Periodic Solutions of the Compressible Euler Equations and the Nonlinear Theory of Sound

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CONCLUDE: 2x2 Shock formation first proved by Riemann in 1860, and made definitive in the Glimm-Lax decay result of 1970, is not indicative of what generically happens in 3×3 compressible Euler.

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CONCLUDE: 2x2 Shock formation first proved by Riemann in 1860, and made definitive in the Glimm-Lax decay result of 1970, is not indicative of what generically happens in 3×3 compressible Euler.

★ There really is a nonlinear theory of musical tones!

COMPRESSIBLE EULER EQUATIONS

The compressible Euler equations consist of three coupled nonlinear PDE's that can be interpreted as the continuum version of

Newton's Laws of Motion

(1) Conservation of Mass: (Continuity Equation)

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" The time-rate of change of momentum equals minus gradient of the pressure"

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…neglecting any form of dissipation, like viscosity and heat conduction…

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Most interestingly…

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…shock-waves introduce a canonical dissipation into this zero dissipation limit…

Compressible Euler Equations:
Eulerian Coordinates:

- For wave propagation in X-direction :

$$\text{Compressible Euler} \Longleftrightarrow \begin{cases} \rho_t + (\rho u)_X = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p)_X = 0 & \text{(Mo)} \\ E_t + \{(E+P)u\}_X = 0 & \text{(En)} \end{cases}$$

- System **(Ma)**, **(Mo)**, **(En)** describes the time evolution of a compressible fluid...

$\rho = \frac{mass}{vol}$=density

$u =$ velocity

p=pressure

$E = \frac{energy}{vol} = \rho e + \frac{1}{2}\rho u^2$

$e = \frac{energy}{mass}$ =specific internal energy

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Lagrangian Coordinates:

Change to material coordinate co-moving with the fluid

$$x = \int_0^X \rho(\xi)\, d\xi$$

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An equation of state relating ρ, p, e
is required to close the equations…

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A consequence is the "adiabatic constraint"

$\left(\text{Second Law of Thermodynamics}\right) +$ (Ma),(Mo),(En) ⟹ (Ent)

$$(\rho s)_t + (\rho s u)_X = 0 \iff s_t = 0 \qquad \text{(Ent)}$$

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"pressure and velocity are constant across contact discontinuities"

A LONG STANDING OPEN PROBLEM

The existence of space and time periodic solutions of compressible Euler has been an open problem since the time of Euler and Riemann.

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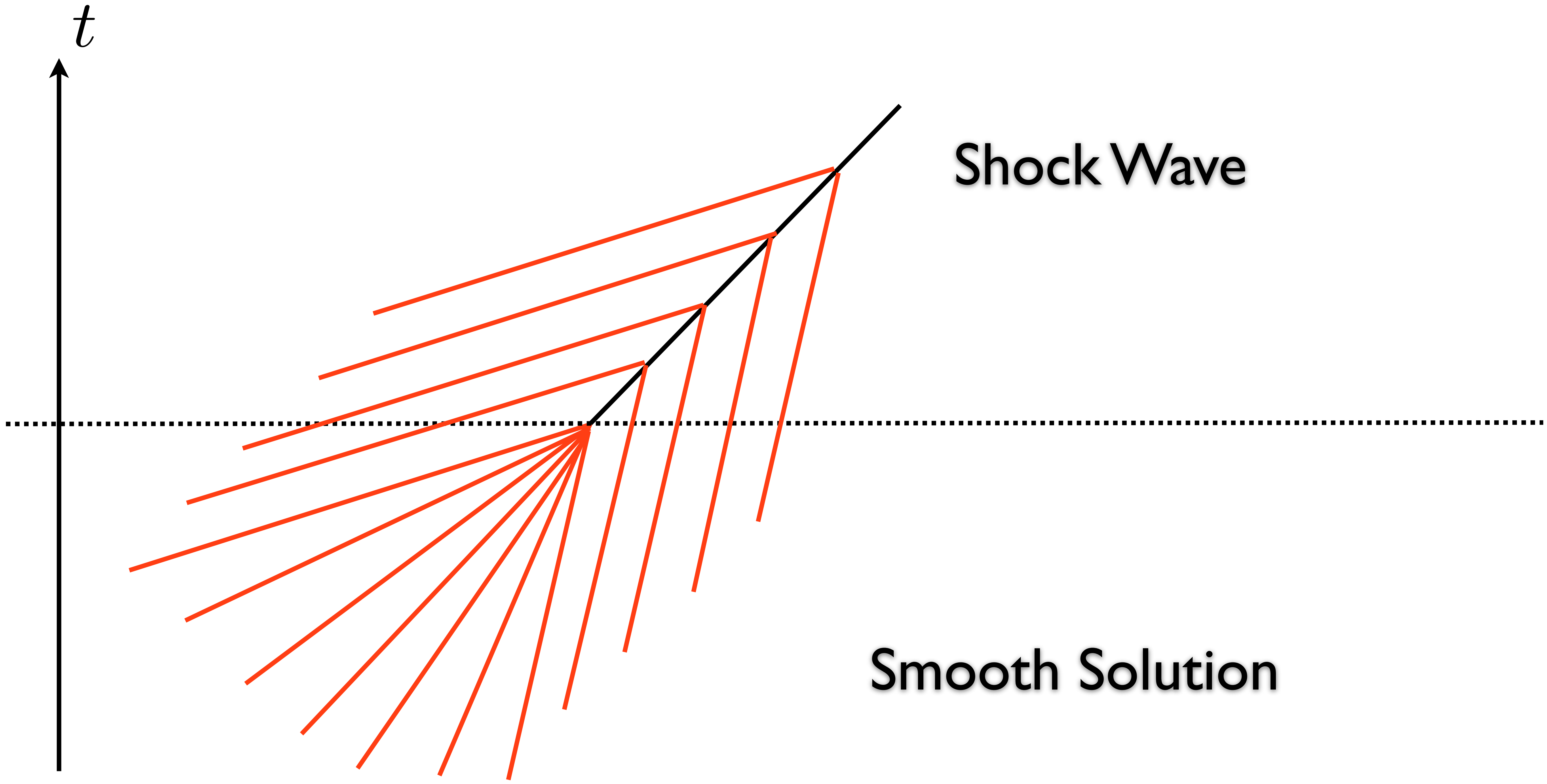
The existence of space and time periodic solutions of compressible Euler has been an open problem since the time of Euler and Riemann.

For most of the history of fluid mechanics it was believed that periodic solutions could not exist due to shock-formation.

