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Causal dissipation for the relativistic dynamics of ideal gases

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We derive a general class of relativistic dissipation tensors by requiring that, combined with the relativistic Euler equations, they form a second-order system of partial differential equations which is symmetric hyperbolic in a second-order sense when written in the natural Godunov variables that make the Euler equations symmetric hyperbolic in the first-order sense. We show that this class contains a unique element representing a causal formulation of relativistic dissipative fluid dynamics which (i) is equivalent to the classical descriptions by Eckart and Landau to first order in the coefficients of viscosity and heat conduction and (ii) has its signal speeds bounded sharply by the speed of light. Based on these properties, we propose this system as a natural candidate for the relativistic counterpart of the classical Navier–Stokes equations.

1. Introduction

In the absence of dissipation, relativistic fluid dynamics is governed by the Euler equations

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}=0, \quad \frac{\partial}{\partial x^\beta}N^\beta=0, \tag{1.1}$$

which are a five-field theory: its constituents, the energy-momentum tensor and the particle number density current,

$$T^{\alpha\beta}=(\rho+p)U^\alpha U^\beta+pg^{\alpha\beta}, \quad N^\beta=nU^\beta,$$

are given in terms of the fluid's velocity U^α, energy density ρ, pressure p and particle number density n.[1] As the velocity satisfies $U^\alpha U_\alpha=-1$ and of the three scalars

[1]We work with the Minkowski metric $g^{\alpha\beta}$ of signature $(-,+,+,+)$.

n, ρ, p two suffice to determine the thermodynamic state, the state description (U^α, ρ, p, n) has five degrees of freedom, while the Euler equations (1.1) are five partial differential equations that properly determine the spatio-temporal evolution of these five fields from general initial data.

Despite various theories that were suggested over the last almost eight decades, the proper modelling of *dissipation*, i.e. viscosity and heat conduction, is debated until today, both from the point of view of theoretical justification and that of usability in numerical computations; cf. e.g. [1–7] and references therein. Some otherwise ingenious proposals lack causality and/or well-posedness proofs, different ones depend on a number of parameters not all values of which appear to be clear, and still others seem difficult to apply with precision as they resort to very large numbers of state variables. While we cannot do justice to the rich and interesting history here, we point out that after the theory that Israel and Stewart gave in the Seventies [8–10], the clearest and cleanest consistent theory currently available is relativistic Extended Thermodynamics, which has been developed by Müller, Ruggeri and co-workers [1,11–15]. This theory, which is based on an infinite hierarchy of first-order hyperbolic systems of partial differential equations, is beautifully exposited in Choquet–Bruhat's book [16] as Section 11 of Chapter X, by Tommaso Ruggeri.

The present paper deals with the question of whether dissipative relativistic fluid dynamics can be properly modelled by a *causal* theory of the form

$$\frac{\partial}{\partial x^\beta}(T^{\alpha\beta} + \Delta T^{\alpha\beta}) = 0, \quad \frac{\partial}{\partial x^\beta}(N^\beta + \Delta N^\beta) = 0, \tag{1.2}$$

with dissipation tensors $\Delta T^{\alpha\beta}, \Delta N^\beta$ that are linear in the space–time gradients of the above-mentioned five fields ('relativistic Navier–Stokes'). As the coefficients associated with the highest, namely second, order derivatives depend on the fields, such systems are *quasi-linear*. The classical descriptions given by Eckart [17] and by Landau [18] are quasi-linear five-field theories.[2] They have strong physical justifications, in particular from extended thermodynamics [11] and kinetic theory [19]. However, they are not causal, in the sense that signals can travel at arbitrarily high speeds [2,3,10], and their mathematical type as systems of partial differential equations (PDEs) is unclear. Regarding the latter, many authors speak of parabolic behaviour, but neither the Eckart system nor the Landau system seems to be parabolic in a clear mathematical sense, and no theorems seem available on the existence of solutions for the associated initial-value problems. This makes it also uncertain to which extent the two descriptions can be relied on in numerical computations. The purpose of this paper is to propose a new quasi-linear second-order five-field theory which is intimately related to the Eckart description and the Landau description but, in contrast with both, has the advantages of being causal and permitting a consistent mathematical solution theory.

Beginning with Friedrichs' 1954 paper [20], applied mathematics has identified *symmetric hyperbolicity* as a basic requirement of fundamental equations in finite-speed-of-propagation continuum mechanics. First-order equations like those governing the flow of inviscid compressible fluids or magnetohydrodynamics, both classical and relativistic, were shown early on to be symmetric hyperbolic [21–23].[3] While the characteristic symmetry of coefficient matrices had often been achieved through ad hoc transformations, Godunov showed that it automatically occurs for first-order systems when one uses a particular choice of variables that is deeply motivated from physical considerations [25]. In 1976, Hughes, Kato and Marsden introduced a notion of symmetric hyperbolicity for second-order equations and showed that both classical nonlinear elasticity and the vacuum Einstein equations were symmetric hyperbolic in this second-order sense. This general situation motivated us to think that a five-field theory (1.2) of dissipative relativistic fluid dynamics should be a second-order symmetric hyperbolic system in the same Godunov variables that symmetrize the first-order Euler equations (1.1). In this paper, we first establish a wide mathematical class of quasi-linear second-order five-field theories which have this property, and then show that this class contains a unique element which (i) is equivalent

[2] ΔN^β is indeed zero for the Eckart description, while $U_\alpha \Delta T^{\alpha\beta}$ vanishes in the Landau description.

[3] Also the above-mentioned systems of extended thermodynamics are of this type, cf. [12,13,24].

to the Eckart and Landau equations to leading order in the small dissipation coefficients η, ζ, κ that quantify shear viscosity, bulk viscosity and heat conduction, and (ii) has its signal speeds bounded *sharply* by the speed of light. Based on these properties, we propose this system as the causal relativistic version of the classical Navier–Stokes–Fourier equations.[4]

In §2, we state the complete ingredients of this system concisely. Section 3 steps back to establish the mentioned mathematical class of symmetric hyperbolic systems from equivariant 'dissipation pairs' $(-\Delta T^{\alpha\beta}, -\Delta N^{\beta})$, and then shows that our proposed system is a limiting case of the original Hughes–Kato–Marsden class. In §4, we first develop some 'algebra' that characterizes a group of general first-order equivalence transformations between five-field theories, and then use it to show that our theory is first-order equivalent with the Eckart formulation and the Landau formulation. In §5, we show that our dissipation operator is causal in the sense that its Fourier–Laplace modes travel at the speed of light or slower speeds. Section 6 discusses the compatibility of our theory with the second law of thermodynamics. Finally, we show in §7 that our theory has the classical Navier–Stokes–Fourier (NSF) equations as its limit for $c \to \infty$.

For concreteness, we assume throughout the paper that the fluid consists of particles of mass $m > 0$ and its internal energy $e = e(n, s)$, with which

$$\rho = n(m + e(n, s)) \quad \text{and} \quad p = n^2 e_n(n, s),$$

is given by

$$e = kn^{\gamma-1} \exp\left(\frac{s}{c_v}\right) = c_v\theta, \quad \text{where } 1 < \gamma \leq 2, \tag{1.3}$$

i.e. the case of the ideal gas. Extensions of the results to other massive non-barotropic fluids should be obvious with small adaptations.

This paper is a follow-up to the authors' earlier study of the same questions in the context of the (massless) pure radiation fluid [26]; cf. also [27]. In the case of barotropic fluids, i.e. fluids whose thermodynamic state can be characterized by *one* scalar, it suffices to consider only the energy–momentum equations $(1.1)_1$ resp. $(1.2)_1$, and one speaks of four-field theories. For four-field theories to which the one presented in this paper is analogous, we refer the reader to [28]. Interesting quasi-linear second-order four-field theories were previously suggested by Lichnerowicz [29] and Choquet-Bruhat [16].

2. The new theory

Besides using U^α, n, p, ρ, we describe, as in [26], the local state of the fluid also by variables that symmetrize the Euler equations (1.1), i.e. the Godunov variables [25]

$$\psi^\alpha = \frac{U^\alpha}{\theta}, \quad \psi = \frac{h}{\theta} - s,$$

where θ and s denote the local temperature and specific entropy and

$$h = \frac{\rho + p}{n}$$

the specific enthalpy of the fluid.

We assume fixed choices of coefficients of viscosity and heat conduction,[5]

$$\eta > 0, \quad \zeta \geq 0 \quad \text{and} \quad \kappa = \frac{\chi\theta^2}{h} > 0, \tag{2.1}$$

[4]Other theories are conceivable which share all these properties except for having their signal speeds bounded by smaller values than the speed of light. This option will not be pursued in this paper.

