight line $(p - p_0)/(\tau_0 - \tau) = (p_1 - p_0)/(\tau_0 - \tau_1) = b^2$ intersects the adiabatic $S = S_0$ at a point $Z = Z_A$, Z_1 lying between Z_0 and Z_A, and $m_0 < b^2 < m_1 = - dp/d\tau(Z_1)\big|_{S=S_1}$

To prove conditions (A)-(D) of Section **1**, consider first the mapping $(\tau, p) \to (\tau, \theta)$ defined by $\theta = \theta(\tau, p)$, $\tau = \tau$. This mapping is 1-1 as well as continuous, for, if both (τ_1, p_1) and (τ_2, p_2) map into the same point (τ, θ), then $\tau_1 = \tau = \tau_2$, and, since $\theta_p > 0$ in the convex (τ, p) plane, it follows also that $p_1 = p_2$. Consequently, the mapping is topological and the $Z(\tau, \theta)$ plane is simply connected.

In the following, let $Z_0(\tau_0, p_0)$, $Z_1(\tau_1, p_1)$, be points for which $H(Z_1, Z_0) = 0$, with $p_1 > p_0$, $\tau_1 < \tau_0$; (either $p_1 > p_0$, $\tau_1 < \tau_0$, or $p_1 < p_0$, $\tau_1 > \tau_0$). Let

$$bM(\tau, p) \equiv p + b^2(\tau - a), \qquad (1/b)L(\tau, p) \equiv e(\tau, p) - \tfrac{1}{2}b^2(\tau - a)^2 - c,$$

$b > 0$, a, c being determined by relations (4), (5). The functions $M(\tau, \theta)$, $L(\tau, \theta)$ are defined in the obvious way:

$$M(\tau, \theta) \equiv M(\tau, p(\tau, \theta)), \qquad L(\tau, \theta) \equiv L(\tau, p(\tau, \theta)).$$

Condition (A) is easily proved, for

$$bM_\theta(\tau, \theta) = p_\theta(\tau, \theta) > 0, \qquad \text{(by Ia)},$$

and

$$(1/b)L_\theta(\tau, \theta) = e_\theta(\tau, \theta) = \theta S_\theta(\tau, \theta), \qquad \text{(by V)}$$

$$= S_p(\tau, p)p_\theta(\tau, \theta) > 0,$$

so that $L_\theta(\tau, \theta) > 0$.

To prove (B), we observe from (a) that $Z_0(\tau_0, p_0)$, $Z_1(\tau_1, p_1)$ are the only simultaneous solutions of $L(\tau, p) = 0$, $M(\tau, p) = 0$ (Weyl's Theorem 7), and therefore $Z_0(\tau_0, \theta_0)$, $Z_1(\tau_1, \theta_1)$ are the only solutions of $L(\tau, \theta) = 0$, $M(\tau, \theta) = 0$. Also, $M(\tau, p) = 0$ only on the straight line M joining Z_0 and Z_1, and since the (τ, p) plane is convex, M consists of only one component. The set in the (τ, θ) plane on which $M(\tau, \theta) = 0$ must therefore also consist of a single curve, namely, the image of M. Letting $\mathcal{L}$ designate the set of points (τ, p) on which $L(\tau, p) = 0$, it remains only to show that the component $L \subset \mathcal{L}$, which passes through Z_1 also passes through Z_0. Since

$$(1/b)L_p(\tau, p) = e_p(\tau, p) = \theta S_p(\tau, p) > 0,$$

the curve L is representable as a single valued function $p = l(\tau)$, with slope,

$$(15) \qquad dl(\tau)/d\tau = -(e_\tau(\tau, p) - b^2(\tau - a))/e_p(\tau, p)$$

$$= -(\theta S_\tau(\tau, b) - p - b^2(\tau - a))/\theta S_p(\tau, p), \quad ((\tau, p) \in L).$$

In particular, at Z_1, L has the slope,

$$-S_\tau/S_p(\tau_1, p_1) = dp/d\tau(Z_1)\big|_{S=S_1} = -m_1 < -b^2.$$

Consequently, L, for increasing τ, enters the convex region D contained between the adiabatic $S = S_0$ and the straight line M (Fig. 3). Since D

lies below M, $p + b^2(\tau - a) < 0$ in D, so that, by (15), $l(\tau)$ is monotonically decreasing function of τ in this region. The curve L cannot terminate in D, must therefore intersect either M, which can only occur at Z_0, or the adiabatic, $S = S_0$.

