Homework-I

Temple

Math 280 Shock Waves W2013

1. Show that if $\mu = \lambda = 0$, so in the limit of vanishing viscosity, when $k \neq 0$ then our initial energy equation

$$E_t + div((E + p)\mathbf{u}) = div(\sigma \cdot \mathbf{u}) + k\Delta T,$$

is equivalent to

$$\rho \frac{De}{Dt} = -p\, div(\mathbf{u}) + k\Delta T,$$

where $\sigma = \lambda(div\mathbf{u})I + 2\mu D$, and D is the 3×3 symmetric part of $\nabla\mathbf{u}$.

2. Verify the identity:

$$Div(\rho\mathbf{u}) = \nabla\rho \cdot \mathbf{u} + \rho Div(\mathbf{u}).$$

3. Derive a sharp estimate for the blowup time for u_x, (as we did in class for Burgers Equation) for the general scalar conservation law $u_t + f(u)_x = 0$, assuming $f''(u) > \delta > 0$, i.e., f is convex.

4. Show:

$$\frac{D(fg)}{Dt} = \frac{Df}{Dt} g + f \frac{Dg}{Dt}$$

5. Assume the continuity equation (MA) $\rho_t + div(\rho\mathbf{u}) = 0$. Derive the identity

$$(\rho f)_t + div(\rho f \mathbf{u}) = \rho \frac{Df}{Dt}.$$

6. Show that σ^{ij} symmetric and A_{ij} antisymmetric implies $\sigma^{ij}A_{ij} = 0$.

7. Let σ denote a 3×3 symmetric stress tensor and assume (MA). Show that the momentum equation can be rewritten as
$$\rho \frac{Du}{Dt} = div(\sigma).$$
(Divergence taken on each row of σ.)

8. Show:
$$\frac{1}{2}\frac{D\mathbf{u}^2}{Dt} = \rho\mathbf{u}\frac{D\mathbf{u}}{Dt} = \mathbf{u} \cdot div(\sigma) = div(\sigma\mathbf{u}) - tr(\sigma \cdot D)$$

9. For Euler, show that $\rho\frac{Du}{Dt} = -\nabla p$.

10. Show that the compressible Euler equations are *reversible* in the sense that if $p(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t), e(\mathbf{x}, t)$ solve the compressible Euler equations
$$\rho_t + div(\rho\mathbf{u}) = 0 \tag{1}$$
$$(\rho\mathbf{u})_t + div(\rho\mathbf{u} \otimes \mathbf{u} + p) \tag{2}$$
$$E_t + div((E + p)\mathbf{u}) = 0, \tag{3}$$
then so does $p(\mathbf{x}, -t), -\mathbf{u}(\mathbf{x}, -t), e(\mathbf{x}, -t)$. Explain why this really expresses "reversibility".

11. The sound speed for compressible Euler is $\sigma = \sqrt{\frac{\partial p(\rho,s)}{\partial \rho}}$. Use the formulas for a polytropic (γ-law) gas to show

$$\sigma^2 = \gamma(\gamma - 1)c_v T.$$

That is: *The sound speed is proportional to the temperature.*

12. Show that the second law of thermodynamics, namely, $de = T ds - p\, dv$ is an *exact differential*, implies that

$$ds = \frac{de}{T} + \frac{pdv}{T}$$

defines a function $s(e, v)$ such that

$$\frac{\partial s}{\partial e}(e, v) = \frac{1}{T} \quad \text{and} \quad \frac{\partial s}{\partial v}(e, v) = \frac{p}{T}.$$

13. Show that the second law of thermodynamics ($de = T ds - pdv$ is exact) implies the following formulation in terms of the material derivative:

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{Dv}{Dt}.$$

14. Show that a strictly hyperbolic matrix (a matrix with n real and distinct eigenvalues) has a basis of eigenvectors. Prove that the left eigenvalues (corresponding to the left eigenvectors) are equal to the right eigenvalues.

15. Verify the following tensor relations:

(a) Show that $div(fI) = \nabla f$.

(b) Show that $div(D) = \frac{1}{2}\Delta\mathbf{u} + \frac{1}{2}\nabla(div(\mathbf{u})$

(c) Show $tr(D) = div(\mathbf{u})$.

(d) Show $< \sigma, D >= \sum_{ij} \sigma^{ij} D^{ij} = tr(\sigma \cdot D)$

(e) Show $tr\left((divu)D\right) = (div\mathbf{u})^2$.

(f) Show $\mathbf{u} \cdot div(\sigma) = div(\sigma \cdot \mathbf{u}) - < \nabla\mathbf{u}, \sigma >$ where $\nabla\mathbf{u} = D + A$ and $< A, \sigma >= 0$.

(g) If conservation of mass holds, then

$$\frac{d}{dt}\int_{\Omega(t)} \rho f dV = \int_{\Omega(t)} \rho \frac{Df}{Dt} dV.$$

16. Let (λ, R) be a genuinely nonlinear characteristic field for a conservation law, and let $u(\lambda)$ parameterize the integral curve of R between $\lambda_L = \lambda(u_L) > \lambda(u_R) = \lambda_R$. Show that the simple wave that starts from initial data $u_0(x)$ shocks at time t_0,

$$u_0(x) = \begin{cases} u_L & x < -\lambda_L t_0 \\ u(-x/t_0) & -\lambda_L t_0 \le x \le -\lambda_R t_0 \\ u_R & x \ge -\lambda_R t_0 \end{cases}$$

17. The eigenpairs of Euler's equations for the Lagrangian form

$$v_t - u_x = 0$$

$$u_t + p_x = 0$$

$$s_t = 0$$

are

$$(\lambda_1 = -\sqrt{-p_v}, R_1 = (1, \sqrt{-p_v}, 0),$$
$$(\lambda_2 = 0, R_2 = (p_s, 0, -p_v),$$
$$(\lambda_3 = +\sqrt{-p_v}, R_1 = (1, -\sqrt{-p_v}, 0).$$

Find the eigenpairs for the Lagrangian formulation in which the energy equation $\mathcal{E}_t + (pu)_x = 0$ is taken in place of $s_t = 0$.

18. For compressible Euler, we proved that the eigenfamilies are 1- and 3-eigenfamilies genuinely nonlinear so long as $p_{vv}(v, s) \neq 0$. Verify this in the case of a polytropic equation of state.