[5]For heat conduction, we use either one of the two symbols κ and χ.

which can be taken to be general functions of the thermodynamic variables. Our proposal is to use[6]

$$-\Delta T^{\alpha\beta} \equiv \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta}\left(\frac{\partial U_\alpha}{\partial x^\beta}+\frac{\partial U_\beta}{\partial x^\alpha}-\frac{2}{3}\eta_{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}\right)+\tilde\zeta\,\Pi^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}$$
$$+\sigma\left(U^\alpha U^\beta\frac{\partial U^\gamma}{\partial x^\gamma}-U^\alpha U^\delta\frac{\partial U^\beta}{\partial x^\delta}-U^\beta U^\delta\frac{\partial U^\alpha}{\partial x^\delta}\right) \tag{2.2}$$

and

$$-\Delta N^\beta \equiv \tilde\kappa g^{\beta\gamma}\frac{\partial\psi}{\partial x^\gamma}+\tilde\sigma(U^\beta\Pi^{\gamma\delta}-U^\delta\Pi^{\beta\gamma})\frac{\partial U_\gamma}{\partial x^\delta}, \tag{2.3}$$

or, written as matrices with respect to the fluid's rest frame,[7]

$$-\Delta T|_0=\begin{pmatrix}\sigma\nabla\cdot\mathbf{u} & -\sigma\dot{\mathbf{u}}^\top\\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u}+\tilde\zeta\nabla\cdot\mathbf{u}\mathbf{I}\end{pmatrix} \tag{2.4}$$

and

$$-\Delta N|_0=(-\tilde\kappa\dot\psi+\tilde\sigma\nabla\cdot\mathbf{u},\ \tilde\kappa\nabla\psi-\tilde\sigma\dot{\mathbf{u}}). \tag{2.5}$$

The coefficients are given by

$$\sigma=\tfrac{4}{3}\eta+\tilde\zeta,\quad \tilde\zeta=\zeta+\tilde\zeta_1+\tilde\zeta_2 \tag{2.6}$$

with[8]

$$\tilde\zeta_1=(\gamma-1)^2\left(\frac{m^2}{h\theta}\right)\kappa,\quad \tilde\zeta_2=(\gamma-1)\left(1-\frac{m}{h}\right)\sigma \tag{2.7}$$

and

$$\tilde\kappa=\frac{\kappa}{h},\quad \tilde\sigma=\frac{\sigma}{h}. \tag{2.8}$$

3. Symmetric hyperbolicity

In the pioneering paper [30], Hughes *et al.* have identified a class of second-order symmetric hyperbolic systems and used Kato's abstract theory of 'evolution equations of hyperbolic type' [31] to establish well-posedness for the initial-value problem of its members in appropriate Sobolev spaces. Our proposed theory (1.2), (2.2) and (2.3) itself is not literally a second-order symmetric hyperbolic system in the sense of [30], but it is a uniform limit of such systems (see corollary 3.2 below); this connection is so close that we find it proper to still call it a (mixed-order) symmetric hyperbolic system.

To explain all this and show from which perspective we found the theory, we now start, in analogy to our approach in [26], from the general equivariant forms of tensors $-\Delta T^{\alpha\beta}$ and (now also:) $-\Delta N^\beta$ that are linear in the gradients of the state variables. These forms are

$$-\Delta T^{\alpha\beta}\equiv U^\alpha U^\beta P+(\Pi^{\alpha\gamma}U^\beta+\Pi^{\beta\gamma}U^\alpha)Q_\gamma+\Pi^{\alpha\beta}R+\Pi^{\alpha\gamma}\Pi^{\beta\delta}W_{\gamma\delta} \tag{3.1}$$

with

$$P=\tau U^\gamma\frac{\partial\theta}{\partial x^\gamma}+\sigma\frac{\partial U^\gamma}{\partial x^\gamma}+\iota U^\gamma\frac{\partial\psi}{\partial x^\gamma},\quad Q_\gamma\equiv\nu\frac{\partial\theta}{\partial x^\gamma}+\mu U^\delta\frac{\partial U_\gamma}{\partial x^\delta}+\upsilon\frac{\partial\psi}{\partial x^\gamma},$$
$$R=\omega U^\gamma\frac{\partial\theta}{\partial x^\gamma}+\hat\zeta\frac{\partial U^\gamma}{\partial x^\gamma}+\varsigma U^\gamma\frac{\partial\psi}{\partial x^\gamma},\quad W_{\alpha\beta}\equiv\eta\left(\frac{\partial U_\alpha}{\partial x^\beta}+\frac{\partial U_\beta}{\partial x^\alpha}-\frac{2}{3}g_{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma}\right)$$

and $$-\Delta N^\beta\equiv U^\beta\hat P+\Pi^{\beta\delta}\hat Q_\delta \tag{3.2}$$

[6]We use $\Pi^{\alpha\beta}=g^{\alpha\beta}+U^\alpha U^\beta$.

[7]We write $\mathbf{u}$ for the three-velocity with respect to the fluid's rest frame at a given point. (While $\mathbf{u}=0$ at that point, its gradient is free.) $\dot{}$ means derivative with respect to x^0, ∇ derivatives with respect to (x^1,x^2,x^3), all in the rest frame.

[8]This implies that $\sigma=((4/3)\eta+\zeta+\tilde\zeta_1)/(1-(\gamma-1)(1-m/h))$.

with

$$\hat{P} = \hat{\tau} U^\delta \frac{\partial \theta}{\partial x^\delta} + \hat{\sigma} \frac{\partial U^\delta}{\partial x^\delta} + \hat{\iota} U^\delta \frac{\partial \psi}{\partial x^\delta}, \quad \hat{Q}_\delta \equiv \hat{\nu} \frac{\partial \theta}{\partial x^\delta} + \hat{\mu} U^\epsilon \frac{\partial U_\delta}{\partial x^\epsilon} + \hat{\upsilon} \frac{\partial \psi}{\partial x^\delta}. \tag{3.3}$$

The Hughes–Kato–Marsden class is characterized by properties that the fields of coefficients of the second-order derivatives must satisfy. Certain matrices composed from these fields must be symmetric and positive (respectively negative) definite [30] (cf. Section 4 of [26] for the covariant formulation we use).

Correspondingly, we write the second-order parts of

$$-\frac{\partial}{\partial x^\beta}(\Delta T^{\alpha\beta}) \quad \text{and} \quad -\frac{\partial}{\partial x^\beta}(\Delta N^\beta)$$

as

$$B^{\alpha\beta g\delta} \frac{\partial^2 \psi_g}{\partial x^\beta \partial x^\delta} \quad \text{and} \quad B^{4\beta g\delta} \frac{\partial^2 \psi_g}{\partial x^\beta \partial x^\delta},$$

respectively, where the index g runs from 0 through 4 and ψ_4 stands for ψ, and must study properties of the $\beta\delta$-symmetrized coefficients

$$\tilde{B}^{a\beta g\delta} \equiv \tfrac{1}{2}(B^{a\beta g\delta} + B^{a\delta g\beta}), \quad \beta, \delta = 0,1,2,3 \quad a, g = 0,1,2,3,4.$$

The goal of this section is to prove the following theorem and corollary.

Theorem 3.1.

(i) *Under the assumptions*

$$(\sigma + \mu) = (\omega + \nu)\theta, \quad \hat{\tau}\theta^2 = \iota, \quad \hat{\nu}\theta^2 = \upsilon, \quad (\hat{\sigma} + \hat{\mu})\theta = \upsilon + \varsigma, \tag{3.4}$$

the coefficients $\tilde{B}^{a\beta g\delta}$ *are symmetric in* $a, g \in \{0,1,2,3,4\}$ *and*

$$B^{a\beta g\delta} U_\beta U_\delta, \quad B^{a\beta g\delta} N_\beta N_\delta$$

correspond to the 5×5 *matrices*

$$\begin{pmatrix} \tau\theta^2 & 0 & \iota \\ 0 & \mu\theta\delta^{ij} & 0 \\ \iota & 0 & \hat{\iota} \end{pmatrix}, \quad \begin{pmatrix} \nu\theta^2 & 0 & \upsilon \\ 0 & \eta\theta\delta ij + (\frac{1}{3}\eta + \hat{\zeta})\theta N^i N^j & 0 \\ \upsilon & 0 & \hat{\upsilon} \end{pmatrix}. \tag{3.5}$$

(ii) *If moreover*

$$\mu, \hat{\iota}, \tau < 0, \quad \iota^2 < \hat{\iota}\tau\theta^2 \tag{3.6}$$

and

$$\eta, \hat{\upsilon}, \nu > 0, \quad \hat{\zeta} \geq 0, \quad \upsilon^2 < \hat{\upsilon}\nu\theta^2, \tag{3.7}$$

then system (1.2) with (3.1), (3.2) is symmetric hyperbolic in the sense of Hughes et al. [30], *pointwise with respect to the fluid's rest frame.*

Corollary 3.2.

(i) *Our proposed system (1.2) with (2.2), (2.3) results by choosing*

$$\tau = \nu = \omega = \hat{\tau} = \hat{\nu} = \varsigma = \iota = \upsilon = 0$$

and

$$\mu = -\sigma, \quad \hat{\sigma} = -\hat{\mu} = \tilde{\sigma}, \quad \hat{\zeta} = \tilde{\zeta}, \quad \hat{\upsilon} = -\hat{\iota} = \tilde{\kappa}.$$

(ii) *It is a uniform limit of a family of systems that are symmetric hyperbolic in the sense of Hughes et al.* [30].

We collect the information leading to these assertions step by step.

1. Coefficients with $a=\alpha$, $g=\gamma\in\{0,1,2,3\}$. Expressing derivatives as

$$\frac{\partial\theta}{\partial x^\delta}=\theta^2U^\gamma\frac{\partial\psi_\gamma}{\partial x^\delta},\quad \frac{\partial U^\sigma}{\partial x^\delta}=\theta\Pi^{\sigma\gamma}\frac{\partial\psi_\gamma}{\partial x^\delta}, \tag{3.8}$$

we see that

$$\begin{aligned}B^{\alpha\beta\gamma\delta}=&+U^\alpha U^\beta(\tau\theta^2U^\gamma U^\delta+\sigma\theta\Pi^{\gamma\delta})\\&+\Pi^{\alpha\beta}(\omega\theta^2U^\gamma U^\delta+\hat{\zeta}\theta\Pi^{\gamma\delta})\\&+\nu\theta^2(\Pi^{\alpha\delta}U^\beta+\Pi^{\beta\delta}U^\alpha)U^\gamma\\&+\mu\theta(\Pi^{\alpha\gamma}U^\beta+\Pi^{\beta\gamma}U^\alpha)U^\delta\\&+\eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta}+\Pi^{\alpha\delta}\Pi^{\beta\gamma}-(2/3)\Pi^{\alpha\beta}\Pi^{\gamma\delta}).\end{aligned}$$