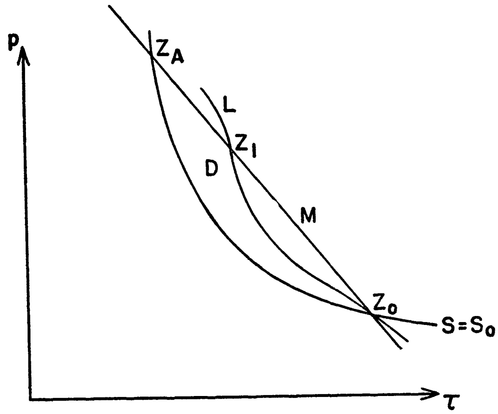

FIG. 3.

But on $S = S_0$,

$$(1/b)\,(dL(\tau, p)/d\tau)\Big)_{S=S_0} = de/d\tau\Big)_{S=S_0} - b^2(\tau - a)$$
$$= [-p - b^2(\tau - a)]_{S=S_0} > 0, \text{ for } \tau_A < \tau < \tau_0;$$

hence $L(\tau, p) \neq 0$ on $S = S_0$ between Z_0 and Z_1. L must therefore pass through Z_0 as well as Z_1. This proves that condition (B) is satisfied.

For condition (C),

$$(1/b)L_\tau(\tau, \theta) = e_\tau(\tau, \theta) - b^2(\tau - a) = \theta S_\tau(\tau, \theta) - p - b^2(\tau - a);$$

but from V, we have, $S_\tau(\tau, \theta) = p_\theta(\tau, \theta) > 0$,

$$\therefore\ (1/b)L_\tau(\tau, \theta) = \theta p_\theta(\tau, \theta) - p - b^2(\tau - a) > 0 \text{ on } L, \text{ for } \tau_1 \leqq \tau \leqq \tau_0.$$

(D) follows from the relations,

$$M_\tau/M_\theta(\tau, \theta) - L_\tau/L_\theta(\tau, \theta) = [M_\tau/M_p(\tau, p) - L_\tau/L_p(\tau, p)]\theta_p(\tau, p)$$
$$= (b^2 - m_0)\theta_p(Z_0) > 0 \quad \text{at } Z_0$$
$$= (b^2 - m_1)\theta_p(Z_1) < 0 \quad \text{at } Z_1.$$

This completes the proof that a Weyl fluid satisfies the conditions (A)-(D) which were assumed in obtaining Theorems 1-5.

BIBLIOGRAPHY.

[1] H. Weyl, " Shock waves in arbitrary fluids," *Communications on Pure and Applied Applied Mathematics*, vol. 2 (1949), pp. 103-122.

[2] R. Becker, " Stosswelle und Detonation," *Zeitschrift für Physik*, vol. 8 (1921-22), pp. 321-362.

[3] L. H. Thomas, " Note on Becker's theory of the shock front," *Journal of Chemical Physics*, vol. 12 (1944), pp. 449-452.

[4] Lord Rayleigh, " Aerial plane waves of finite amplitude," *Proceedings of the Royal Society of London*, Ser. A 84 (1910), pp. 247-284.

[5] A. E. Puckett and H. J. Stewart, " The thickness of a shock wave in air," *Quarterly of Applied Mathematics*, vol. 7 (1950), pp. 457-463.

[6] M. Morduchow and P. Libby, " On a complete solution of the one-dimensional flow equations of a viscous, heat-conducting, compressible gas," *Journal of the Aeronautical Sciences*, vol. 16 (1949), pp. 674-684, 704.

[7] R. Courant and K. Friedrichs, *Supersonic flow and shock waves*, Interscience, New York (1948).

[8] G. I. Taylor and J. W. Maccoll, *The Mechanics of compressible fluids*, vol. III of W. F. Durand, *Aerodynamic Theory*, Springer, Berlin (1934), pp. 218-222.

[9] P. Duhem, *Recherches sur l'hydrodynamique*, Gauthier-Villars, Paris (1903); Première partie, Ch. I. = Annales de la Faculté des Sciences Toulouse, (2) vol. 3 (1901), pp. 315-377.