SHOCK-WAVES

Shock-waves produce increase of entropy and dissipation…



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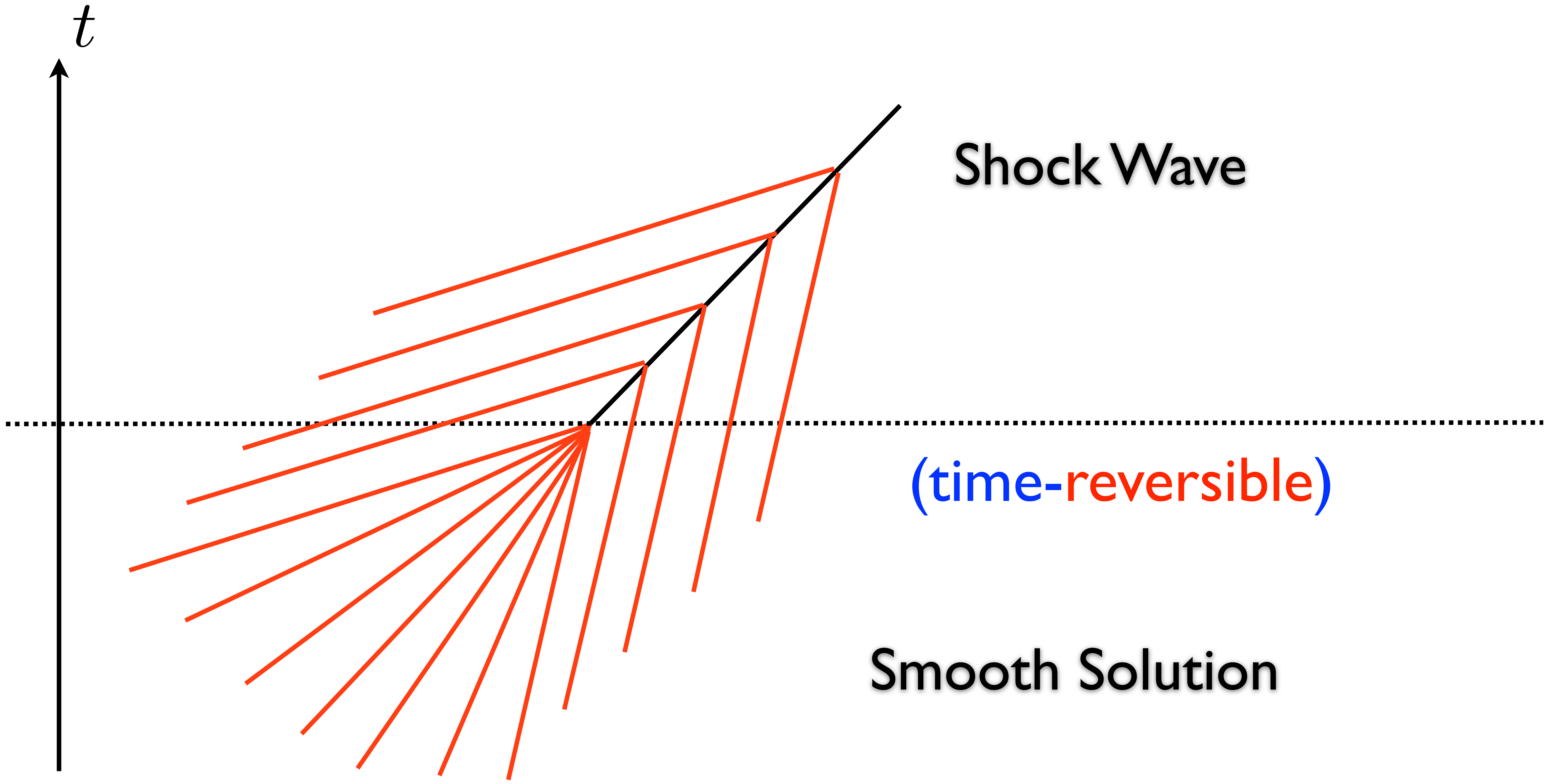

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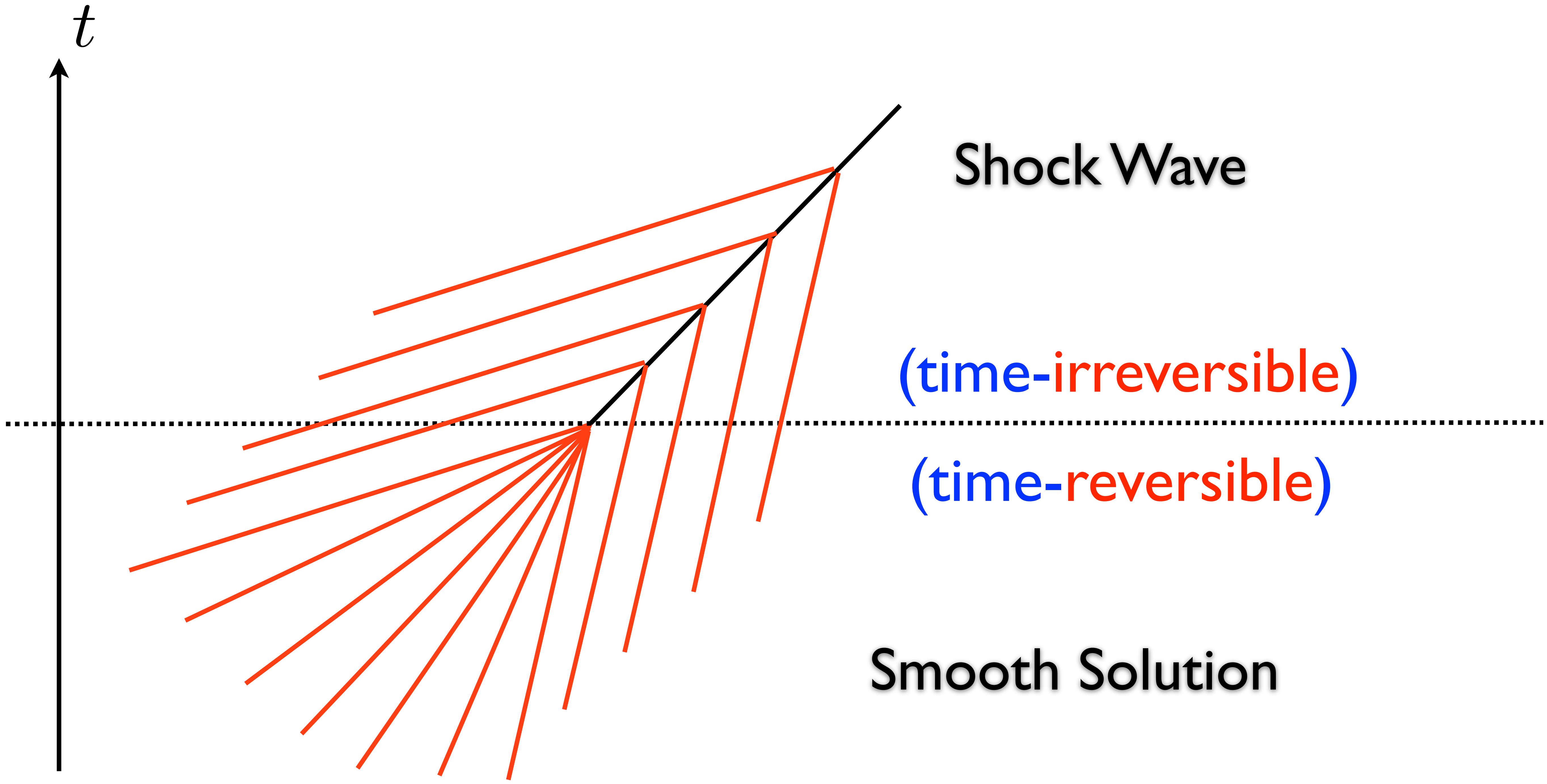