Lemma 3.3. *$\tilde{B}^{\alpha\beta\gamma\delta}$ is symmetric in α,γ if and only if*

$$(\sigma+\mu)=(\omega+\nu)\theta. \tag{3.9}$$

Proof. To see this, note that

$$\begin{aligned}2\tilde{B}^{\alpha\beta\gamma\delta}=&+U^\alpha U^\beta(\tau\theta^2U^\gamma U^\delta+\sigma\theta\Pi^{\gamma\delta})\\&+\Pi^{\alpha\beta}(\omega\theta^2U^\gamma U^\delta+\hat{\zeta}\theta\Pi^{\gamma\delta})\\&+\nu\theta^2(\Pi^{\alpha\delta}U^\beta+\Pi^{\beta\delta}U^\alpha)U^\gamma\\&+\mu\theta(\Pi^{\alpha\gamma}U^\beta+\Pi^{\beta\gamma}U^\alpha)U^\delta\\&+\eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta}+\Pi^{\alpha\delta}\Pi^{\beta\gamma}-(2/3)\Pi^{\alpha\beta}\Pi^{\gamma\delta})\\&+U^\alpha U^\delta(\tau\theta^2U^\gamma U^\beta+\sigma\theta\Pi^{\gamma\beta})\\&+\Pi^{\alpha\delta}(\omega\theta^2U^\gamma U^\beta+\hat{\zeta}\theta\Pi^{\gamma\beta})\\&+\nu\theta^2(\Pi^{\alpha\beta}U^\delta+\Pi^{\beta\delta}U^\alpha)U^\gamma\\&+\mu\theta(\Pi^{\alpha\gamma}U^\delta+\Pi^{\delta\gamma}U^\alpha)U^\beta\\&+\eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta}+\Pi^{\alpha\beta}\Pi^{\delta\gamma}-(2/3)\Pi^{\alpha\delta}\Pi^{\gamma\beta}),\end{aligned}$$

the antisymmetric part of which,

$$\begin{aligned}&2(\tilde{B}^{\alpha\beta\gamma\delta}-\tilde{B}^{\gamma\beta\alpha\delta})\\&\quad=(\sigma+\mu)\theta[(U^\alpha U^\beta g^{\gamma\delta}+U^\alpha U^\delta g^{\gamma\beta})-(g^{\alpha\beta}U^\gamma U^\delta+g^{\alpha\delta}U^\beta U^\gamma)]\\&\qquad+(\omega+\nu)\theta^2[(g^{\alpha\beta}U^\gamma U^\delta+g^{\alpha\delta}U^\beta U^\gamma)-(U^\alpha U^\beta g^{\gamma\delta}+U^\alpha U^\delta g^{\gamma\beta})]\\&\quad=[(\sigma+\mu)\theta-(\omega+\nu)\theta^2)][(U^\alpha U^\beta g^{\gamma\delta}+U^\alpha U^\delta g^{\gamma\beta})-(g^{\alpha\beta}U^\gamma U^\delta+g^{\alpha\delta}U^\beta U^\gamma)]\\&\quad=[(\sigma+\mu)\theta-(\omega+\nu)\theta^2)][(U^\alpha U^\beta g^{\gamma\delta}-U^\gamma U^\beta g^{\alpha\delta})+(U^\alpha U^\delta g^{\gamma\beta}-U^\gamma U^\delta g^{\alpha\beta}],\end{aligned}$$

vanishes if and only if (3.9) holds. ∎

Independently of (3.9), direct computation shows the following, which holds for any N^β satisfying

$$N^\beta U_\beta=0,\quad N^\beta N_\beta=1. \tag{3.10}$$

Lemma 3.4. *One has*

$$\tilde{B}^{\alpha\beta\gamma\delta}U_\beta U_\delta=\tau\theta^2U^\alpha U^\gamma+\mu\theta\Pi^{\alpha\gamma},$$

which corresponds to the 4×4 matrix

$$\begin{pmatrix}\tau\theta^2 & 0\\ 0 & \mu\theta\delta^{ij}\end{pmatrix},$$

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and

$$\tilde{B}^{\alpha\beta\gamma\delta}N_\beta N_\delta = \nu\theta^2 U^\alpha U^\gamma + \eta\theta(\Pi^{\alpha\gamma} + (1/3)N^\alpha N^\gamma) + \hat{\zeta}\theta N^\alpha N^\gamma,$$

which corresponds to

$$\begin{pmatrix} \nu\theta^2 & 0 \\ 0 & \eta\theta\delta ij + ((1/3)\eta + \hat{\zeta})\theta N^i N^j \end{pmatrix}.$$

2. *Coefficients with* $a = 4$ *or* $g = 4$.

Lemma 3.5. *One has*

$$\tilde{B}^{\alpha\beta 4\delta} = (\iota U^\alpha U^\beta U^\delta + \upsilon \Pi^{\beta\delta} U^\alpha) + (1/2)(\upsilon + \varsigma)(\Pi^{\alpha\delta} U^\beta + \Pi^{\alpha\beta} U^\delta) \tag{3.11}$$

and

$$B^{\alpha\beta 4\delta} U_\beta U_\delta = \iota U^\alpha \quad \text{and} \quad B^{\alpha\beta 4\delta} N_\beta N_\delta = \upsilon U^\alpha \tag{3.12}$$

Proof. One readily finds

$$B^{\alpha\beta 4\delta} = \iota U^\alpha U^\beta U^\delta + \upsilon(\Pi^{\alpha\delta} U^\beta + \Pi^{\beta\delta} U^\alpha) + \varsigma \Pi^{\alpha\beta} U^\delta,$$

from which (3.11) and (3.12) follow directly. ■

Lemma 3.6. *We have*

$$\tilde{B}^{4\beta\gamma\delta} = (\hat{\tau}\theta^2 U^\beta U^\delta U^\gamma + \hat{\nu}\theta^2 \Pi^{\beta\delta} U^\gamma) + (1/2)(\hat{\sigma} + \hat{\mu})\theta(U^\beta \Pi^{\delta\gamma} + U^\delta \Pi^{\beta\gamma}) \tag{3.13}$$

as well as

$$B^{4\beta\gamma\delta} U_\beta U_\delta = \hat{\tau}\theta^2 U^\gamma, \quad B^{4\beta\gamma\delta} N_\beta N_\delta = \hat{\nu}\theta^2 U^\gamma. \tag{3.14}$$

Proof. Using (3.8) in (3.2), we find

$$\begin{aligned} -\Delta N^\beta = &((\hat{\tau}\theta^2 U^\beta U^\delta U^\gamma + \hat{\nu}\theta^2 \Pi^{\beta\delta} U^\gamma) + (\hat{\sigma}\theta U^\beta \Pi^{\delta\gamma} + \hat{\mu}\theta U^\delta \Pi^{\beta\gamma}))\frac{\partial \psi_\gamma}{\partial x^\delta} \\ &+ (\hat{\iota} U^\beta U^\delta + \hat{\upsilon}\Pi^{\beta\delta})\frac{\partial \psi}{\partial x^\delta} \end{aligned}$$

and thus

$$B^{4\beta\gamma\delta} = (\hat{\tau}\theta^2 U^\beta U^\delta U^\gamma + \hat{\nu}\theta^2 \Pi^{\beta\delta} U^\gamma) + (\hat{\sigma}\theta U^\beta \Pi^{\delta\gamma} + \hat{\mu}\theta U^\delta \Pi^{\beta\gamma}),$$

from which (3.13) and (3.14) readily follow. ■

Lemma 3.7. *One has*

$$\tilde{B}^{4\beta\gamma\delta} = \tilde{B}^{\gamma\beta 4\delta}$$

if and only if

$$\hat{\tau}\theta^2 = \iota, \quad \hat{\nu}\theta^2 = \upsilon \tag{3.15}$$

and

$$(\hat{\sigma} + \hat{\mu})\theta = \upsilon + \varsigma.$$

Proof. A short computation shows that

$$\begin{aligned} \tilde{B}^{4\beta\gamma\delta} - \tilde{B}^{\gamma\beta 4\delta} = &2(\hat{\tau}\theta^2 U^\beta U^\delta U^\gamma + \hat{\nu}\theta^2 \Pi^{\beta\delta} U^\gamma) + (\hat{\sigma} + \hat{\mu})\theta(U^\beta \Pi^{\delta\gamma} + U^\delta \Pi^{\beta\gamma}) \\ &- 2(\iota U^\beta U^\delta U^\gamma + \upsilon \Pi^{\beta\delta} U^\gamma) + (\upsilon + \varsigma)(\Pi^{\gamma\delta} U^\beta + \Pi^{\gamma\beta} U^\delta). \end{aligned}$$

■

Lemma 3.8. *One has*

$$B^{4\beta 4\delta} = \hat{\iota} U^\beta U^\delta + \hat{\upsilon}\Pi^{\beta\delta}, \tag{3.16}$$

and

$$B^{4\beta 4\delta} U_\beta U_\delta = \hat{\iota}, \quad B^{4\beta 4\delta} N_\beta N_\delta = \hat{\upsilon}.$$

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Proof of theorem 3.1.

(i) The first assertion is clear from lemmas 3.3 to 3.8, the second one follows as $B^{\alpha\beta g\delta}U^{\beta}U^{\delta}$ and $B^{\alpha\beta g\delta}N_{\beta}N_{\delta}$ are in any case represented by the matrices

$$\begin{pmatrix} \tau\theta^2 & 0 & \iota \\ 0 & \mu\theta\delta^{ij} & 0 \\ \hat{\tau}\theta^2 & 0 & \hat{\iota} \end{pmatrix}, \quad \begin{pmatrix} \nu\theta^2 & 0 & \upsilon \\ 0 & \eta\theta\delta ij + (\frac{1}{3}\eta + \hat{\zeta})\theta N^i N^j & 0 \\ \hat{\nu}\theta^2 & 0 & \hat{\upsilon} \end{pmatrix}.$$

(ii) cf. [26] for this notion. The assertion follows as these conditions make the first matrix in (3.5) negative definite and the second one positive definite.

