It has already been proved (Theorem 2, §1) that the functions

$$\pi(x) - s_\pi(x), \qquad \nu(x) - s_\nu(x)$$

are continuous and increasing. From this it follows that the difference

$$\varphi(x) = f(x) - s(x)$$

is a continuous function of finite variation.

The result just obtained can be stated in the following form.

THEOREM 7. *Every function of finite variation can be written as the sum of its saltus function and a continuous function of finite variation.*

§ 4. HELLY'S PRINCIPLE OF CHOICE

In this section, we take up a theorem due to E. Helly which has many important applications. We first prove two lemmas.

LEMMA 1. *Let an infinite family of functions $H = \{f(x)\}$ be defined on $[a, b]$. If all the functions of the family are bounded by one and the same number*

$$|f(x)| \leqslant K, \tag{1}$$

then, for any denumerable subset E of $[a, b]$, it is possible to find a sequence $\{f_n(x)\}$ in the family H which converges at every point of the set E.

Proof. Let $E = \{x_k\}$. Consider the set

$$\{f(x_1)\}$$

of values taken on by the functions of the family H at the point x_1. By (1), this set is bounded and, by the Bolzano-Weierstrass Theorem, we can select a convergent sequence from it:

$$f_1^{(1)}(x_1),\ f_2^{(1)}(x_1),\ f_3^{(1)}(x_1), \ldots; \quad \lim_{n\to\infty} f_n^{(1)}(x_1) = A_1. \tag{2}$$

Now consider the sequence

$$f_1^{(1)}(x_2),\ f_2^{(1)}(x_2),\ f_3^{(1)}(x_2), \ldots$$

of values taken on by the functions of the set $\{f_n^{(1)}(x)\}$ at the point x_2. This sequence is also bounded, and we can apply the Bolzano-Weierstrass Theorem to it. This gives a convergent subsequence

$$f_1^{(2)}(x_2),\ f_2^{(2)}(x_2),\ f_3^{(2)}(x_2), \ldots, \quad \lim_{n\to\infty} f_n^{(2)}(x_2) = A_2, \tag{3}$$

selected from $\{f_n^{(1)}(x_2)\}$. It is essential to note that the relative order of two functions $f_n^{(2)}$ and $f_m^{(2)}$ in the sequence (3) is the same as in the sequence (2). Continuing this process indefinitely, we construct a denumerable set of convergent sequences:

$$f_1^{(1)}(x_1),\ f_2^{(1)}(x_1),\ f_3^{(1)}(x_1), \ldots, \quad \lim_{n\to\infty} f_n^{(1)}(x_1) = A_1.$$

$$f_1^{(2)}(x_2),\ f_2^{(2)}(x_2),\ f_3^{(2)}(x_2), \ldots, \quad \lim_{n\to\infty} f_n^{(2)}(x_2) = A_2.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_1^{(k)}(x_k),\ f_2^{(k)}(x_k),\ f_3^{(k)}(x_3), \ldots, \quad \lim_{n\to\infty} f_n^{(k)}(x_k) = A_k.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where each sequence of numbers is a subsequence of the preceding one, and in which the order of elements has not been altered. We now form the sequence of diagonal elements of the infinite matrix just constructed, *i.e.*, the sequence

$$\{f_n^{(n)}(x)\} \qquad (n=1, 2, 3, \ldots).$$

This sequence converges at every point of the set E. In fact, for every fixed k, the sequence

$$\{f_n^{(n)}(x_k)\} \qquad (n \geqslant k)$$

is a subsequence of $\{f_n^{(k)}(x_k)\}$ and converges to A_k.

LEMMA 2. *Let $F=\{f(x)\}$ be an infinite family of increasing functions, defined on the segment $[a, b]$. If all functions of the family are bounded by one and the same number,*

$$|f(x)| \leqslant K, \quad f \in F, \; a \leqslant x \leqslant b,$$

then there is a sequence of functions $\{f_n(x)\}$ in F which converges to an increasing function $\varphi(x)$ at every point of $[a, b]$.

Proof. Apply Lemma 1 to $\{f(x)\}$, taking for the set E the set consisting of all rational points of $[a, b]$, together with the point a if it is irrational. We thus find a sequence of functions of the family F,

$$F_0=\{f^{(n)}(x)\}$$

such that

$$\lim_{n \to \infty} f^{(n)}(x_k) \tag{4}$$

exists and is finite at every point $x_k \in E$.

We now define a function $\psi(x)$ by the following procedure. First, we define

$$\psi(x_k)=\lim_{n \to \infty} f^{(n)}(x_k) \qquad (x_k \in E)$$

for all $x_k \in E$. This defines $\psi(x)$ only on E, of course. It is easy to see that $\psi(x)$ is an increasing function on E, that is, if $x_i, x_k \in E$ and $x_k < x_i$, then

$$\psi(x_k) \leqslant \psi(x_i).$$

For $x \in [a, b]-E$, we define $\psi(x)$ by the relation

$$\psi(x)=\sup_{x_k<x}\{\psi(x_k)\} \qquad (x_k \in E).$$

It is obvious that $\psi(x)$ is an increasing function on the closed interval $[a, b]$ and that the set of points Q where $\psi(x)$ is discontinuous is at most denumerable.

