On the Essential Regularity of Singular Connections in Geometry

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UC-Davis

Nonlinear Analysis Seminar
National Taiwan Normal University
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All Joint Work With: Moritz Reintjes

The question we address:

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"How do we determine, apriori, whether a singularity in a gravitational metric tensor in General Relativity is essential or removable by coordinate transformation?"

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Virtually "everyone" asks this question after reading the derivation of the Schwarzschild's solution in General Relativity…

Karl Schwarzschild (1873-1914):

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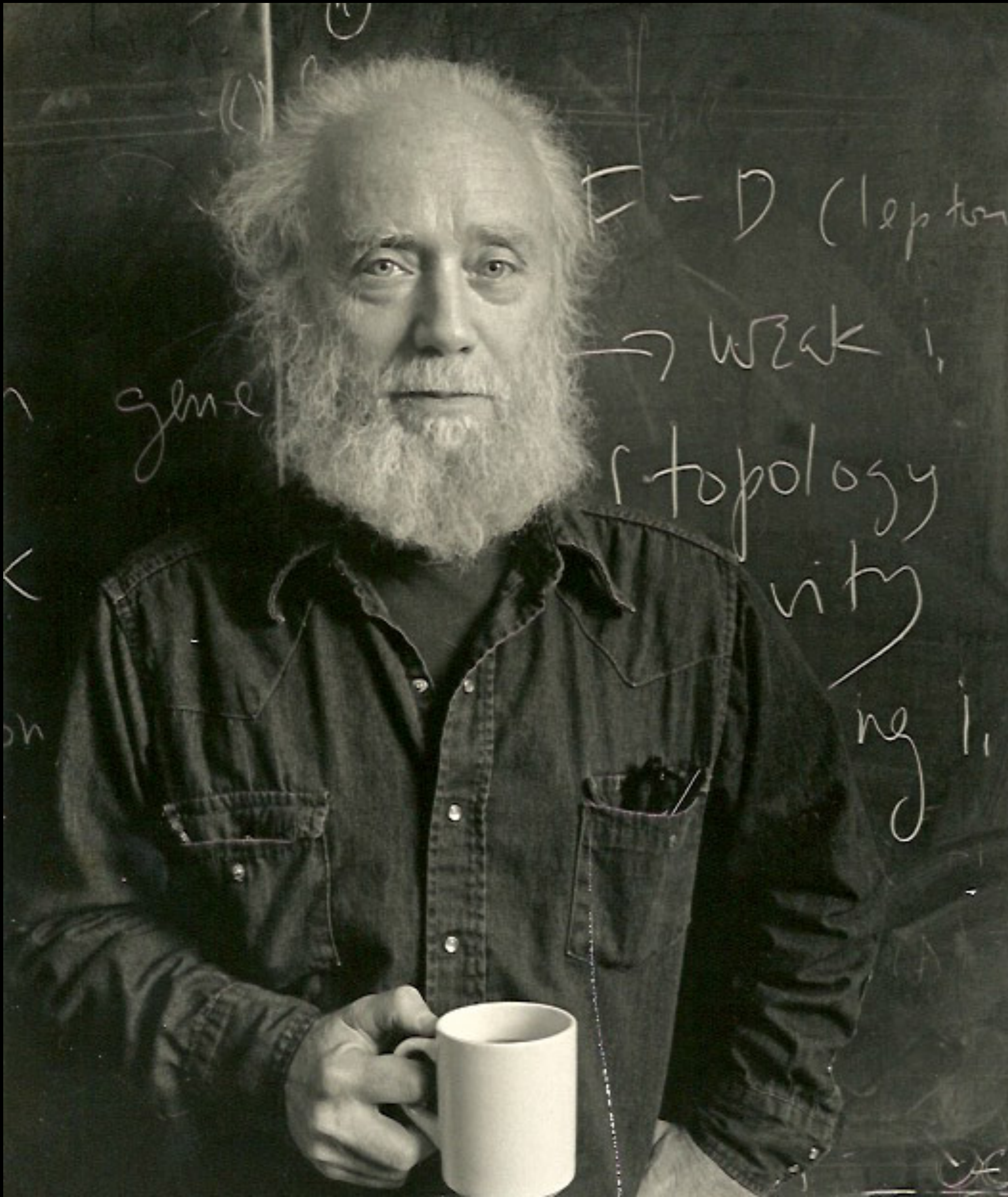
Question: Is this essential or removable by coordinate transformation?

Answer: Eddington-Finkelstein early 1920's:

Sir Arthur Eddington (1882-1944):

David Finkelstein (1929-2016):



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No singularity in the new coordinates $r \neq 0$!

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Q: How could you tell ahead of time the Schwarzschild singularity is removable?

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…implying Black Holes are physically possible…

Q: How could you tell ahead of time the Schwarzschild singularity is removable?

And what procedure provides the regularizing coordinate transformations?