■

Proof of corollary 3.2.

(i) Choosing $\tau = \nu = \omega = \hat{\tau} = \hat{\nu} = \varsigma = \iota = \upsilon = 0$ and $\mu = -\sigma, \hat{\mu} = -\hat{\sigma}$ yields

$$P = \sigma \frac{\partial U^{\gamma}}{\partial x^{\gamma}}, \quad Q_{\gamma} \equiv -\sigma U^{\delta} \frac{\partial U_{\gamma}}{\partial x^{\delta}}, \quad R = \hat{\zeta} \frac{\partial U^{\gamma}}{\partial x^{\gamma}}$$

and

$$\hat{P} = \hat{\sigma} \frac{\partial U^{\delta}}{\partial x^{\delta}} + \hat{\iota} U^{\delta} \frac{\partial \psi}{\partial x^{\delta}}, \quad \hat{Q}_{\delta} \equiv -\hat{\sigma} U^{\epsilon} \frac{\partial U_{\delta}}{\partial x^{\epsilon}} + \hat{\upsilon} \frac{\partial \psi}{\partial x^{\delta}}.$$

Choosing $\hat{\zeta} = \tilde{\zeta}$ and easily confirming that

$$U^{\alpha}U^{\beta}P + (\Pi^{\alpha\gamma}U^{\beta} + \Pi^{\beta\gamma}U^{\alpha})Q_{\gamma} = \sigma \left(U^{\alpha}U^{\beta} \frac{\partial U^{\gamma}}{\partial x^{\gamma}} - U^{\alpha}U^{\delta} \frac{\partial U^{\beta}}{\partial x^{\delta}} - U^{\beta}U^{\delta} \frac{\partial U^{\alpha}}{\partial x^{\delta}} \right),$$

we arrive at

$$-\Delta T^{\alpha\beta} \equiv \eta \Pi^{\alpha\gamma}\Pi^{\beta\delta} \left(\frac{\partial U_{\alpha}}{\partial x^{\beta}} + \frac{\partial U_{\beta}}{\partial x^{\alpha}} - \frac{2}{3}\eta_{\alpha\beta} \frac{\partial U^{\gamma}}{\partial x^{\gamma}} \right) + \tilde{\zeta} \Pi^{\alpha\beta} \frac{\partial U^{\gamma}}{\partial x^{\gamma}}$$
$$+ \sigma \left(U^{\alpha}U^{\beta} \frac{\partial U^{\gamma}}{\partial x^{\gamma}} - U^{\alpha}U^{\delta} \frac{\partial U^{\beta}}{\partial x^{\delta}} - U^{\beta}U^{\delta} \frac{\partial U^{\alpha}}{\partial x^{\delta}} \right),$$

which is (2.2).
Choosing $\hat{\upsilon} = -\hat{\iota} = \tilde{\kappa}$ and $\hat{\sigma} = \tilde{\sigma}$, we reach

$$-\Delta N^{\beta} = \tilde{\kappa} \left(-U^{\beta}U^{\delta} \frac{\partial \psi}{\partial x^{\delta}} + \Pi^{\beta\delta} \frac{\partial \psi}{\partial x^{\delta}} \right) + \tilde{\sigma} \left(U^{\beta} \frac{\partial U^{\delta}}{\partial x^{\delta}} - \Pi^{\beta\delta}U^{\epsilon} \frac{\partial U_{\delta}}{\partial x^{\epsilon}} \right)$$

which is (2.3).

(ii) Resetting, relative to the choices made just above,

$$\nu = -\tau = -\omega = \varepsilon\theta^{-2},$$

we recover it as the limit, as $\varepsilon \to 0$, of systems for which $B^{\alpha\beta g\delta}U_{\beta}U_{\delta}$ and $B^{\alpha\beta g\delta}N_{\beta}N_{\delta}$ are given by

$$\begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & -\sigma\theta\delta^{ij} & 0 \\ 0 & 0 & -\tilde{\kappa} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \eta\theta\delta ij + (\frac{1}{3}\eta + \tilde{\zeta})\theta N^i N^j & 0 \\ 0 & 0 & \tilde{\kappa} \end{pmatrix}. \tag{3.17}$$

■

4. Connection with the theories of Eckart and Landau

Theorem 3.1 and Corollary 3.2 position our theory with respect to symmetric hyperbolicity, but they do not explain the specific choices (2.2), (2.3) of $-\Delta T^{\alpha\beta}$ and $-\Delta N^{\beta}$ and, in particular, our selection (2.6), (2.7), (2.8) of the coefficients $\tilde{\zeta}, \tilde{\kappa}, \tilde{\sigma}$. We deduce them now.

For this purpose, we consider the space $\mathcal{F}_5$ of all pairs of linear gradient forms

$$\left.\begin{aligned}\Delta T^{\alpha\beta} &= T_U^{\alpha\beta\gamma\delta}\frac{\partial U_\gamma}{\partial x^\delta} + T_n^{\alpha\delta}\frac{\partial n}{\partial x^\delta} + T_\rho^{\alpha\delta}\frac{\partial \rho}{\partial x^\delta} + T_p^{\alpha\delta}\frac{\partial p}{\partial x^\delta}\\ \text{and}\qquad \Delta N^{\beta} &= N_U^{\gamma\delta}\frac{\partial U_\gamma}{\partial x^\delta} + N_n^{\delta}\frac{\partial n}{\partial x^\delta} + N_\rho^{\delta}\frac{\partial \rho}{\partial x^\delta} + N_p^{\delta}\frac{\partial p}{\partial x^\delta},\end{aligned}\right\} \tag{4.1}$$

and express the smallness of dissipation by giving them a common small factor $\epsilon > 0$, i.e. we consider $(\Delta T^{\alpha\beta}, \Delta N^\beta) \in \mathcal{F}_5$ as representing the five-field theory

$$\left.\begin{aligned}&\frac{\partial}{\partial x^\beta}(T^{\alpha\beta} + \epsilon\Delta T^{\alpha\beta}) = 0\\ \text{and}\qquad &\frac{\partial}{\partial x^\beta}(N^{\beta} + \epsilon\Delta N^{\beta}) = 0.\end{aligned}\right\} \tag{4.2}$$

We characterize a group of transformations that establishes formal equivalences between different elements of $\mathcal{F}_5$ up to $O(\epsilon^2)$, and then show that our theory lies in the same equivalence class as Eckart's and Landau's.

Physically speaking, the idea is that instead of working with the quantities U^α, n, ρ, p which appear in the evolution equations (4.2) one might alternatively base one's considerations on certain local spatio-temporally anisotropic averages $\tilde{U}^\alpha, \tilde{n}, \tilde{\rho}, \tilde{p}$. This is expressed through an ansatz[9]

$$\left.\begin{aligned}U^\alpha &= \tilde{U}^\alpha + \varepsilon\Delta\tilde{U}^\alpha + O(\varepsilon^2),\\ n &= \tilde{n} + \varepsilon\Delta\tilde{n} + O(\varepsilon^2),\\ \rho &= \tilde{\rho} + \varepsilon\Delta\tilde{\rho} + O(\varepsilon^2)\\ \text{and}\qquad p &= \tilde{p} + \varepsilon\Delta\tilde{p} + O(\varepsilon^2),\end{aligned}\right\} \tag{4.3}$$

in which

$$\left.\begin{aligned}\Delta\tilde{U}^\alpha &= \tilde{U}_U^{\alpha\gamma\delta}\frac{\partial \tilde{U}_\gamma}{\partial x^\delta} + \tilde{U}_n^{\alpha\delta}\frac{\partial \tilde{n}}{\partial x^\delta} + \tilde{U}_\rho^{\alpha\delta}\frac{\partial \tilde{\rho}}{\partial x^\delta} + \tilde{U}_p^{\alpha\delta}\frac{\partial \tilde{p}}{\partial x^\delta},\\ \Delta\tilde{n} &= \tilde{n}_U^{\gamma\delta}\frac{\partial \tilde{U}_\gamma}{\partial x^\delta} + \tilde{n}_n^{\delta}\frac{\partial \tilde{n}}{\partial x^\delta} + \tilde{n}_\rho^{\delta}\frac{\partial \tilde{\rho}}{\partial x^\delta} + \tilde{n}_p^{\delta}\frac{\partial \tilde{p}}{\partial x^\delta},\\ \Delta\tilde{\rho} &= \tilde{\rho}_U^{\gamma\delta}\frac{\partial \tilde{U}_\gamma}{\partial x^\delta} + \tilde{\rho}_n^{\delta}\frac{\partial \tilde{n}}{\partial x^\delta} + \tilde{\rho}_\rho^{\delta}\frac{\partial \tilde{\rho}}{\partial x^\delta} + \tilde{\rho}_p^{\delta}\frac{\partial \tilde{p}}{\partial x^\delta}\\ \text{and}\qquad \Delta\tilde{p} &= \tilde{p}_U^{\gamma\delta}\frac{\partial \tilde{U}_\gamma}{\partial x^\delta} + \tilde{p}_n^{\delta}\frac{\partial \tilde{n}}{\partial x^\delta} + \tilde{p}_\rho^{\delta}\frac{\partial \tilde{\rho}}{\partial x^\delta} + \tilde{p}_p^{\delta}\frac{\partial \tilde{p}}{\partial x^\delta}\end{aligned}\right\} \tag{4.4}$$

are further linear gradient forms whose coefficients $\tilde{U}_U^{\alpha\gamma\delta}, \ldots, \tilde{p}_p^\delta$ depend on $\tilde{U}^\alpha, \tilde{n}, \tilde{\rho}, \tilde{p}$.