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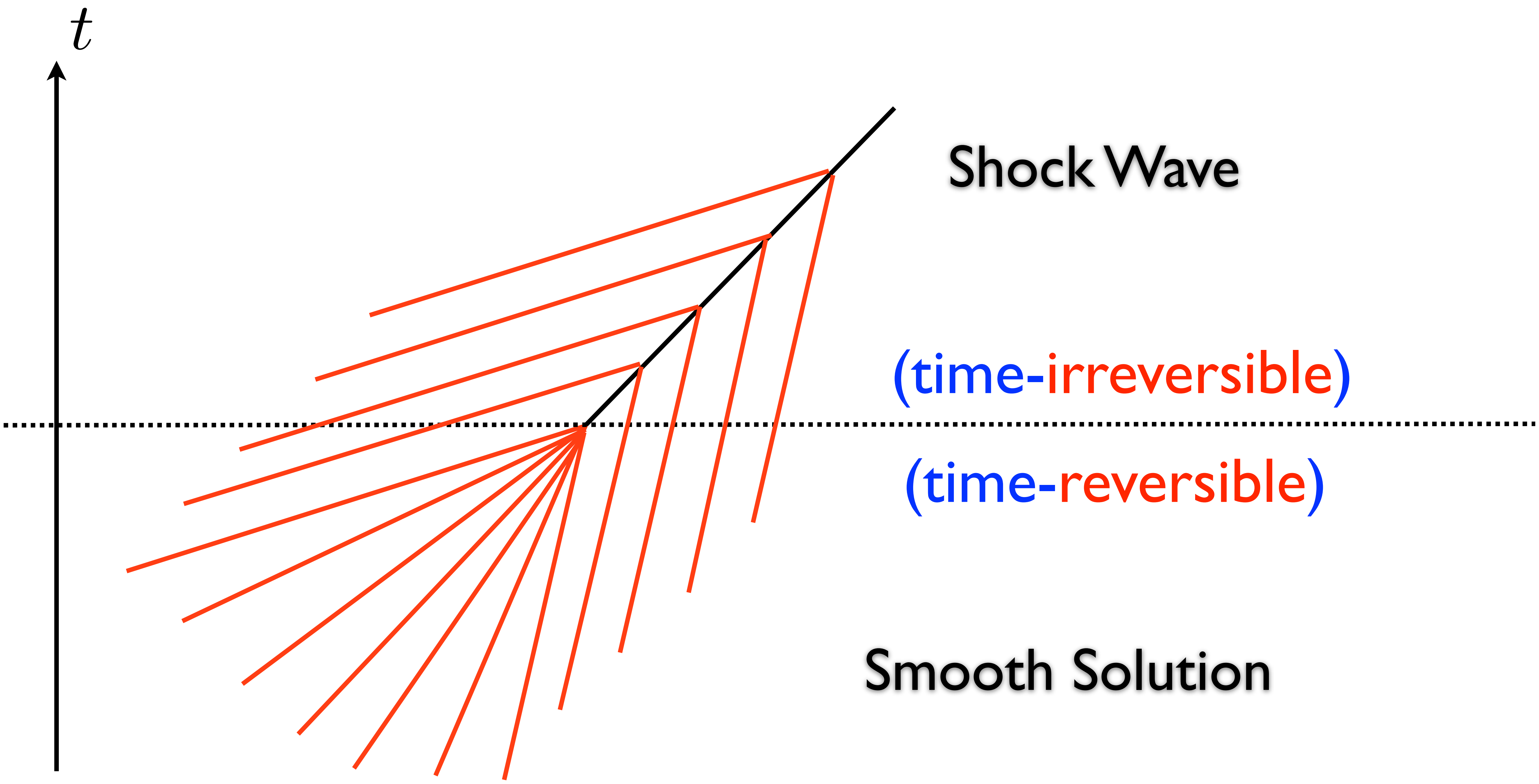

Shock-waves are inconsistent with space-time periodicity…

The Difficulty in a Nutshell

- The compressible Euler Equations form a system of 3-coupled nonlinear conservation laws of form---

$$u_t + f(u)_x = 0$$

- Basic warmup problem: scalar Burgers Equation:

$$\boxed{u_t + \frac{1}{2}(u^2)_x = 0}$$

$$u_t + uu_x = 0$$

$$\Downarrow$$

$$\nabla_{(1,u)}u(x,t) = 0$$

$$\Downarrow$$

“u=const. along lines of speed u”

$$\Downarrow$$

“inconsistent with time-periodic evolution”

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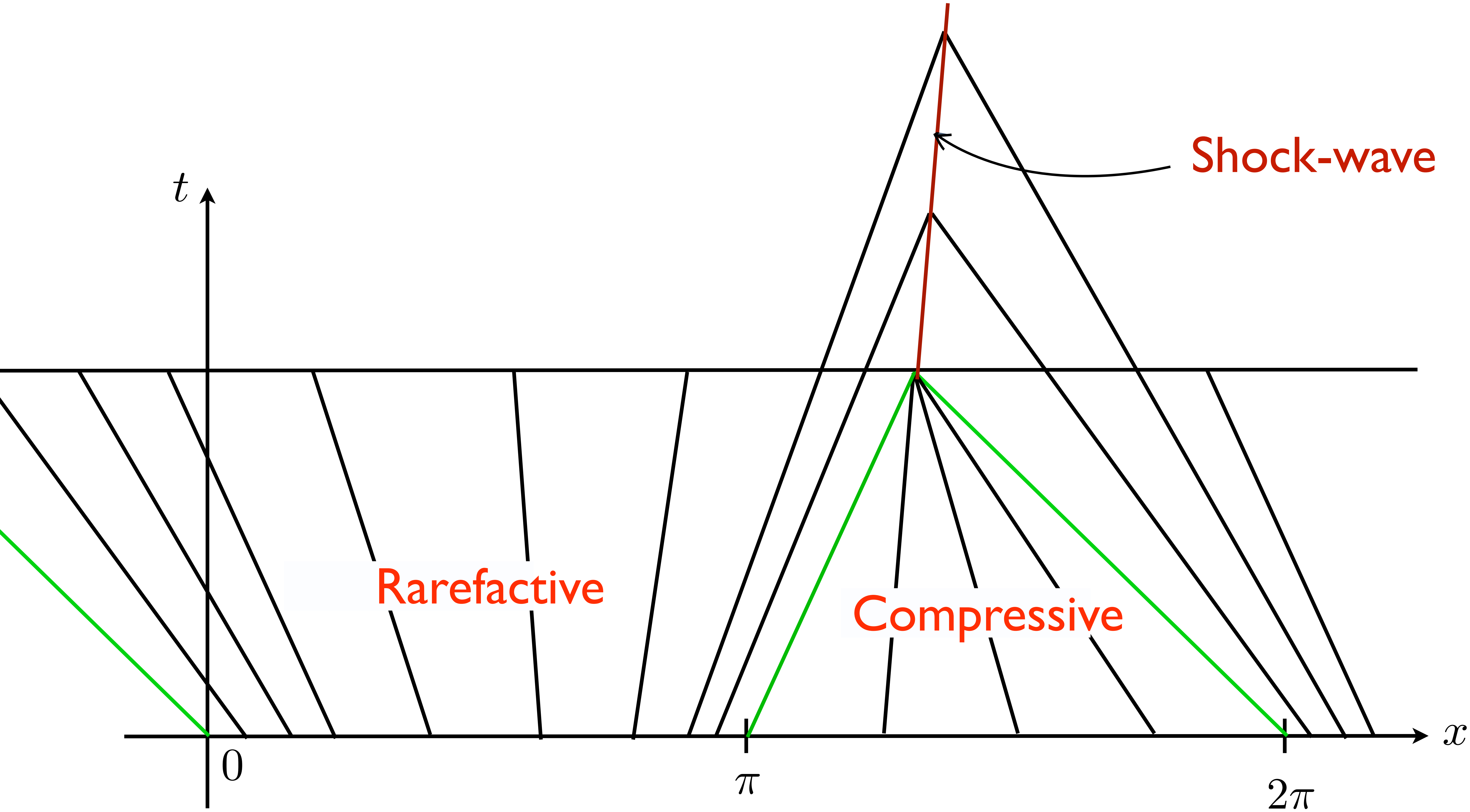