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Answer: Yes. By theory of the RT-equations

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Everything depends on regularity of the Riemann Curvature Tensor

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curvature in L^p, $p > n/2$ (in progress)*

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* $p > n/2$ sufficient for GR shock-waves,

not Black Holes*

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Our theory relies on an existence theory for the RT-equations

The RT-equations are a nonlinear elliptic system of equations with matrix valued differential forms as unknowns…

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…the unknowns are the Jacobians J of coordinate transformations which regularize the components of a connection Γ… coupled to the unknown regularized connection $\tilde{\Gamma}$.

A connection is the most general construct in geometry which has a Riemann Curvature Tensor..

The RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\
\Delta J &= \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \\
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We get this by solving the RT-equations:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), \tag{1}$$

$$\Delta J = \delta(J\,\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{2}$$

$$d\vec{A} = \overrightarrow{\text{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\text{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \tag{3}$$

$$\delta\vec{A} = v, \tag{4}$$

Key: " δ comes after d ":

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They are constructed from the Euclidean metric of the coordinate system in which components are given…

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They are constructed from the Euclidean metric of the coordinate system in which components are given…

Hence they are elliptic independent of metric signature…

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(Locally, in a neighborhood of every point.)

Our existence theorem for the RT-equations is based on elliptic regularity in L^p-spaces:

Theoretical papers on the RT-equations:

M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry I: Connections on tangent bundles*, Meth. Appl. Anal., Vol. 29, No. 4, 303-396, (2023)

M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry II: Connections on vector bundles*, Adv. Theo. Math. Phys. Volume 27, Number 3, 623–684, (2023)

M. Reintjes and B. Temple, *Optimal regularity and Uhlenbeck compactness for General Relativity and Yang-Mills Theory*, (2022), Proc. R. Soc. A 479: 20220444

M. Reintjes and B. Temple, *How to smooth a crinkled map of spacetime: Uhlenbeck compactness for L^∞ connections and optimal regularity for general relativistic shock waves by the Reintjes-Temple-equations*, Proc. R. Soc. A (2022) 476: 20200177

M. Reintjes and B. Temple, *The regularity transformation equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity*, Adv. Theor. Math. Phys 24.5, (2020), 1203-1245.

M. Reintjes and B. Temple, *Shock Wave Interactions and the Riemann-flat Condition: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames in General Relativity*, Arch. Rat. Mech. Anal. 235 (2020), 1873-1904.

The Riemann-flat condition:

Assume $\Gamma, R \in L^{\infty}$.

Then: There exists a $C^{1,1}$ coordinate transformation which smooths Γ to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ st

$$Riem(\Gamma + \tilde{\Gamma}) = 0.$$

Applications of the RT-equations:

M. Reintjes, B. Temple, *On weak solutions to the geodesic equation in the presence of curvature bounds*, Jour. Diff Eqns, Vol. 392, 306-324 (2024)

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Topic of this talk…

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Theorem (RT): Assume

$$\Gamma \in L^{2p} \text{ and } Riem(\Gamma) \in L^p, \; p > n/2$$

in a given coordinate system x. Then there always exist coordinate transformations

$$x \to y$$

such that in y-coordinates

$$\Gamma \in W^{1,p}, \; Riem(\Gamma) \in L^p$$

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Connection one full derivative above curvature…

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For metric connections:

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Theorem (RT): Assume

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$$g \in W^{2,p}, \ Riem(\Gamma) \in L^p$$

The existence of such coordinate transformations rules out "regularity singularities" in General Relativity.

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(We called this optimal regularity because we did not realize that J could regularize the curvature as well...so "optimal regularity" was not the "essential regularity" of the connection...)

The extra derivative implies compactness for sequences $\{\Gamma_i\}_{k=1}^{\infty}$ of affine connections:

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Theorem (RT): Assume

$$\Gamma_i \in L^\infty \quad \text{and} \quad Riem(\Gamma_i) \in L^p, \quad p > n$$

with uniform bounds. Then there exist a convergent subsequence in y-coordinates such that

$$\Gamma_i \to \Gamma \quad \text{strongly in } L^p, \text{ weakly in } W^{1,p}$$

Same for smooth solutions:

If: $\Gamma, Riem(\Gamma) \in W^{m,p}, m \geq 1$.

Then: $x \to y$ gives

$$\Gamma \in W^{m+1,p}, Riem(\Gamma) \in W^{m,p}$$

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Vector Bundle version of the RT-equations

"Same Theorems"

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Vector Bundle version of the RT-equations

⇨ Same Theorems

Both compact and non-compact Lie Groups:

$$\begin{aligned} \Delta\tilde{\mathcal{A}} &= \delta d\mathcal{A} - \delta\big(dU^{-1}\wedge dU\big) \\ \Delta U &= U\delta\mathcal{A} - (U^T\eta)^{-1}\langle dU^T;\eta dU\rangle \end{aligned}$$

$\mathcal{A} \equiv$ Non-optimal Connection

$U \equiv$ Gauge Transformation to optimal regularity…(we do case $SO(r,s)$)

This extends Uhlenbeck compactness to connections on vector bundles:

Theorem (RT): Assume

$(A_i, \Gamma_i) \in L^\infty$ and $dA \in L^1, d\Gamma \in L^p$, $p > n$

with uniform bounds. Then there exist a convergent subsequence such that under a gauge and coordinate transformation,

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Extends important results of Kazden-DeTurck and Uhlenbeck for (positive definite) Riemannian metrics and compact Lie groups…

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…to arbitrary connections on vector bundles allowing for compact and non-compact Lie groups…

…including the Lorentzian metrics and affine connections of General Relativity…

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Riemann (1854): "On the Hypotheses which lie at the Foundation of Geometry "

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Define weak solutions by "coordinate transformation" instead of "multiply by test function and integrate by parts"

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M. Reintjes, *Strong Cosmic Censorship with bounded curvature*, Class. and Quant. Grav., Volume 41, Number 17, (July 2024)

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Plan for Remainder of Talk:

- Review Riemann's theory of Curvature…
- State our necessary and sufficient condition for essential regularity of connections in geometry…
- Explain how these results follow from our existence theory for the RT-equations…
- Describe the RT-equations and how we discovered them…

Introduction

The Riemann Curvature Tensor

In Riemann's Theory of Curvature:

Metrical properties of a space are given by a

Riemannian metric g : $ds = g_{ij}dx^i dx^j$

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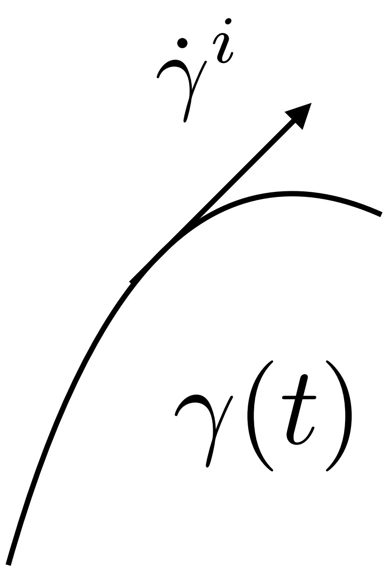

$$\|\dot\gamma\| = \sqrt{g_{ij}\dot\gamma^i\dot\gamma^j}$$

$$L = \int ds = \int_{t_0}^{t} \|\dot\gamma\| dt$$

g transforms like a bilinear form under $x \to y$

$$g_{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$$ (components)

$$g_y = J^t g_x J$$ (nxn matrices)

For (positive definite) Riemannian metrics, we recover flat Euclidean space locally…

$$g_{ij}(p) = \delta_{ij} + O(|p - p_0|^2)$$

I.e. $g_{ij} = \delta_{ij}$ Implies $ds^2 = dx_1^2 + \cdots + dx_n^2$

For metrics of signature $\delta(r, s)$, we locally recover flat (Minkowski) space…

$$g_{ij}(p) = \delta_{ij}(r, s) + O(|p - p_0|^2)$$

I.e. $g_{ij} = \delta_{ij}(r, s)$ Implies

$$ds^2 = -dx_1^2 - \cdots - dx_r^2 + dx_{r+1}^2 + \cdots dx_{r+s}^2$$

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— $Riem(\Gamma)$ measures second derivative Taylor errors but transforms by first derivative Jacobians…

Riemann Curvature Tensor:

—transforms by 1st derivative Jacobians

$$R^{\alpha}_{\beta\gamma\delta} = R^{i}_{jkl} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{k}}{\partial y^{\beta}} \frac{\partial x^{l}}{\partial y^{\gamma}} \frac{\partial x^{j}}{\partial y^{\delta}} \qquad \text{(tensor)}$$

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—But measures 2nd derivatives in the Taylor series

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—Thm (Riemann): $R \equiv 0$ iff $O(|p-p_0|)^2 \equiv 0$

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Christoffel Symbols

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$$Commutator$$

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R does NOT bound ALL the derivatives of Γ !

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Co-derivatives $\delta\Gamma$ are uncontrolled (pointwise)

View Γ as a matrix valued 1-form:

$$\Gamma \equiv \Gamma_k dx^k \equiv \left(\Gamma^i_j\right)_k dx^k$$

Then: $$R = d\Gamma + \Gamma \wedge \Gamma$$

R is a "Curl" plus a "Commutator"

as nxn matrices expressed as wedge product

RT-Equations: Equations for Jacobians that lift the regularity of connection Γ to one derivative above $d\Gamma$

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A), \\
\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\
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…controls all derivatives of Γ under coordinate transformation.

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Metric has a singularity in $g_{11}(x)$ at $r = 2M$

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So far we have a theory for $p > n$

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To globalize, start with a connection Γ defined by its components in an an atlas $\mathcal{A}$ of coordinate charts $x : \Omega_x \to \mathbb{R}^n$ which in turn define manifold $\mathcal{M}$

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Definition: An atlas $\mathcal{A}$ is said to have regularity $W^{s,p}$ if all of its transition maps on the overlaps of coordinate neighborhoods have regularity $W^{s,p}$.