For any fixed element (4.1) of $\mathcal{F}_5$, substituting (4.3), (4.4) in (4.2), and writing

$$\tilde{T}^{\alpha\beta} = (\tilde{\rho} + \tilde{p})\tilde{U}^\alpha\tilde{U}^\beta + \tilde{p}g^{\alpha\beta}, \quad \tilde{N}^\beta = \tilde{n}\tilde{U}^\beta,$$

and analogously $\Delta\tilde{T}^{\alpha\beta}, \Delta\tilde{N}^\beta$, will result in modifications $\delta\Delta\tilde{T}^{\alpha\beta}$ and $\delta\Delta\tilde{N}^\beta$ in the equations of motion

$$\left.\begin{aligned}&\frac{\partial}{\partial x^\beta}(\tilde{T}^{\alpha\beta} + \epsilon(\Delta\tilde{T}^{\alpha\beta} + \delta\Delta\tilde{T}^{\alpha\beta})) = 0\\ \text{and}\qquad &\frac{\partial}{\partial x^\beta}(\tilde{N}^{\beta} + \epsilon(\Delta\tilde{N}^{\beta} + \delta\Delta\tilde{N}^{\beta})) = 0\end{aligned}\right\} \tag{4.5}$$

for $\tilde{U}^\alpha, \tilde{n}, \tilde{\rho}, \tilde{p}$. The new ingredients will be of the form

$$\delta\Delta\tilde{T}^{\alpha\beta} = \tilde{\Delta}\tilde{T}^{\alpha\beta} + O(\epsilon), \quad \delta\Delta\tilde{N}^{\beta} = \tilde{\Delta}\tilde{N}^{\beta} + O(\epsilon)$$

with a unique element $(\tilde{\Delta}\tilde{T}^{\alpha\beta}, \tilde{\Delta}\tilde{N}^\beta)$ of $\mathcal{F}_5$.

[9]Note the symmetry of this class with respect to quantities with versus quantities without ˜. 'Gradient theories' are a standard concept of continuum mechanics. For their use in relativistic fluid dynamics, cf. [4,6].

Definition 4.1. We call the assignment

$$\mathcal{F}_5 \to \mathcal{F}_5, \quad (\Delta T^{\alpha\beta}, \Delta N^\beta) \mapsto (\Delta \tilde{T}^{\alpha\beta} + \tilde{\Delta}\tilde{T}^{\alpha\beta}, \Delta \tilde{N}^\beta + \tilde{\Delta}\tilde{N}^\beta)$$

the *first-order equivalence* generated by the gradient transformation (4.3), (4.4).

We consider first-order equivalences of three kinds, the first of which corresponds to changes of what is often referred to as 'flow frames'.

1. *Velocity shifts.* Only the velocity transforms, while $\Delta\tilde{n} = \Delta\tilde{\rho} = \Delta\tilde{p} = 0$. One finds that

$$\delta\Delta\tilde{T}^{\alpha\beta} = (\tilde{\rho} + \tilde{p})(\tilde{U}^\alpha(\Delta\tilde{U}^\beta) + (\Delta\tilde{U}^\alpha)\tilde{U}^\beta)) + O(\varepsilon) \quad \text{and} \quad \delta\Delta N^\beta = \tilde{n}\Delta\tilde{U}^\beta + O(\varepsilon). \tag{4.6}$$

   To respect unitarity of the velocity, such transformations are constrained by the condition

$$\tilde{U}_\alpha(\Delta\tilde{U}^\alpha) = 0. \tag{4.7}$$

2. *Thermodynamic shifts.* Only n, ρ, p transform while $\Delta U^\alpha = 0$. One finds that

$$\delta\Delta\tilde{T}^{\alpha\beta} = (\Delta\tilde{\rho})\tilde{U}^\alpha\tilde{U}^\beta + (\Delta\tilde{p})\tilde{\Pi}^{\alpha\beta} + O(\varepsilon) \quad \text{and} \quad \delta\Delta\tilde{N}^\beta = (\Delta\tilde{n})\tilde{U}^\beta + O(\varepsilon). \tag{4.8}$$

   Such transformations are constrained by the condition

$$\left(\frac{1}{\gamma - 1}\right)\Delta p = \Delta\rho - m\Delta n, \tag{4.9}$$

   which expresses the necessity that both (n, ρ, p) and $(\tilde{n}, \tilde{\rho}, \tilde{p})$ satisfy the equation of state

$$\left(\frac{1}{\gamma - 1}\right)p = \rho - mn.$$

3. *Eulerian gradient re-expressions.* These are modifications of $\Delta T^{\alpha\beta}, \Delta N^\beta$ which do not change the fields. We call any pair of covariant linear gradient expressions

$$S^{\alpha\beta} = S^{\alpha\beta\gamma\delta}\frac{\partial\psi^\gamma}{\partial x^\delta} + S^{\alpha\beta\gamma}\frac{\partial\psi}{\partial x^\delta}, \quad M^\beta = M^{\beta\gamma\delta}\frac{\partial\psi^\gamma}{\partial x^\delta} + M^{\beta\gamma}\frac{\partial\psi}{\partial x^\delta}$$

   for which

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta} = \frac{\partial}{\partial x^\beta}N^\beta = 0 \implies S^{\alpha\beta} = M^\beta \equiv 0$$

   holds as an implication for arbitrary (ψ^α, ψ) an *Eulerian constraint*. For any Eulerian constraint $(S^{\alpha\beta}, M^\beta)$, arbitrary solutions of (4.2) satisfy (4.5) with $\tilde{T}^{\alpha\beta} = T^{\alpha\beta}$, $\Delta\tilde{T}^{\alpha\beta} = \Delta T^{\alpha\beta}$, $\tilde{N}^\beta = N^\beta$, $\Delta\tilde{N}^\beta = \Delta N^\beta$ and

$$\delta\Delta\tilde{T}^{\alpha\beta} = S^{\alpha\beta} + O(\varepsilon), \quad \delta\Delta\tilde{N}^\beta = M^\beta + O(\varepsilon).$$

   The following is obvious.

Lemma 4.2. *Velocity shifts, thermodynamic shifts and Eulerian gradient re-expressions form a group of first-order equivalences on* $\mathcal{F}_5$.

While we have intentionally carried out the above considerations in covariant form, the practical use of such transformations is more nicely handled in a rest frame notation. For this purpose, we represent 'dissipation pairs' $(-\Delta T^{\alpha\beta}, -\Delta N^\beta)$ in the form

$$\begin{pmatrix} -\Delta T^{00}|_0, & -\Delta T^{0j}|_0 \\ -\Delta T^{i0}|_0, & -\Delta T^{ij}|_0 \\ -\Delta N^0|_0, & -\Delta N^j|_0 \end{pmatrix} \in \begin{pmatrix} \Lambda & \Lambda^{1\times 3} \\ \Lambda^{3\times 1} & \Lambda^{3\times 3} \\ \Lambda & \Lambda^{1\times 3} \end{pmatrix},$$

where Λ denotes any real-valued linear form in the gradients of the state variables $\mathbf{u}, \theta, \psi$ and $\Lambda^{n\times m}$ an $n \times m$-matrix of such objects.

In this notation, a velocity shift is any assignment

$$\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} * & * + \Delta\mathbf{u}^\top \\ * + \Delta\mathbf{u}, & * \\ *, & * + \left(\dfrac{1}{h}\right)\Delta\mathbf{u}^\top \end{pmatrix}$$

with some $\Delta\mathbf{u} \in \Lambda^{3\times 1}$; note that the factor $1/h = n/(\rho + p)$ in the last line corresponds to (4.6). A thermodynamic shift is any assignment

$$\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} * + \Delta\rho & * \\ * & * + \Delta p\,\mathbf{I} \\ * + \Delta n & * \end{pmatrix}$$

with some triple $(\Delta n, \Delta\rho, \Delta p) \in \Lambda^3$ that satisfies (4.9), and an Eulerian gradient re-expression is any transition

$$\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} \tilde{*} & \tilde{*} \\ \tilde{*} & \tilde{*} \\ \tilde{*} & \tilde{*} \end{pmatrix}$$

for which each respective entrywise transition

$$\tilde{}: * \mapsto \tilde{*}$$

replaces, if anything, one gradient form Λ by another gradient form $\tilde{\Lambda}$ with the property that the values of Λ and $\tilde{\Lambda}$ agree on arbitrary Eulerian gradients, i.e. gradients that can be realized as those of solutions to the Euler equations (1.1).

For example, due to the Eulerian constraint

$$\chi(\nabla\theta + \theta\dot{\mathbf{u}}^\top) = -\kappa\nabla\psi, \tag{4.10}$$

the re-expression

$$\begin{pmatrix} 0 & \chi\left(\nabla\theta + \theta\dot{\mathbf{u}}^\top\right) \\ \chi\left(\nabla^\top\theta + \theta\dot{\mathbf{u}}\right) & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -\kappa\nabla\psi \\ -\kappa\nabla\psi^\top & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & 0 \end{pmatrix}$$

is an equivalent recasting of the Eckart description (for the Eckart description, cf. e.g. [32], p. 55).

The velocity shift

$$\begin{pmatrix} 0 & -\kappa\nabla\psi \\ -\kappa(\nabla\psi)^\top & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & \left(\dfrac{\kappa}{h}\right)\nabla\psi \end{pmatrix} \tag{4.11}$$

completes the equivalence bridge towards the Landau description. The latter is also known (cf. [18, p. 514] as what one can get via the further transformation

$$\begin{pmatrix} 0 & 0 \\ 0 & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & \left(\dfrac{\kappa}{h}\right)\nabla\psi \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & -\left(\dfrac{\chi}{h}\right)\left(\nabla\theta - \left(\dfrac{\theta}{nh}\right)\nabla p\right) \end{pmatrix};$$

this is a re-expression by virtue of (4.10) and the further Eulerian constraint

$$\dot{\mathbf{u}}^\top = -\frac{1}{nh}\nabla p.$$

The purpose of this section is to show the following.