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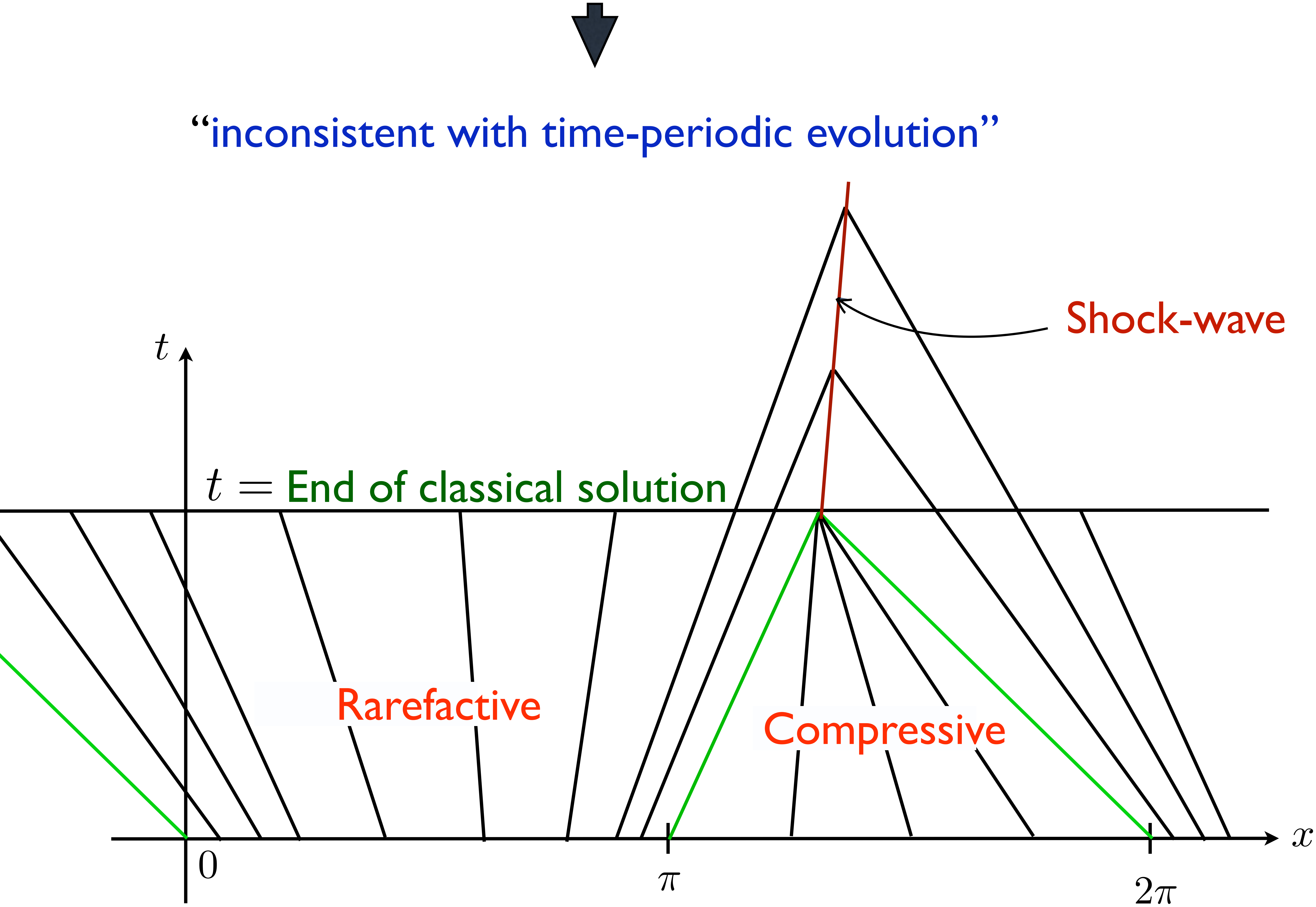

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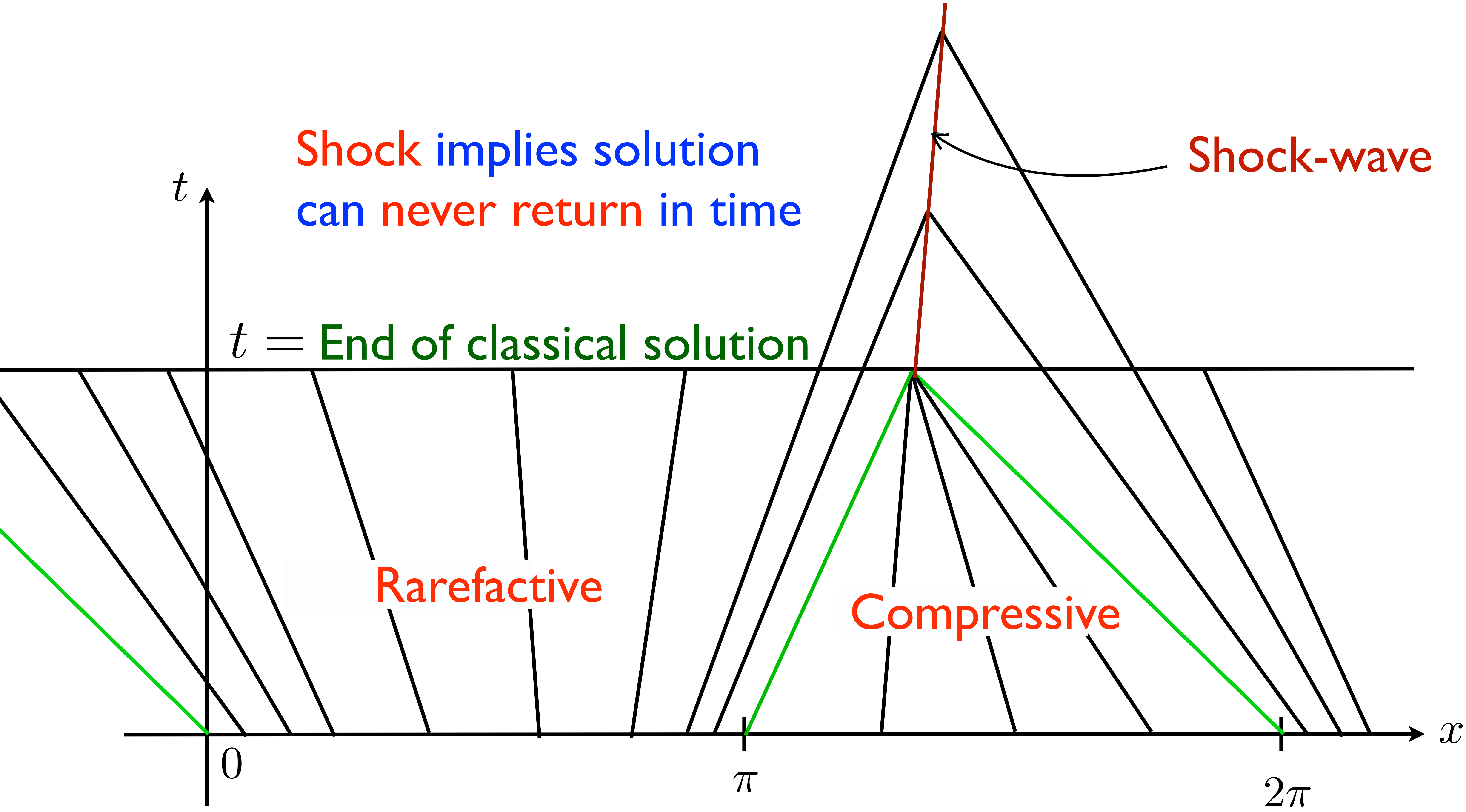