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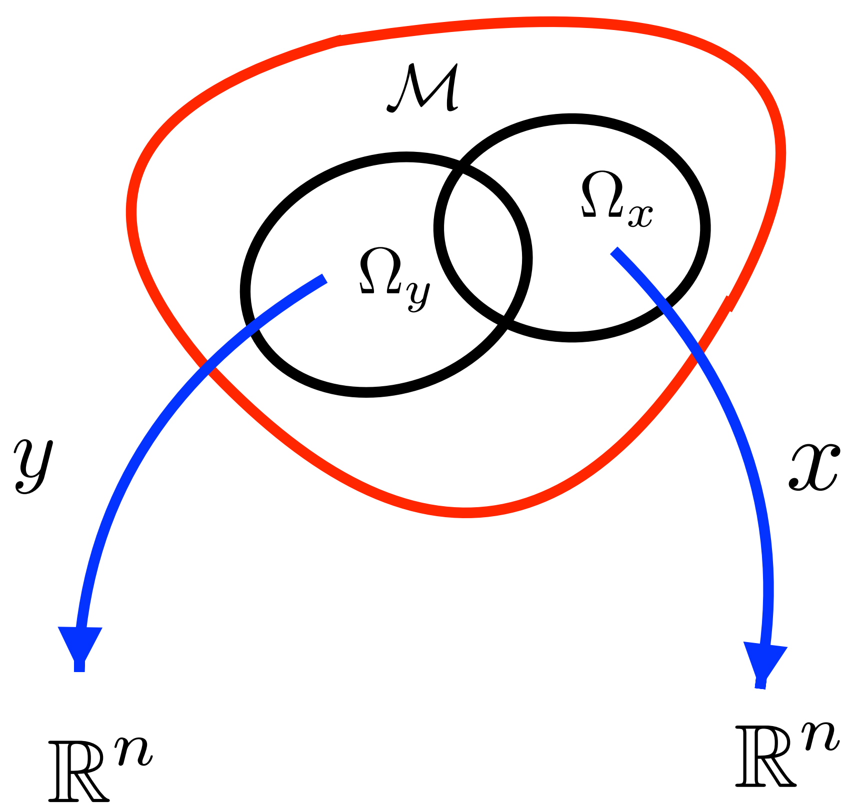

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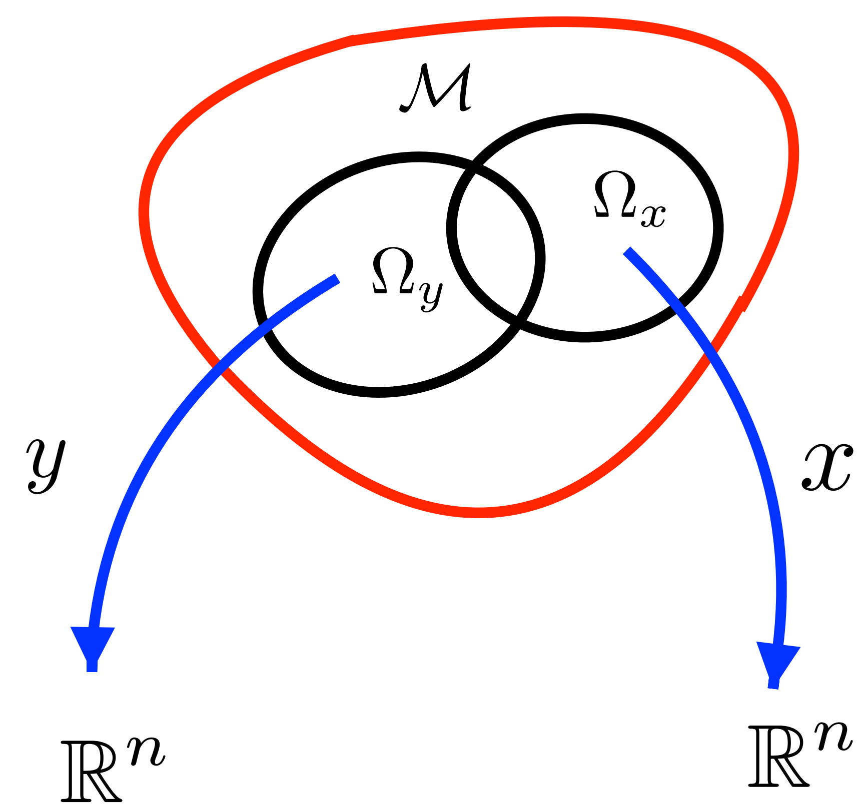