Theorem 4.3. *Our formulation (2.2)–(2.8) is first-order equivalent to the Eckart and Landau descriptions.*

Proof. Starting from the Landau description, in its form on the r.h.s. of (4.11), we first perform a velocity shift

$$\begin{pmatrix} 0 & 0 \\ 0 & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & \left(\dfrac{\kappa}{h}\right)\nabla\psi \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & \left(\dfrac{\kappa}{h}\right)\nabla\psi - \left(\dfrac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

corresponding to a shift vector

$$\Delta\mathbf{u} = -\sigma\dot{\mathbf{u}},$$

tentative inasmuch as σ is to be determined later. The purpose of this velocity shift consists in creating a system of wave equations 'in the velocity part'. However, regarding $(-\Delta T^{\alpha\beta})_{,\beta}$, this has induced a disturbing isolated mixed derivative $-\nabla(\sigma\dot{\mathbf{u}})$ in the 'θ equation'. We compensate for this by a thermodynamic shift

$$\begin{pmatrix} 0 & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + \zeta\nabla\cdot\mathbf{u}\,\mathbf{I} \\ 0 & \left(\dfrac{\kappa}{h}\right)\nabla\psi - \left(\dfrac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + (\zeta + \tilde{\zeta}_2)\nabla\cdot\mathbf{u}\,\mathbf{I} \\ \left(\dfrac{\sigma}{h}\right)\nabla\cdot\mathbf{u} & \left(\dfrac{\kappa}{h}\right)\nabla\psi - \left(\dfrac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

with

$$\Delta\rho = \sigma\nabla\cdot\mathbf{u}, \quad \Delta n = \frac{\sigma}{h}\nabla\cdot\mathbf{u}, \quad \Delta p = (\gamma-1)\left(1 - \frac{m}{h}\right)\sigma\nabla\cdot\mathbf{u} = \tilde{\zeta}_2\nabla\cdot\mathbf{u}$$

that leads to cancelling mixed derivatives of $\mathbf{u}$ both in the 'θ equation' and the 'ψ equation'. Note that this explains the choice of $\tilde{\zeta}_2$ in (2.7).

Next, we use another thermodynamic shift, with

$$\Delta\rho = 0, \quad \Delta n = -\frac{\kappa}{h}\dot{\psi}, \quad \Delta p = (\gamma-1)m\frac{\kappa}{h}\dot{\psi},$$

i.e.

$$\rightarrow \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + ((\zeta + \tilde{\zeta}_2)\nabla\cdot\mathbf{u} + (\gamma-1)m\left(\dfrac{\kappa}{h}\right)\dot{\psi})\,\mathbf{I} \\ -\left(\dfrac{\kappa}{h}\right)\dot{\psi} + \left(\dfrac{\sigma}{h}\right)\nabla\cdot\mathbf{u} & \left(\dfrac{\kappa}{h}\right)\nabla\psi - \left(\dfrac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

to generate a wave equation in the 'ψ part'.

Finally, as Eulerian gradients satisfy

$$\dot{\psi} = (\gamma-1)m\theta^{-1}\nabla\cdot\mathbf{u},$$

the further transition

$$\rightarrow \begin{pmatrix} \sigma\nabla\cdot\mathbf{u} & -\sigma\dot{\mathbf{u}}^\top \\ -\sigma\dot{\mathbf{u}} & \eta\mathbf{S}\mathbf{u} + (\zeta + \tilde{\zeta}_1 + \tilde{\zeta}_2)\nabla\cdot\mathbf{u}\,\mathbf{I} \\ -\left(\dfrac{\kappa}{h}\right)\dot{\psi} + \left(\dfrac{\sigma}{h}\right)\nabla\cdot\mathbf{u} & \left(\dfrac{\kappa}{h}\right)\nabla\psi - \left(\dfrac{\sigma}{h}\right)\dot{\mathbf{u}} \end{pmatrix}$$

is a re-expression, if $\tilde{\zeta}_1$ is chosen as in (2.7).

We have arrived at (2.2), (2.3). ■

The only thing left to decide from this point of view is which value to give to the free parameter σ, which is still up to us. Our choice $(2.6)_1$ is explained in the next section.

5. Causality of the dissipation operator

Whether our theory is causal can be understood by looking at the linearization of (1.2) at an arbitrarily fixed homogeneous reference state $\bar{\psi}^e = (\bar{\psi}^\epsilon, \bar{\psi})$ in the fluid's rest frame and studying the associated Fourier–Laplace modes. Our arguments parallelize the development in [26].

Written in the Godunov variables (ψ^α, ψ), the linearized equations will be of the form

$$A^{a\beta g}\frac{\partial \psi_g}{\partial x^\beta} = B^{a\beta g\delta}\frac{\partial^2 \psi_g}{\partial x^\beta \partial x^\delta} \tag{5.1}$$

with $B^{a\beta g\delta}$ as in §3 and

$$A^{a\beta g} = \frac{\partial T^{a\beta}}{\partial \psi_g}(\bar{\psi}^e), \quad a, g = 0, 1, 2, 3, 4,$$

where we use $T^{4\beta}$ to denote N^β.

System (5.1) permits a Fourier–Laplace mode

$$\hat{\psi}^e \exp(\lambda x^0 + \mathrm{i}\xi_j x^j) \quad \text{with } \lambda \in \mathbb{C},\ \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \tag{5.2}$$

if and only if the dispersion relation

$$0 = \pi(\lambda, \boldsymbol{\xi}) \equiv \det((\lambda A^{a0g} + \mathrm{i}\xi_j A^{ajg} + \lambda^2 B^{a0g0} - \xi_j\xi_k B^{ajgk})_{a,g=0,1,2,3,4}) \tag{5.3}$$

holds. As $\mathrm{Im}(\lambda)/|\boldsymbol{\xi}|$ is the speed at which a mode travels, we call our theory causal if

$$\mathrm{Im}(\lambda) \le |\boldsymbol{\xi}| \quad \text{for any solution } (\lambda, \boldsymbol{\xi}) \text{ of (5.3).}$$

Numerical studies of π show that the system is causal, for arbitrary reference states, arbitrary values of the dissipation coefficients $\eta > 0, \zeta \ge 0, \kappa > 0$, and arbitrary values of the parameters $\gamma \in (1, 2], c_v > 0$ and $m > 0$. We leave a rigorous analysis of this question to a later publication.

Here we focus on the high-frequency limit

$$B^{a\beta g\delta}\frac{\partial^2 \psi_g}{\partial x^\beta \partial x^\delta} = 0 \tag{5.4}$$

of (5.1) to check the causality of our dissipation tensor as such. Because of the degeneracy, the associated dispersion relation is

$$0 = \pi_\infty(\lambda, \boldsymbol{\xi}) \equiv \det(\lambda^2 B^{\check{a}0\check{g}0} - \xi_j\xi_k B^{\check{a}j\check{g}k}), \tag{5.5}$$

where $B^{\check{a}0\check{g}0}$ and $B^{\check{a}0\check{g}k}$ are the lower right 4×4 blocks of the 5×5 matrices B^{a0g0} and B^{ajgk}, with indices $\check{a}, \check{g}$ running only from 1 to 4.

Continuing from the end of the previous section, we assume that $\tilde{\zeta}$ has been fixed according to $(2.6)_2$, but σ remains free to be chosen. The following is the main point of this section.

Lemma 5.1. *The theory is causal in the high-frequency limit,*

$$\mathit{Im}(\lambda) \le |\boldsymbol{\xi}| \ \textit{ for any solution } (\lambda, \boldsymbol{\xi}) \textit{ of (5.5),}$$

if and only if

$$\sigma \ge \tfrac{4}{3}\eta + \tilde{\zeta}.$$

Proof. From §3 we know that $\lambda^2 B^{a0g0}$ and $\xi_j\xi_k B^{ajgk}$ are given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma\theta\lambda^2\delta^{jk} & 0 \\ 0 & 0 & -\tilde{\kappa}\lambda^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta\theta|\boldsymbol{\xi}|^2\delta^{jk} + ((1/3)\eta + \tilde{\zeta})\theta\xi^j\xi^k & 0 \\ 0 & 0 & \tilde{\kappa}|\boldsymbol{\xi}|^2 \end{pmatrix}, \tag{5.6}$$

whence $\lambda^2 B^{\check{a}0\check{g}0}$ and $\xi_j\xi_k B^{\check{a}j\check{g}k}$ are

$$\begin{pmatrix} -\sigma\theta\lambda^2\delta^{jk} & 0 \\ 0 & -\tilde{\kappa}\lambda^2 \end{pmatrix}, \quad \begin{pmatrix} \eta\theta|\boldsymbol{\xi}|^2\delta^{jk} + ((1/3)\eta + \tilde{\zeta})\theta\xi^j\xi^k & 0 \\ 0 & \tilde{\kappa}|\boldsymbol{\xi}|^2 \end{pmatrix}. \tag{5.7}$$

The dispersion relation splits as

$$\pi_\infty(\lambda, \boldsymbol{\xi}) = \pi^L_\infty(\lambda, \boldsymbol{\xi})\pi^T_\infty(\lambda, \boldsymbol{\xi})$$

with

$$\pi^L_\infty(\lambda, \xi) = \tilde{\kappa}\theta(\lambda^2 + |\boldsymbol{\xi}|^2)(\sigma\lambda^2 + ((4/3)\eta + \tilde{\zeta})|\boldsymbol{\xi}|^2)$$

and

$$\pi_\infty^T(\lambda,\xi) = (\sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2).$$

While the roots of π_∞^L correspond to longitudinal modes, i.e. the amplitude vector $(\hat{\psi}^{\check{e}})_{e=1,2,3,4}$ of the mode belongs to the invariant subspace $\mathbb{C}\boldsymbol{\xi} \times \mathbb{C}$ of the matrix $\lambda^2 B^{\check{a}0\check{g}0} - \xi_j\xi_k B^{\check{a}j\check{g}k}$, the roots of π_∞^T are associated with transverse modes, for which $(\hat{\psi}^{\check{e}})_{e=1,2,3,4}$ lies in the complementing invariant subspace $\{\boldsymbol{\xi}\}^\perp \times \{0\}$.