Shock implies solution can never return in time

$t =$ End of classical solution

This always happens in the 2x2 p-system obtained by closing the first two Euler equations with

$$\boxed{p = p(\rho)}$$

$$p\text{-system}\quad \begin{cases} \rho_t + (\rho u)_x = 0 & \text{(Ma)} \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0 & \text{(Mo)} \end{cases}$$

Theorem: (Riemann1860) Shock waves generically form in the isentropic and isothermal 2x2 systems...

Theorem: (Lax 1964) Genuinely nonlinear 2x2 systems like the p-system always form shock-waves when there is compression initially...

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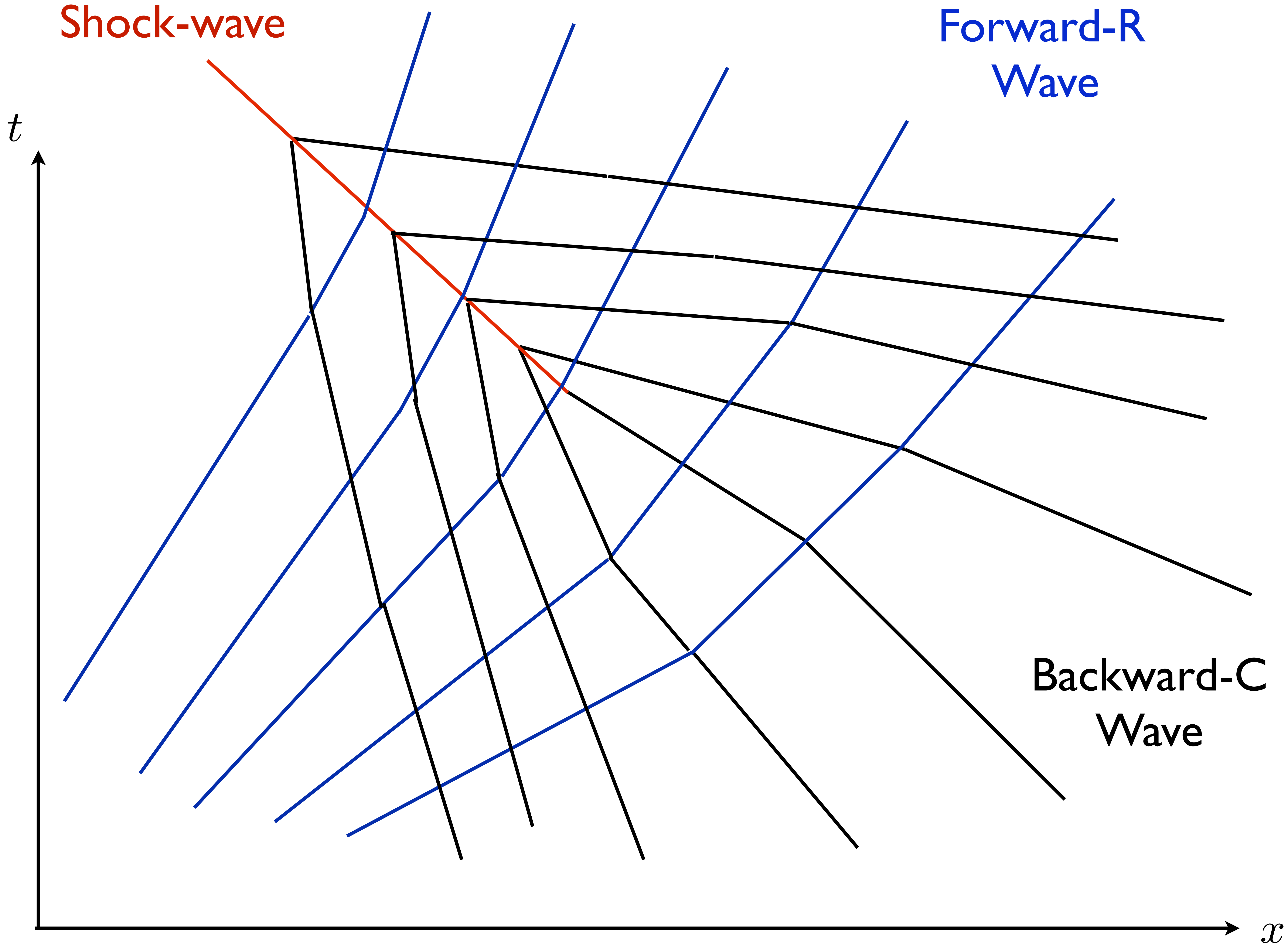

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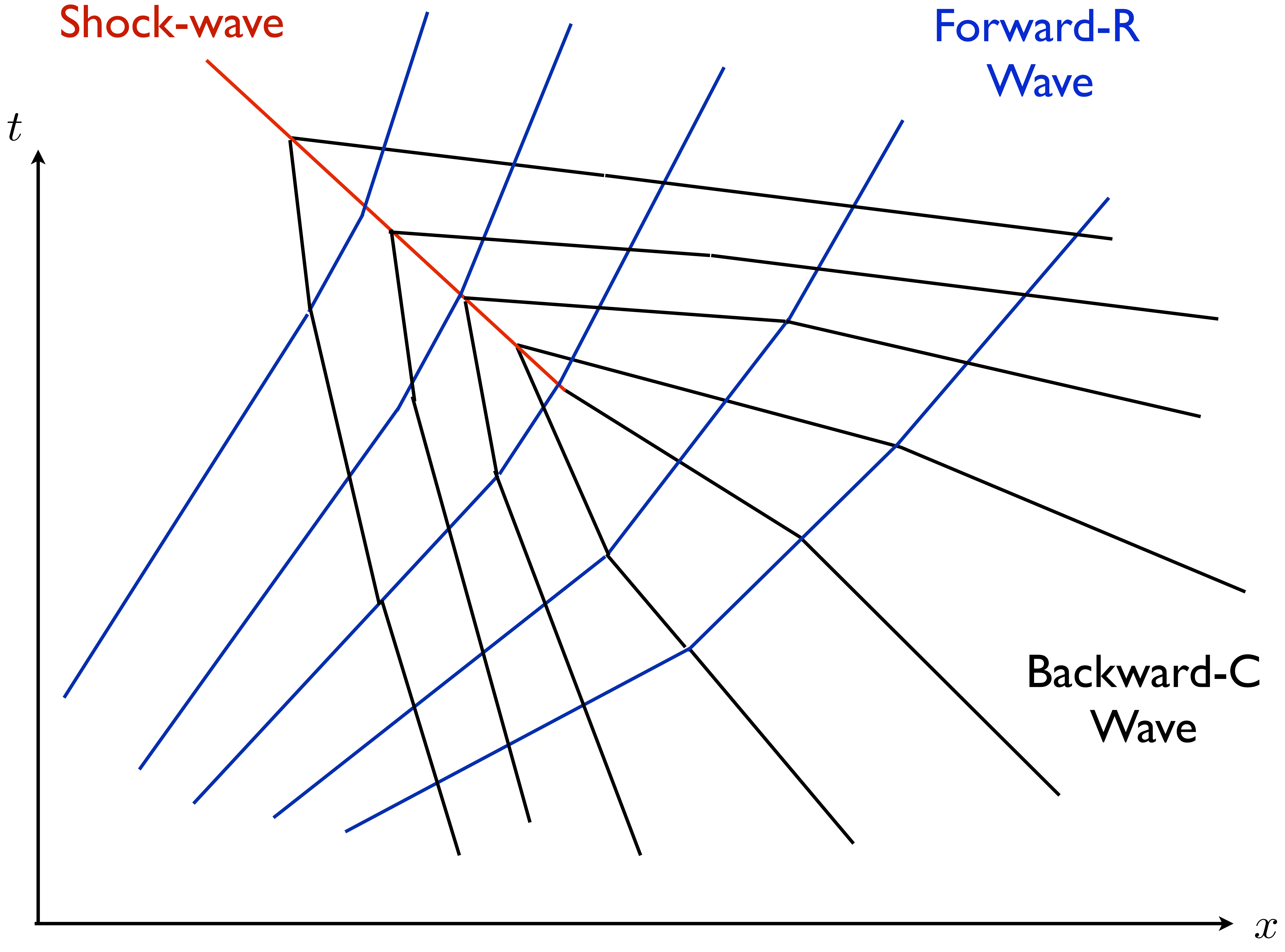