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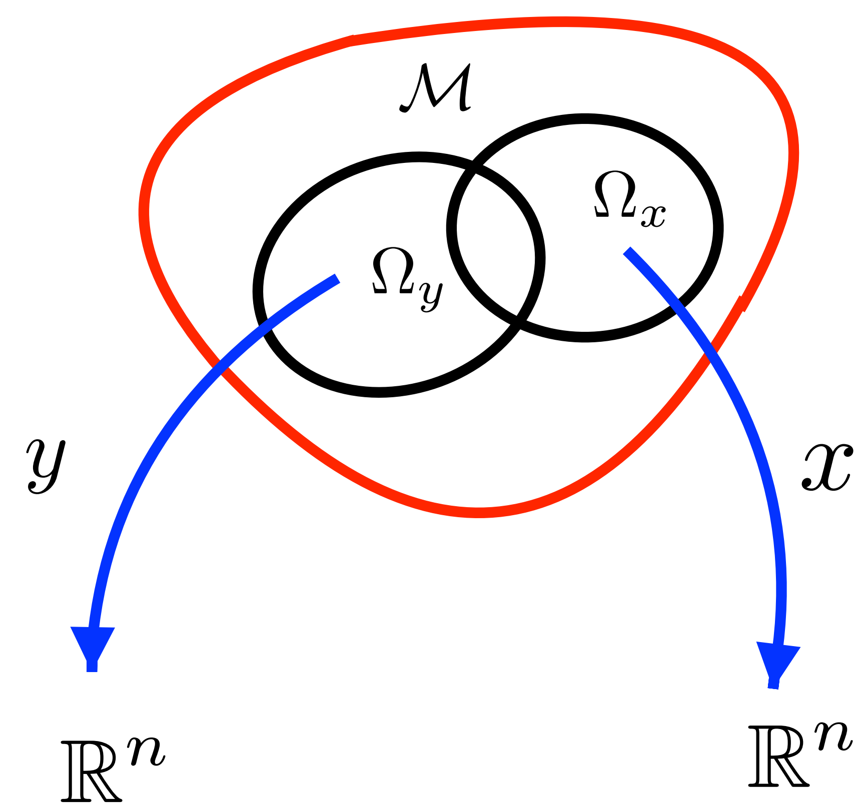

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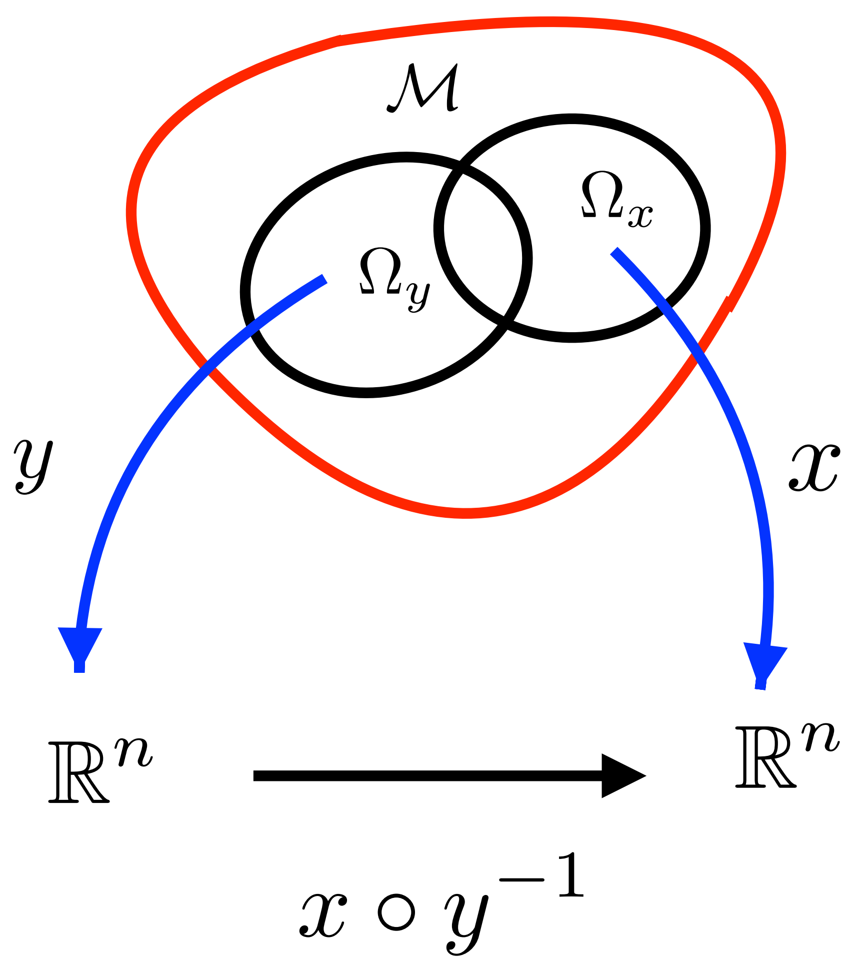

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Moreover, any such maximal atlas $\mathcal{A}^{\max}(s,p)$ contains subatlases of arbitrary higher regularity, including C^∞ subatlases, and given any chart in $\mathcal{A}$ there exist a C^∞ atlas which contains this chart.

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Similarly: regularity of the curvature:

$Riem(\Gamma_x) \in W^{s,p}(\Omega_x)$ if components have regularity $W^{s,p}$ in chart (x, Ω).