The longitudinal part π_∞^L has the roots

$$\lambda_{\text{heat}}^{\pm} = \pm\mathrm{i}|\boldsymbol{\xi}|$$

and

$$\lambda_{\text{visc}}^{L,\pm} = \pm\mathrm{i}\sqrt{\frac{(4/3)\eta + \tilde{\zeta}}{\sigma}}|\boldsymbol{\xi}|,$$

the transverse part π_∞^T the double roots

$$\lambda_{\text{visc}}^{T,\pm} = \pm\mathrm{i}\sqrt{\frac{\eta}{\sigma}}|\boldsymbol{\xi}|. \qquad \blacksquare$$

The lemma says that our choice $(2.6)_1$ is the limiting causal case.

Corollary 5.2. *For $\sigma = (\frac{4}{3})\eta + \tilde{\zeta}$, all longitudinal modes travel at the speed of light.*

In other words, the choice $(2.6)_1$ is the unique one for which the dissipation operator is causal, while the propagation speeds of all dissipation-attenuated longitudinal modes of the full system (5.1) tend, in the limit of large wavenumbers, to exactly the speed of light.

6. Entropy production and vanishing viscosity limit

In this section we study the compatibility of our relativistic NSF equations with the second law of thermodynamics. We start by computing the divergence of the original entropy current.

Lemma 6.1. *For any solution of* (1.2), *the entropy density s satisfies*

$$(nsU^\beta)_{,\beta} = \frac{U_\alpha}{\theta}(\Delta T^{\alpha\beta})_{,\beta} + \psi(\Delta N^\beta)_{,\beta}.$$

Proof. From $T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + pg^{\alpha\beta}$ we find

$$\begin{aligned} U_\alpha(T^{\alpha\beta})_{,\beta} &= U^\beta p_{,\beta} - (nhU^\beta)_{,\beta} \\ &= U^\beta(p_{,\beta} - nh_{,\beta}) - h(nU^\beta)_{,\beta} \\ &= U^\beta(p_{,\beta} - nh_{,\beta}) + h(\Delta N^\beta)_{,\beta}. \end{aligned}$$

As the first law,

$$\theta\,\mathrm{d}s = pd\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right),$$

gives

$$\begin{aligned} U^\beta(p_{,\beta} - nh_{,\beta}) &= -nU^\beta\left(p\left(\frac{1}{n}\right)_{,\beta} + \left(\frac{\rho}{n}\right)_{,\beta}\right) \\ &= -nU^\beta\theta s_{,\beta} \\ &= -\theta[(nsU^\beta)_{,\beta} - s(nU^\beta)_{,\beta} \\ &= -\theta[(nsU^\beta)_{,\beta} + s(\Delta N^\beta)_{,\beta}], \end{aligned}$$

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combining indeed yields

$$-U_\alpha(\Delta T^{\alpha\beta})_{,\beta} = U_\alpha(T^{\alpha\beta})_{,\beta} = \theta(-(nsU^\beta)_{,\beta} + \psi(\Delta N^\beta)_{,\beta}).$$

■

Similarly to [32, p. 54], we redefine the entropy current as

$$S^\beta \equiv nsU^\beta - \frac{U_\alpha}{\theta}\Delta T^{\alpha\beta} - \psi\,\Delta N^\beta, \tag{6.1}$$

and consistently use

$$\mathcal{Q} \equiv S^\beta_{,\beta} = \left(\frac{U_\alpha}{\theta}\right)_{,\beta}(-\Delta T^{\alpha\beta}) + \psi_{,\beta}(-\Delta N^\beta) \tag{6.2}$$

as net local entropy production. $\mathcal{Q}$ is a quadratic form in the gradients of U^α, θ and ψ. As in [26], we use non-negativity of $\mathcal{Q}$ on Eulerian gradients as our criterion for thermodynamic admissibility of our proposed equations of motion (1.2).

Technically, the following is the main result of this section.

Theorem 6.2. *Entropy production is non-negative on Eulerian gradients.*

The following properties are useful.

Lemma 6.3. *Expressed in the rest frame of the reference state, Eulerian gradients satisfy*

$$\dot{\mathbf{u}} = -(nh)^{-1}\nabla p, \quad \dot\theta = -(\gamma-1)\theta\nabla\cdot\mathbf{u}, \quad \dot\psi = (\gamma-1)m\theta^{-1}\nabla\cdot\mathbf{u}. \tag{6.3}$$

Proof. In the rest frame, the linearized equations read

$$\left.\begin{aligned}\dot\rho + (\rho+p)\nabla\cdot\mathbf{u} &= 0,\\ (\rho+p)\dot{\mathbf{u}} + \nabla p &= 0\\ \text{and} \qquad \dot n + n\nabla\cdot\mathbf{u} &= 0.\end{aligned}\right\} \tag{6.4}$$

$(6.4)_2$ is $(6.3)_1$. In view of $\dot s = 0$ and as the representations

$$\theta = \hat\theta(n,s) = \frac{k}{c_v}n^{\gamma-1}\exp\left(\frac{s}{c_v}\right), \quad \psi = \hat\psi(n,s) = \frac{m}{\hat\theta(n,s)} + \gamma c_v - s$$

satisfy

$$n\hat\theta_n = (\gamma-1)\theta, \quad n\hat\psi_n = -(\gamma-1)m\theta^{-1},$$

$(6.3)_2$ and $(6.3)_3$, follow from

$$\dot\theta = \hat\theta_n\dot n + \hat\theta\dot s, \quad \dot\psi = \hat\psi_n\dot n + \hat\psi\dot s.$$

and $(6.4)_3$. ■

From (2.4) and (2.5), we determine the entropy production as

$$\begin{aligned}\mathcal{Q} &\equiv -\frac{1}{\theta^2}\frac{\partial\theta}{\partial x^0}\Delta T^{00}|_0 - \frac{1}{\theta^2}\left(\frac{\partial\theta}{\partial x^i} + \theta\frac{\partial u_i}{\partial x^0}\right)\Delta T^{i0}|_0 - \frac{1}{\theta}\frac{\partial u_i}{\partial x^j}\Delta T^{ij}|_0 - \frac{\partial\psi}{\partial x^0}\Delta N^0|_0 - \frac{\partial\psi}{\partial x^j}\Delta N^j|_0\\ &= \frac{\sigma}{\theta^2}\dot\theta\nabla\cdot\mathbf{u} - \frac{\sigma}{\theta^2}(\nabla\theta + \theta\dot{\mathbf{u}})\cdot\dot{\mathbf{u}} + \mathcal{Q}_3 + \dot\psi(-\tilde\kappa\dot\psi + \tilde\sigma\nabla\cdot\mathbf{u}) + \nabla\psi\cdot(\tilde\kappa\nabla\psi - \tilde\sigma\dot{\mathbf{u}})\\ &\equiv \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 + \mathcal{Q}_5,\end{aligned}$$

where

$$\mathcal{Q}_3 = \frac{1}{\theta}\frac{\partial u_i}{\partial x^j}\left(\eta\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} - \frac{2}{3}\frac{\partial u^l}{\partial x^l}\delta^{ij}\right) + \tilde\zeta\frac{\partial u^l}{\partial x^l}\delta^{ij}\right) = \frac{\eta}{2\theta}\|\mathcal{S}\mathbf{u}\|^2 + \frac{\tilde\zeta}{\theta}(\nabla\cdot\mathbf{u})^2.$$

As

$$\nabla\theta + \theta\dot{\mathbf{u}} = -\frac{\theta^2}{h}\nabla\psi,$$

we find

$$\mathcal{Q}_2 + \mathcal{Q}_5 = \tilde\kappa|\nabla\psi|^2.$$

Using (6.3) and, notably, (2.6), (2.7), we get

$$\mathcal{Q}_1 + \mathcal{Q}_4 = \left(\frac{\sigma}{\theta}(\gamma-1)\left(\frac{m}{h}-1\right) - \frac{\kappa}{h}(\gamma-1)^2\left(\frac{m}{\theta}\right)^2\right)(\nabla\cdot\mathbf{u})^2 = -\frac{\tilde{\zeta}_1+\tilde{\zeta}_2}{\theta}(\nabla\cdot\mathbf{u})^2$$

and finally

$$\mathcal{Q} = \frac{\eta}{2\theta}\|\mathcal{S}\mathbf{u}\|^2 + \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2 + \frac{\kappa}{h}|\nabla\psi|^2. \tag{6.5}$$

Theorem 6.2 means that entropy production is non-negative to leading order in the small dissipation coefficients. As in §4 we express the situation through a small extra factor ϵ that multiplies η, ζ, κ and state this point as follows.

Corollary 6.4. *For solutions to* (4.2), *the entropy production is*

$$\mathcal{Q} = \epsilon\left(\frac{\eta}{2\theta}\|\mathcal{S}\mathbf{u}\|^2 + \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2 + \frac{\kappa}{h}|\nabla\psi|^2\right) + O(\epsilon^2).$$

Remark. In the non-causal descriptions of Eckart and Landau, entropy production is automatically non-negative on arbitrary gradients [17,18,26]. Owing to the hyperbolic character of our theory, it is not difficult to fabricate initial data (including temporal derivatives), far from the near-Eulerian regime, that make the above form $\mathcal{Q}$ negative. Note that this does *not* necessarily imply that solutions corresponding to these data would violate the second law. Recall that there is room to (re)define the entropy current, and it could also be that in such cases the second law is perfectly valid, while this simply is not obvious from the perspective of the particular entropy current (6.1) on which the definition of this specific version $\mathcal{Q}$ of net entropy production is based. In any case, our system is made for the near-Eulerian regime, in accordance with the fundamental vanishing-viscosity concept that underlies the mathematical theory of systems of conservation laws [33,34].