We show next that

$$\lim_{n \to \infty} f^{(n)}(x_0)=\psi(x_0) \tag{5}$$

at every point x_0 where $\psi(x)$ is continuous. Let ε be any positive number, and let x_k and x_i be points of E such that

$$x_k < x_0 < x_i, \quad \psi(x_i)-\psi(x_k) < \frac{\varepsilon}{2}.$$

Fixing the points x_k and x_i, select a natural number n_0 such that for $n > n_0$,

$$|f^{(n)}(x_k)-\psi(x_k)| < \frac{\varepsilon}{2}, \quad |f^{(n)}(x_i)-\psi(x_i)| < \frac{\varepsilon}{2}.$$

It is easy to see that

$$\psi(x_0) - \varepsilon < f^{(n)}(x_k) \leqslant f^{(n)}(x_i) < \psi(x_0) + \varepsilon,$$

for $n > n_0$. Since

$$f^{(n)}(x_k) \leqslant f^{(n)}(x_0) \leqslant f^{(n)}(x_i),$$

we have

$$\psi(x_0) - \varepsilon < f^{(n)}(x_0) < \psi(x_0) + \varepsilon,$$

for $n > n_0$. This proves (5). Thus, the equality

$$\lim_{n \to \infty} f^{(n)}(x) = \psi(x) \tag{6}$$

can fail only on the finite or denumerable set Q, where $\psi(x)$ is discontinuous.

We now apply Lemma 1 to the sequence F_0, taking for the set E the set of those points of Q where (6) is not fulfilled. This yields a subsequence

$$\{f_n(x)\}$$

of F_0, which converges at all points of $[a, b]$ (because at points where the sequence $\{f^{(n)}(x)\}$ converges, all subsequences also converge). Setting

$$\varphi(x) = \lim_{n \to \infty} f_n(x),$$

we obtain a function which is obviously an increasing function.

THEOREM (HELLY'S FIRST THEOREM). *Let an infinite family of functions $F = \{f(x)\}$ be defined on the segment $[a, b]$. If all functions of the family and the total variation of all functions of the family are bounded by a single number*

$$|f(x)| \leqslant K, \quad \overset{b}{\underset{a}{\mathrm{V}}}(f) \leqslant K,$$

then there exists a sequence $\{f_n(x)\}$ in the family F which converges at every point of $[a, b]$ to some function $\varphi(x)$ of finite variation.

Proof. For every function $f(x)$ of the family F, set

$$\pi(x) = \overset{x}{\underset{a}{\mathrm{V}}}(f), \quad \nu(x) = \pi(x) - f(x).$$

Both $\pi(x)$ and $\nu(x)$ are increasing functions. Furthermore,

$$|\pi(x)| \leqslant K, \quad |\nu(x)| \leqslant 2K.$$

Applying Lemma 2 to the family $\{\pi(x)\}$, we find that there is a convergent sequence $\{\pi_k(x)\}$,

$$\lim_{k \to \infty} \pi_k(x) = \alpha(x)$$

in this family. To every function $\pi_k(x)$, there corresponds a function $\nu_k(x)$, extending it to the function $f_k(x)$ of the family F. Applying Lemma 2 to the family $\{\nu_k(x)\}$, we find a convergent subsequence $\{\nu_{k_i}(x)\}$,

$$\lim_{i \to \infty} \nu_{k_i}(x) = \beta(x)$$

of $\{\nu_k(x)\}$. Then the sequence of functions

$$f_{k_i}(x) = \pi_{k_i}(x) - \nu_{k_i}(x),$$

belonging to F, converges to the function

$$\varphi(x) = \alpha(x) - \beta(x).$$

This proves Helly's theorem.

§ 5. CONTINUOUS FUNCTIONS OF FINITE VARIATION

THEOREM 1. *Let a function $f(x)$ of finite variation be defined on the closed interval $[a, b]$. If $f(x)$ is continuous at the point x_0, then the function*

$$\pi(x) = \overset{x}{\underset{a}{\mathrm{V}}}(f)$$

is also continuous at x_0.

Proof. Suppose that $x_0 < b$. We shall show that $\pi(x)$ is continuous on the right at the point x_0. For this purpose, taking an arbitrary $\varepsilon > 0$, we subdivide the segment $[x_0, b]$ by means of the points

$$x_0 < x_1 < \ldots < x_n = b$$

so that

$$V = \sum_{k=0}^{n-1} |f(x_{k+1} - f(x_k)| > \overset{b}{\underset{x_0}{\mathrm{V}}}(f) - \varepsilon. \tag{1}$$

Since the sum V only increases when new points are added, we may suppose that

$$|f(x_1) - f(x_0)| < \varepsilon.$$

It follows from (1) that

$$\overset{b}{\underset{x_0}{\mathrm{V}}}(f) < \varepsilon + \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| < 2\varepsilon + \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| \leqslant 2\varepsilon + \overset{b}{\underset{x_1}{\mathrm{V}}}(f).$$

Hence

$$\overset{x_1}{\underset{x_0}{\mathrm{V}}}(f) < 2\varepsilon,$$

and consequently

$$\pi(x_1) - \pi(x_0) < 2\varepsilon.$$

This implies that

$$\pi(x_0 + 0) - \pi(x_0) < 2\varepsilon.$$

Since ε is arbitrary, we have

$$\pi(x_0 + 0) = \pi(x_0).$$

It can be shown in like manner that $\pi(x_0 - 0) = \pi(x_0)$, *i.e.*, that $\pi(x)$ is continuous on the *left* (if $x_0 > a$) at the point x_0.

COROLLARY. *A continuous function of finite variation can be written as the difference of two continuous increasing functions.*

In fact, if $f(x)$ is a continuous function of finite variation defined on $[a, b]$, then both of its increasing components

$$\pi(x) = \overset{x}{\underset{a}{\mathrm{V}}}(f) \text{ and } \nu(x) = \pi(x) - f(x)$$

are continuous.