$Riem(\Gamma) \in W^{s,p}_{\mathcal{A}}$ if the Riemann curvature tensor has regularity $W^{s,p}$ in each chart of $\mathcal{A}$.

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The largest relevant extension is thus… $\mathcal{A}^{\max}(2,p)$

This leads to the following definition of the essential (highest possible) regularity of a connection…

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Definition: Γ defined on $(\mathcal{M}, \mathcal{A})$ has global essential regularity $m = ess_{\mathcal{M}}(\Gamma) \geq 0$ if there exists a sub-atlas $\mathcal{A}_m$ of $A^{\max}(2,p)$ of $\mathcal{A}$ such that $\Gamma \in W^{m,p}_{\mathcal{A}_m}$, and there does not exist a subatlas $\mathcal{A}_s$ of $A^{\max}(2,p)$ in which Γ, is more regular.

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Theorem (RT-2024): This provides a natural consistent geometric definition of the essential regularity of a connection.

Theorem (RT-2024): Assume $\Gamma \in W^{s,p}_{\mathcal{A}_s}$ is given on $(\mathcal{M}, \mathcal{A}_s)$, $p > n$, $s \geq 0$. Then:

(i) $ess_{\mathcal{M}}(\Gamma) = s$ if and only if $Riem(\Gamma) \in W^{s-1,p}_{\mathcal{A}}$ and $Riem(\Gamma) \notin W^{s',p}_{\mathcal{A}}$, any $s' \geq s$;

(ii) $ess_{\mathcal{M}}(\Gamma) = s + 1$ if and only if $Riem(\Gamma) \in W^{s,p}_{\mathcal{A}}$ and $Riem(\Gamma) \notin W^{s',p}_{\mathcal{A}}$, any $s' > s$;

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Moreover: RT-equations provide iterative method for lifting any connection below essential regularity...

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Note: Curvature is more regular than connection whenever you are two derivatives below essential.

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(iii) $ess_{\mathcal{M}}(\Gamma) \geq s+2$ if and only if $Riem(\Gamma) \in W^{s+1,p}_{\mathcal{A}}$.

(This explains why the Schwarzshild curvature is one order more regular than the connection...)

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…in a neighborhood of every point.

The union of all these local neighborhoods produces a new atlas $\mathcal{A}_{s+1}$ in which

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$Riem(\Gamma_y) \in W^{s,p}$

$J, J^{-1} \in W^{s+1,p}$

$Riem(\Gamma_x) \in W^{s,p}$

We didn't realize…

… J can lift the regularity of curvature as well!

To see this…

Apply a low regularity transformation to a connection starting at its essential regularity…

$$\Gamma \in W^{m,p}_{\mathcal{A}_m} \qquad Riem(\Gamma) \in W^{m-1,p}_{\mathcal{A}_m}$$

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$$J \in W^{s+1,p} \quad s < m$$

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➡ $\Gamma_y \in W^{s,p}_{\mathcal{A}} \qquad \mathcal{A} \in W^{s+2,p}$

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$$\Rightarrow \quad \Gamma_y \in W^{s,p}_{\mathcal{A}} \qquad \mathcal{A} \in W^{s+2,p} \qquad \underbrace{R^{\alpha}_{\beta\gamma\delta}} = \underbrace{R^{i}_{jkl}} \underbrace{\frac{\partial y^{\alpha}}{\partial x^{i}}} \underbrace{\frac{\partial x^{j}}{\partial y^{\beta}}} \underbrace{\frac{\partial x^{k}}{\partial y^{\gamma}}} \underbrace{\frac{\partial x^{l}}{\partial y^{\gamma}}}$$

$J, J^{-1} \in W^{s+1,p}$

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Assume essential regularity:

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$J, J^{-1} \in W^{s+1,p}$

$Riem(\Gamma_x) \in W^{m-1,p}$

$s = m-1 \qquad Riem(\Gamma_y) \in W^{s,p}$

$s < m-1 \qquad Riem(\Gamma_y) \in W^{s+1,p}$

To make this a proof, we need to know there do not exist Jacobians of regularity below $W^{s+1,p}$ which transform $\Gamma_y \to \Gamma_x \in W^{s,p}$

...i.e., by some weird cancellation of terms...

Lemma: The Jacobian J transforms $\Gamma_y \in W^{s,p}$ to $\Gamma_x \in W^{r,p}$, $0 \leq r \leq s$ if and only if the components of J satisfy $J \in W^{r+1,p}$.

"Proof": You can solve for the Jacobian in the connection transformation law...

$$(\Gamma_x)^\mu_{\rho\nu} = (J^{-1})^\mu_\alpha \Big(J^\beta_\rho J^\gamma_\nu \, (\Gamma_y)^\alpha_{\beta\gamma} + \tfrac{\partial}{\partial x^\rho} J^\alpha_\nu \Big)$$

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$$\frac{\partial}{\partial x^\rho} J^\alpha_\nu = J^\alpha_\mu (\Gamma_x)^\mu_{\rho\nu} - J^\beta_\rho J^\gamma_\nu (\Gamma_y)^\alpha_{\beta\gamma}$$

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↑

The regularity of Γ_x

determines regularity of $\frac{\partial}{\partial x^\rho} J^\alpha_\nu \in W^{s,p}$

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↑

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Summary: Everything works because the regularity of a connection controls the regularity of both the atlas… and the Jacobians which transform it…

Conclude:

- Essential regularity is characterized by the condition that the curvature is precisely one derivative below the regularity of its connection.