7. Classical limit

We show that our theory has the classical NSF equations as its limit, as the speed of light tends to infinity.

To understand the limit $c \to \infty$, we put dimensions back in, writing

$$x^0 = ct, \quad u^\alpha = \frac{v^\alpha}{c} \quad \text{and} \quad m \text{ as } mc^2$$

as well as

$$\eta = c\bar{\eta}, \quad \zeta = c\bar{\zeta}, \quad \text{and} \quad \kappa = \frac{\bar{\kappa}}{c^3}$$

and let

$$\bar{\chi} = \left(\frac{m}{\theta^2}\right)\bar{\kappa}.$$

The following is the goal of this section.

Theorem 7.1. *Equations* (1.2) *with* (2.2) and (2.3) *can be written as*

$$\frac{\partial}{\partial t}(nm) + \frac{\partial}{\partial x^j}(nmv^j) = O\left(\frac{1}{c^2}\right),$$

$$\frac{\partial}{\partial t}(nmv^i) + \frac{\partial}{\partial x^j}(nmv^iv^j + p\delta^{ij})$$
$$= \frac{\partial}{\partial x^j}\left(\bar{\eta}\left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3}g^{ij}\frac{\partial v^k}{\partial x^k}\right) + \bar{\zeta}\left(\frac{\partial v^l}{\partial x^l}\right)\right) + O\left(\frac{1}{c^2}\right),$$

$$\frac{\partial}{\partial t}\left(n\left(e+\frac{1}{2}mv^2\right)\right)+\frac{\partial}{\partial x^j}\left(\left(n\left(e+\frac{1}{2}mv^2\right)+p\right)v^j\right)$$
$$=\frac{\partial}{\partial x^j}\left(\bar{\chi}\frac{\partial\theta}{\partial x^j}+\bar{\eta}v_i(\mathcal{S}v)^{ij}+\bar{\zeta}v^j\nabla\cdot v\right)+O\left(\frac{1}{c^2}\right).$$

I.e. taking $c\to\infty$ yields the classical Navier–Stokes–Fourier equations.

We first quickly recapitulate what the *inviscid* terms look like in terms of powers of $1/c$. Substitution yields

$$cmN^{\beta}_{,\beta}=\left(nm\sqrt{1+\left(\frac{v}{c}\right)^2}\right)_t+(nmv^j)_{x^j}$$
$$=\{(nm)_t+(nmv^j)_{x^j}\}+\frac{1}{c^2}\left(\frac{1}{2}nmv^2\right)_t+O\left(\frac{1}{c^4}\right) \tag{7.1}$$

and

$$T^{\alpha\beta}_{,\beta}=\frac{1}{c}\left\{(nmc^2+ne+p)\frac{v^\alpha v^0}{c^2}+pg^{\alpha 0}\right\}_t+\left\{(nmc^2+ne+p)\frac{v^\alpha v^j}{c^2}+pg^{\alpha j}\right\}_{x^j}.$$

In the cases $\alpha\equiv i=1,2,3$,

$$T^{i\beta}_{,\beta}=\frac{1}{c}\left((nmc^2+ne+p)\sqrt{1+\left(\frac{v}{c}\right)^2}\frac{v^i}{c}\right)_t+\left((nmc^2+ne+p)\frac{v^iv^j}{c^2}+p\delta^{ij}\right)_{x^j}+O\left(\frac{1}{c^2}\right)$$
$$=\{(nmv^i)_t+(nmv^iv^j+p\delta^{ij})_{x^j}\}+O\left(\frac{1}{c^2}\right), \tag{7.2}$$

so $T^{i\beta}_{,\beta}$ clearly converges to the momentum part of the classical Euler equations as $c\to\infty$, $i=1,2,3$. Note that only the leading-order part of (7.1) is used to obtain (7.2).

For $\alpha=0$, one finds

$$T^{0\beta}_{,\beta}=\left[\left\{(nmc^2+ne+p)\left(1+\left(\frac{v}{c}\right)^2\right)-p\right\}_t+\left\{(nmc^2+ne+p)\sqrt{1+\left(\frac{v}{c}\right)^2}v^j\right\}_{x^j}\right]+O\left(\frac{1}{c^2}\right)$$
$$=c^2\{(nm)_t+(nmv^j)_{x^j}\}+\left\{(nmv^2+ne)_t+\left(\left(\frac{1}{2}mv^2+ne+p\right)v^j\right)_{x^j}\right\}+O\left(\frac{1}{c^2}\right), \tag{7.3}$$

which together with (7.1) yields

$$c(T^{0\beta}_{,\beta}-mc^2N^{\beta}_{,\beta})=\left(ne+\frac{1}{2}nmv^2\right)_t+\left(\left(ne+p+\frac{1}{2}nmv^2\right)v^j\right)_{x^j}+O\left(\frac{1}{c^2}\right). \tag{7.4}$$

Note that it is not $cT^{0\beta}_{,\beta}$ but $c(T^{0\beta}_{,\beta}-mc^2N^{\beta}_{,\beta})$ that converges to the energy part of the classical Euler equations as $c\to\infty$. The next-to-leading order part in (7.1) 'corrects' the term $(nmv^2)_t$ in the next-to-leading order part in (7.3), to give the term $\frac{1}{2}(nmv^2)_t$ in the classical equation. Something analogous happens in the dissipative part; we identify it in the term E in lemmas 7.2 and 7.3 just below.

Theorem 7.1 is a direct consequence of equations (7.1), (7.2), (7.4) and the following three lemmas.

Lemma 7.2.

$$\frac{\partial}{\partial x^\beta}(-\Delta N^\beta)=\frac{1}{c^3}\frac{\partial}{\partial x^j}\left(\bar{\kappa}\frac{\partial(1/\theta)}{\partial x_j}\right)+\frac{1}{mc^3}E+O\left(\frac{1}{c^4}\right) \tag{7.5}$$

with

$$E=\frac{\partial}{\partial t}(\bar{\sigma}\nabla\cdot v)+\frac{\partial}{\partial x^j}\left(\bar{\sigma}v^j\nabla\cdot v-\bar{\sigma}\left(\frac{\partial v^j}{\partial t}+v_i\frac{\partial v^j}{\partial x^i}\right)\right).$$

Lemma 7.3.

$$\frac{\partial}{\partial x^\beta}(-c\Delta T^{0\beta}) = D + E + O\left(\frac{1}{c^2}\right)$$

with

$$D = \frac{\partial}{\partial x^j}\left(v_i\left(\bar{\eta}\left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3}g^{ij}\frac{\partial v^k}{\partial x^k}\right) + \bar{\zeta}\frac{\partial v^l}{\partial x^l}\right)\delta^{ij}\right). \tag{7.6}$$

Lemma 7.4. *One obtains for $i = 1, 2, 3$:*

$$\frac{\partial}{\partial x^\beta}(-\Delta T^{i\beta}) = \frac{\partial}{\partial x^j}\left(\bar{\eta}\left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3}g^{ij}\frac{\partial v^k}{\partial x^k}\right) + \bar{\zeta}\frac{\partial v^l}{\partial x^l}\delta^{ij}\right) + O\left(\frac{1}{c^2}\right).$$

Similarly to how we got to (7.4), it is the next-to-leading order parts of (7.5) and (7.6) that play together to yield

$$\begin{aligned} c\frac{\partial}{\partial x^\beta}(-\Delta T^{0\beta}_{\ ,\beta} + mc^2\Delta N^\beta) &= D - \frac{\partial}{\partial x^j}\left(m\bar{\kappa}\frac{\partial(1/\theta)}{\partial x_j}\right) \\ &= D + \frac{\partial}{\partial x^j}\left(\bar{\chi}\frac{\partial\theta}{\partial x_j}\right). \end{aligned}$$

Proof of lemma 7.2. Using

$$\frac{h}{c^2} = m + O\left(\frac{1}{c^2}\right) \quad \text{and} \quad \frac{\psi}{c^2} = \frac{m}{\theta} + O\left(\frac{1}{c^2}\right),$$

we find

$$\frac{\partial}{\partial x^j}\left(\tilde{\kappa}\frac{\partial\psi}{\partial x_j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\kappa}{h}\frac{\partial\psi}{\partial x_j}\right) = \frac{1}{c^3}\frac{\partial}{\partial x^j}\left(\bar{\kappa}\frac{\partial(1/\theta)}{\partial x_j}\right) + O\left(\frac{1}{c^5}\right)$$

On the other hand,

$$\frac{\partial}{\partial x^\beta}\left(\tilde{\sigma}(u^\beta\Pi^{\gamma\delta} - u^\delta\Pi^{\beta\gamma})\frac{\partial u_\gamma}{\partial x^\delta}\right) = \frac{1}{mc^3}E + O\left(\frac{1}{c^5}\right).$$

The assertion follows as

$$\frac{\partial}{\partial x^\beta}(-\Delta N^\beta) = \frac{\partial}{\partial x^\beta}\left(\tilde{\kappa}g^{\beta\gamma}\frac{\partial\psi}{\partial x^\gamma} + \tilde{\sigma}(u^\beta\Pi^{\gamma\delta} - u^\delta\Pi^{\beta\gamma})\frac{\partial u_\gamma}{\partial x^\delta}\right).$$

■

We suppress the lengthy, but trivial proofs of lemmas 7.3 and 7.4.

A point worth emphasizing regarding the assertion of lemma 7.4 is the fact that the coefficients $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ leave no traces in the limit. This is due to the fact that they both scale with higher powers of $1/c$ than η and ζ: ζ_1 scales like κ, and, compared with η, ζ, the coefficient $\tilde{\zeta}_2$ also has an extra factor of $1/c^2$ coming from the $1 - m/h$ term in $(2.7)_2$.

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