“Characteristics compresses into shocks like Burgers”

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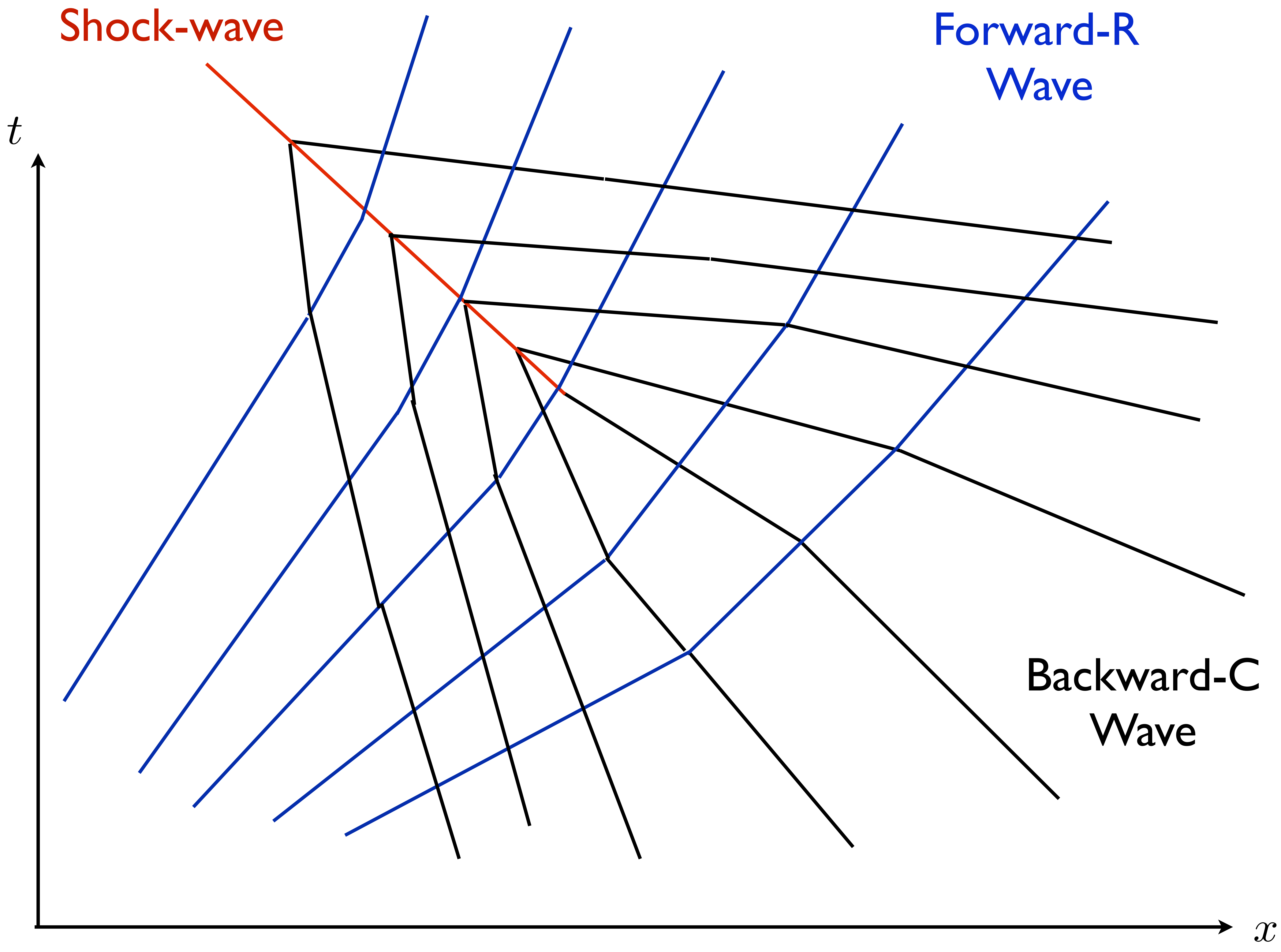

"Periodic solutions decay by shock dissipation at rate 1/t"

History/Background

Periodic Solutions of Compressible Euler

Ref: Tuesdell, Lindsay, Johnson and Cheret…

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Since Euler, it appeared that the linear theory of sound based on modes of vibration, was inconsistent with the nonlinear theory!

History/Background

- 1808-- Poisson developed the method of characteristics.
- 1848-- Challis pointed out that in some cases Poisson's solutions appeared to break down.
- 1848-- Stokes proposed discontinuous solutions to describe shock waves.
- 1860-- Earnshaw introduced simple waves.
- 1860-- Riemann proved that compressive solutions of Euler's equations "always" suffer gradient blowup.

"...portions of the wave where the density decreases in the direction of propagation, will accordingly become increasingly more narrow as it progresses, and finally goes over into compression shocks.

History/Background

After Riemann…

…shock-waves became the central issue in the study of the compressible Euler equations…

- Latter part 19'th century-- got thermodynamics and the roles of entropy and energy straight.
- 1880's-- Rankine-Hugoniot gave correct treatment of discontinuous solutions and entropy condition at shocks.
- 1957-- Lax formulated the general theory of conservation laws.

History/Background

- 1964-- Lax proved finite time blow-up in derivatives for 2x2 (isentropic) systems.

  P.D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, Jour. Math. Physics, Vol. 5, pp. 611-613 (1964).

- 1965-- Glimm's celebrated existence theory- represented smooth solutions by using weak-shocks.

  J.Glimm, *Solutions in the large for nonlinear hyperbolic systems of conservation laws*, Comm Pure Appl Math, Vol XVII, 697-715 (1965).

- 1970-- Glimm and Lax periodic solutions of 2x2 GR systems always form shock-waves and decay like 1/t.

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This is where the field was when I started studying conservation laws under Joel Smoller!