- Connection one derivative below essential regularity iff curvature is at precisely the same regularity as the connection.

- Connection two or more derivatives below essential regularity iff the curvature is precisely one derivative more regular than the connection.

- Theorem (RT-2024): Every connection can be lifted to essential regularity $m < \infty$ by a sequence of solutions of the RT-equations.

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Now we know a manifold together with any connection, always does…for $p > n$.

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Hirsch, Theorem 2.9: A C^1-manifold alone does not have enough structure to determine a geometric level of regularity.

Now we know a manifold together with any connection, always does…for $p > n$.

This would apply to Black Hole singularities if we extend our existence theory for the RT-equations to $p < n$.

The Essential Regularity of a Connection

(1) Given a metric or connection, is there a necessary and sufficient condition for determining, apriori, whether a singularity is essential or removable? YES

(2) If so: Is there a canonical procedure for constructing coordinate transformations which remove it? YES

(3) Does every metric/connection have an essential (highest possible) regularity to which it can be lifted? YES

End

Thank you!!

Derivation of the RT-equations

M. Reintjes and B. Temple, *On the optimal regularity implied by the assumptions of geometry I: Connections on tangent bundles*, Meth. Appl. Anal., Vol. 29, No. 4, 303-396, (2023)

M. Reintjes and B. Temple, *The regularity transformation equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity*, Adv. Theor. Math. Phys 24.5, (2020), 1203-1245.

The RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A),\\
\Delta J &= \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big),\\
\delta\vec{A} &= v
\end{aligned}
$$

$$d\vec{J} = 0 \;\text{ on } \partial\Omega.$$

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\end{aligned}
$$

free to be chosen

Here: $\tilde{\Gamma}$ is a matrix valued 1-form, J and A are matrix valued 0-forms, and $\vec{J}$, $\vec{A}$ are vector valued 1-forms as follows:

$$\tilde{\Gamma} \equiv \tilde{\Gamma}^{\mu}_{\nu i} dx^i$$

$$J \equiv J^{\mu}_{\nu} \qquad \vec{J} \equiv J^{\mu}_{i} dx^i \qquad d\vec{J} = Curl(J)$$

$$A \equiv A^{\mu}_{\nu} \qquad \vec{A} \equiv A^{\mu}_{i} dx^i \qquad d\vec{A} = Curl(A)$$

The integrability condition for J is: $Curl(J) = 0$

Two operations on matrix valued forms:

$$\overrightarrow{\mathrm{div}}(\omega)^{\alpha} \equiv \sum_{l=1}^{n} \partial_l \Big((\omega_l^{\alpha})_{i_1,,,i_k} \Big) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

("take divergence in lower matrix component")

$$\langle A \, ; B \rangle_{\nu}^{\mu} \equiv \sum_{i_1 < \ldots < i_k} A^{\mu}_{\sigma \; i_1 \ldots i_k} B^{\sigma}_{\nu \; i_1 \ldots i_k}$$

("matrix valued inner product")

To derive the RT-equations…

The first breakthrough was the Riemann-flat condition…

The Riemann-flat Condition

M. Reintjes and B. Temple, *Shock Wave Interactions and the Riemann-flat Condition: The Geometry behind Metric Smoothing and the Existence of Locally Inertial Frames in General Relativity*, Arch. Rat. Mech. Anal. 235 (2020), 1873-1904.

The Riemann-flat condition:

Assume $\Gamma, R \in L^{\infty}$.

Then: There exists a $C^{1,1}$ coordinate transformation which smooths Γ to $C^{0,1}$ if and only if there exists a tensor $\tilde{\Gamma} \in C^{0,1}$ st

$$Riem(\Gamma + \tilde{\Gamma}) = 0.$$

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$\Gamma \in L^{\infty}$

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$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

$\Gamma \in W^{m,p}$

$\Gamma \in W^{m+1,p}$

Geometric & independent of metric signature…

“Proof”: Assume $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

and

$$\Gamma^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{i}_{jk} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial y^{\beta}} \frac{\partial x^{k}}{\partial y^{\gamma}} + \frac{\partial^2 y^{\alpha}}{\partial x^{\sigma} \partial x^{\tau}} \frac{\partial x^{\sigma}}{\partial y^{\beta}} \frac{\partial x^{\tau}}{\partial y^{\gamma}}$$

L^{∞}

$C^{0,1}$ $C^{0,1}$

$-\tilde{\Gamma}^{\alpha}_{\beta\gamma}$

Riemann-flat

so

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

The “hard” part is: If

$$Riem(\Gamma + \tilde{\Gamma}) = 0$$

…then a smoothing transformation exists. ■

$\tilde{\Gamma}$ continuous implies $\Gamma + \tilde{\Gamma}$ has the same jump discontinuities (shock set) as Γ

First idea: Find Nash-type embedding theorem to extend the shock set to a flat connection

Better Idea: Use the Riemann-flat condition to derive a system of elliptic equations in $\tilde{\Gamma}, J$

"The Regularity Transformation Equations: An elliptic mechanism for smoothing gravitational metrics in General Relativity"

Moritz Reintjes, Blake Temple

Arch. Rat. Mech. Anal. 235 (2020), 1873-1904

Derivation of the RT-equations from Riemann-Flat Condition

Start with the Riemann-flat condition:

Assume: $g \in C^{0,1},\ \Gamma \in L^{\infty},\ J \in C^{0,1}$

Then

$$\underbrace{\Gamma^k_{ij}}_{\Gamma} = \underbrace{(J^{-1})^k_\alpha J^\beta_i J^\gamma_j \Gamma^\alpha_{\beta\gamma}}_{\tilde{\Gamma}} + \underbrace{(J^{-1})^k_\alpha \partial_j J^\alpha_i}_{J^{-1}dJ}$$

Riemann-flat

$$\Gamma - \tilde{\Gamma} = J^{-1}dJ$$

$$Riem(\Gamma - \tilde{\Gamma}) = 0$$

2-Equivalent forms of Riemann-flat condition

The Riemann-flat condition:

$Riem(\Gamma - \tilde{\Gamma}) = 0$ implies

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Augment to first order system…

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma})$$

$$\delta\tilde{\Gamma} = h$$

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δ = co-derivative of Euclidean coord metric

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Augment to first order system…

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"gauge freedom"

Yields 1st order Cauchy-Riemann system for $\tilde{\Gamma}$

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$$\delta\tilde{\Gamma} = h$$

This is not a solvable system!

J

Yields 1st order Cauchy-Riemann system for $\tilde{\Gamma}$

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$$J^{-1} dJ = \Gamma - \tilde{\Gamma} \quad \Longleftrightarrow \quad dJ = J(\Gamma - \tilde{\Gamma})$$

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We now construct a closed system in $(\tilde{\Gamma}, J)$

from these 2 forms of Riemann-flat condition

(They start out as equivalent!)

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to get Poisson equations…

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta d\big(J^{-1} dJ\big) + d(J^{-1}A),$$

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where $h = J^{-1}A$ is free…

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$$d\vec{A} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J\, d\Gamma) - d(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle})$$

This leads to the RT-equations:

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\
\Delta J &= \delta(J \cdot \Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \\
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Another "miraculous cancellation" occurs in in A-equation

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Consider the **A-equation:**

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\Delta J &= \delta(J\cdot\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A,\\
d\vec{A} &= \underbrace{\overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big)}_{d\left(\overrightarrow{\delta(J\,\Gamma)}\right)} - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big),
\end{aligned}
$$

Lemma: (for smooth Γ):

$$d\left(\overrightarrow{\delta(J\,\Gamma)}\right) = \underbrace{\overrightarrow{\mathrm{div}}\left(dJ \wedge \Gamma\right)} + \underbrace{\overrightarrow{\mathrm{div}}\left(J\, d\Gamma\right)}$$

$W^{m-2,p}$ $\qquad$ $W^{m-1,p}$

$d\Gamma$ one derivative smoother than $\delta\Gamma$!

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Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations iff $(\tilde{\Gamma}', J)$ does...

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

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Theorem: If $(\tilde{\Gamma}, J)$ solves the RT-equations iff $(\tilde{\Gamma}', J)$ does...

$$\tilde{\Gamma}' = \Gamma - J^{-1}dJ$$

$$\tilde{\Gamma}' \in W^{m+1,p} \qquad \Gamma \in W^{m,p} \qquad dJ \in W^{m,p}$$

$\tilde{\Gamma}'$ has the same regularity as $\tilde{\Gamma}$!

Theorem: $(\tilde{\Gamma}, J, A)$ is a solution of the RT-equations

$$
\begin{aligned}
\Delta\tilde{\Gamma} &= \delta d\Gamma - \delta d\big(J^{-1}dJ\big) + d(J^{-1}A), & (1)\\
\Delta J &= \delta(J\,\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, & (2)\\
d\vec{A} &= \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), & (3)\\
\delta\vec{A} &= v, & (4)
\end{aligned}
$$

$$
Curl(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0 \;\; \text{on } \partial\Omega,
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Summary: We start with two first order equations both equivalent to the Riemann-flat condition…

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Using the co-derivative of the coordinate Euclidean metric, we convert these in to a coupled system of Poisson equations…

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Using the co-derivative of the coordinate Euclidean metric, we convert these in to a coupled system of Poisson equations…

…and by miraculous cancellations on the RHS, the solutions provide Jacobians which lift the connection to optimal regularity…

End

Thank you!!