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Courant Institute 1980, experts thought Glimm-Lax extended to 3x3 Euler, and weak shocks explained musical tones…

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…experts suggested Glimm-Lax explained the attenuation of sonar signals observed by Navy.

History/Background

- 1974-97 Blow-up results extending Lax to 3x3 systems could not rule out the time- periodic solutions...

F. John, *Formation of singularities in one-dimensional wave propagation*, Comm. Pure Appl. Math., Vol. 27, pp. 377-405 (1974).

T.P. Liu, *Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations*, J. Diff. Eqns, Vol. 33, pp. 92-111 (1979).

Li Ta-Tsien, Zhou Yi and Kong De-Xing, *Global classical solutions for general quasilinear hyperbolic systems with decay initial data*, Nonlinear. Analysis., Theory., Methods. and Applications., Vol. 28, No. 8, pp. 1299-1332 (1997).

Blowup result by Chen/Young (2010).

History/Background

- 1984-88-- Time periodic solutions conjectured to exist based on nonlinear geometric optics…

A. Majda and R. Rosales, *Resonantly interacting weakly nonlinear hyperbolic waves I. A single variable*, Stud. in Appl. Math., 22, pp. 149-179 (1984).

A. Majda, R. Rosales and M.Schonbeck, *A canonical system of integrodifferential equations arising in resonant nonlinear acoustics*, Stud. in Appl. Math., 79, pp. 205-262 (1988).

R.L. Pego, *Some explicit resonating waves in weakly nonlinear gas dynamics*, Stud. in Appl. Math., 79, pp. 263-270 (1988).

(Scalar/Asymptotic Models)

History/Background

- 1996-99-- Rosales et al produced numerical simulations and conjectured the possibility of periodic, or quasi-periodic attractor solutions.

M. Shefter and R. Rosales, *Quasi-periodic solutions in weakly nonlinear gas dynamics*, Studies in Appl. Math., Vol. 103, pp. 279-337 (1999).

D. Vaynblat, *The strongly attracting character of large amplitude nonlinear resonant acoustic waves without shocks. A numerical study.* M.I.T. Dissertation, (1996).

MY LONG TERM PROGRAM with YOUNG

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- Step 1: Identify the mechanism by which time-periodic/shock-free solutions are possible.

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- Step 1: Identify the mechanism by which time-periodic/shock-free solutions are possible.
- Step 2: Find the simplest possible time periodic structure and realize solutions at the linearized level.

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We have now completed Step 3:

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We have now completed Step 3:

I.e. we have a rigorous existence theory!

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We completed Steps 1 and 2 over a decade ago, but to complete Step 3 we needed to construct an iteration scheme and prove convergence in the presence of resonances and small divisors.

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A Nash-Moser-Newton method to expunge resonant linearized operators looked like the only way to go!

But how to expunge?

And how to get uniformity in the small divisors?

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We completed Steps 1 and 2 over a decade ago, but to complete Step 3 we needed to construct an iteration scheme and prove convergence in the presence of resonances and small divisors.

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But how to expunge?

And how to get uniformity in the small divisors?

We formulated a consistent strategy in a scalar model…

INVERSION OF A NON-UNIFORM DIFFERENCE OPERATOR AND A STRATEGY FOR NASH-MOSER

B. TEMPLE AND R. YOUNG

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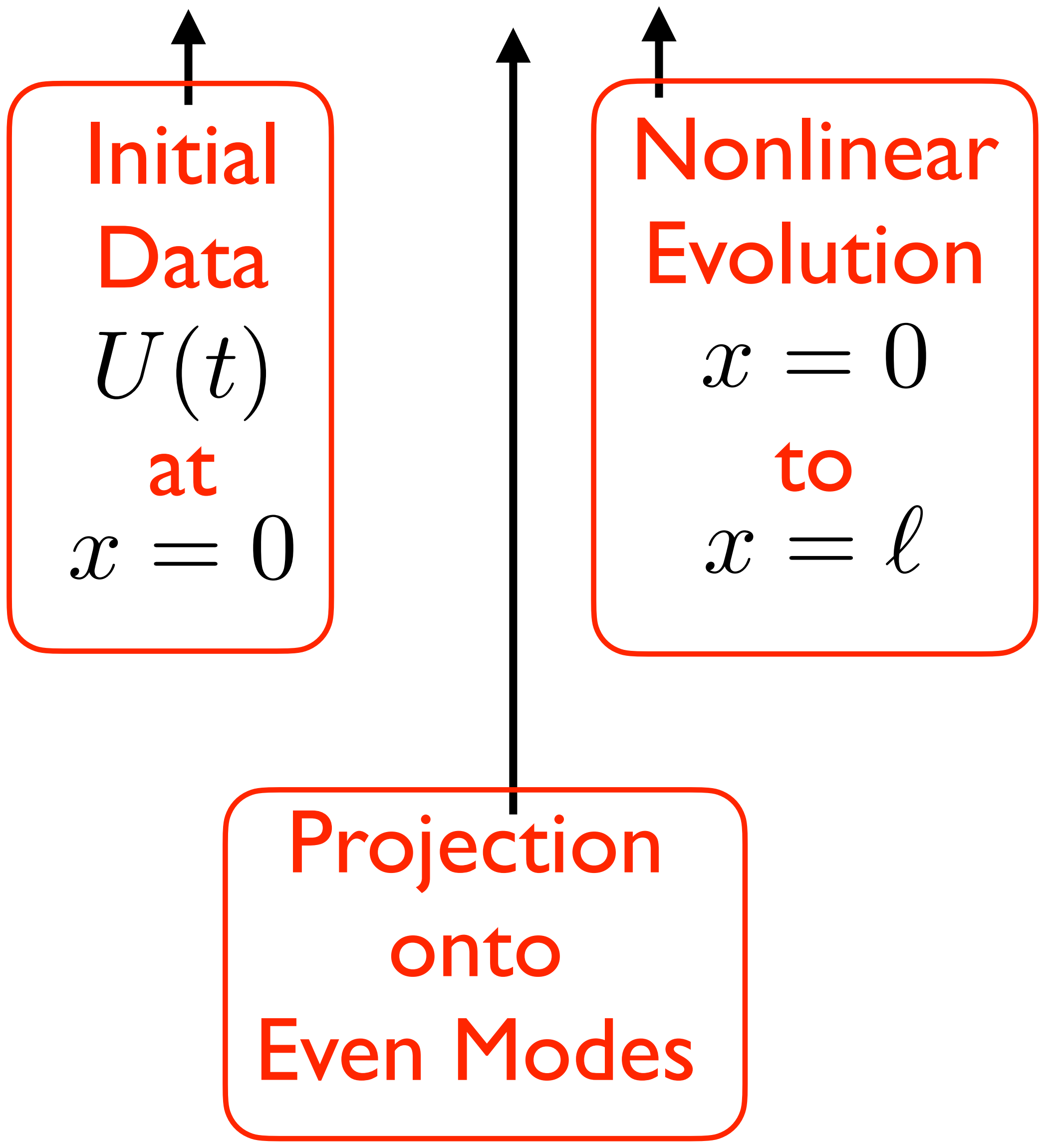

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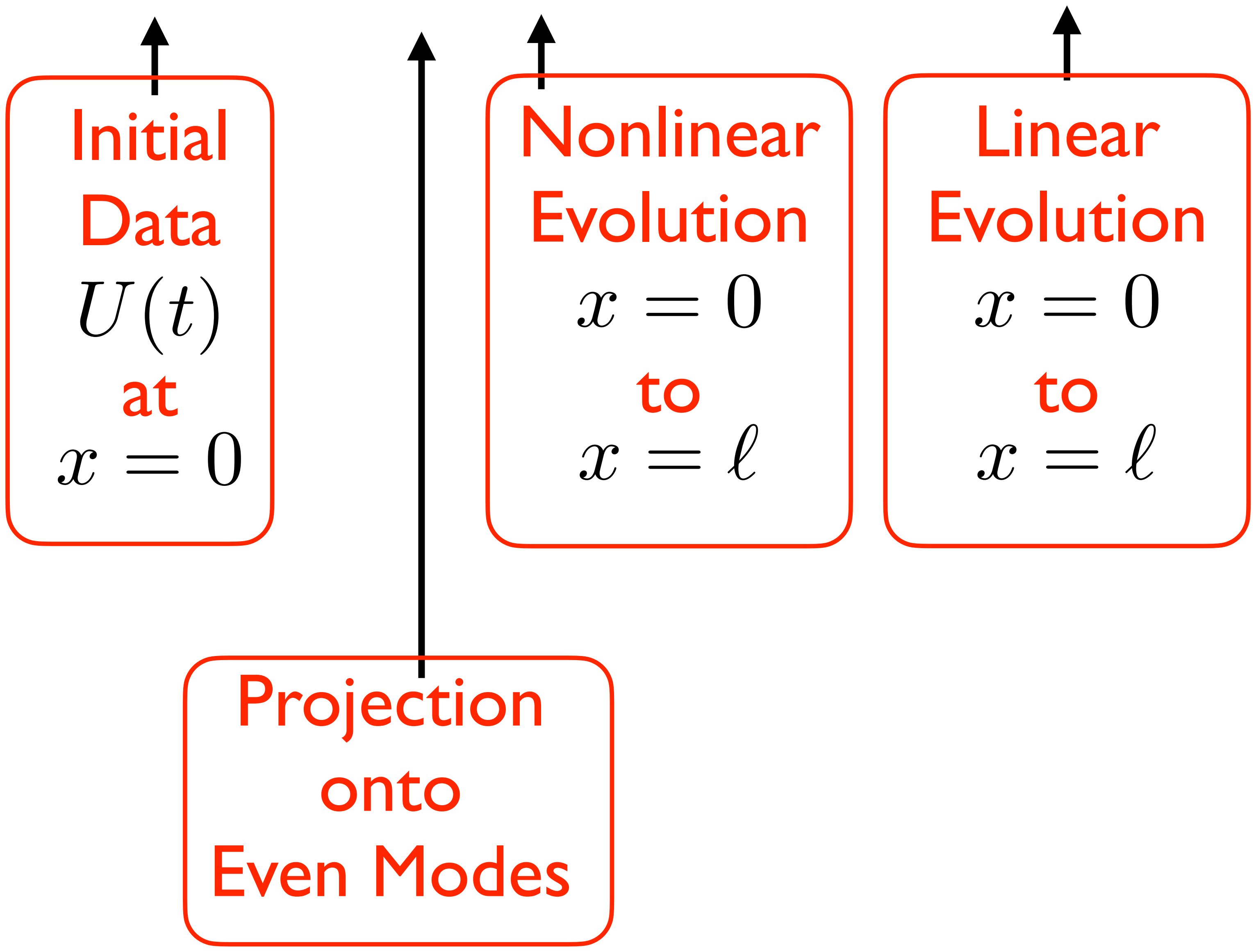

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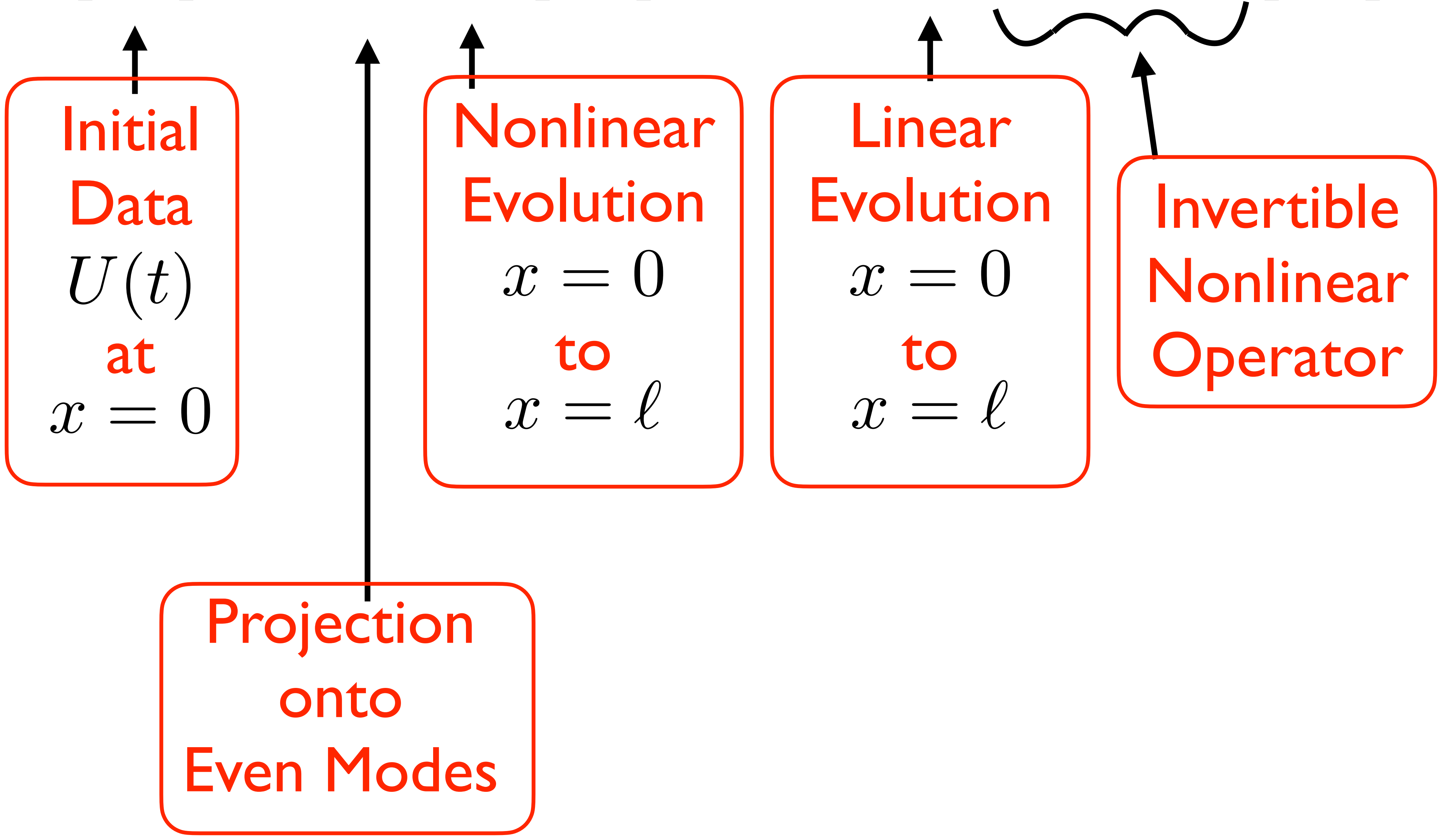

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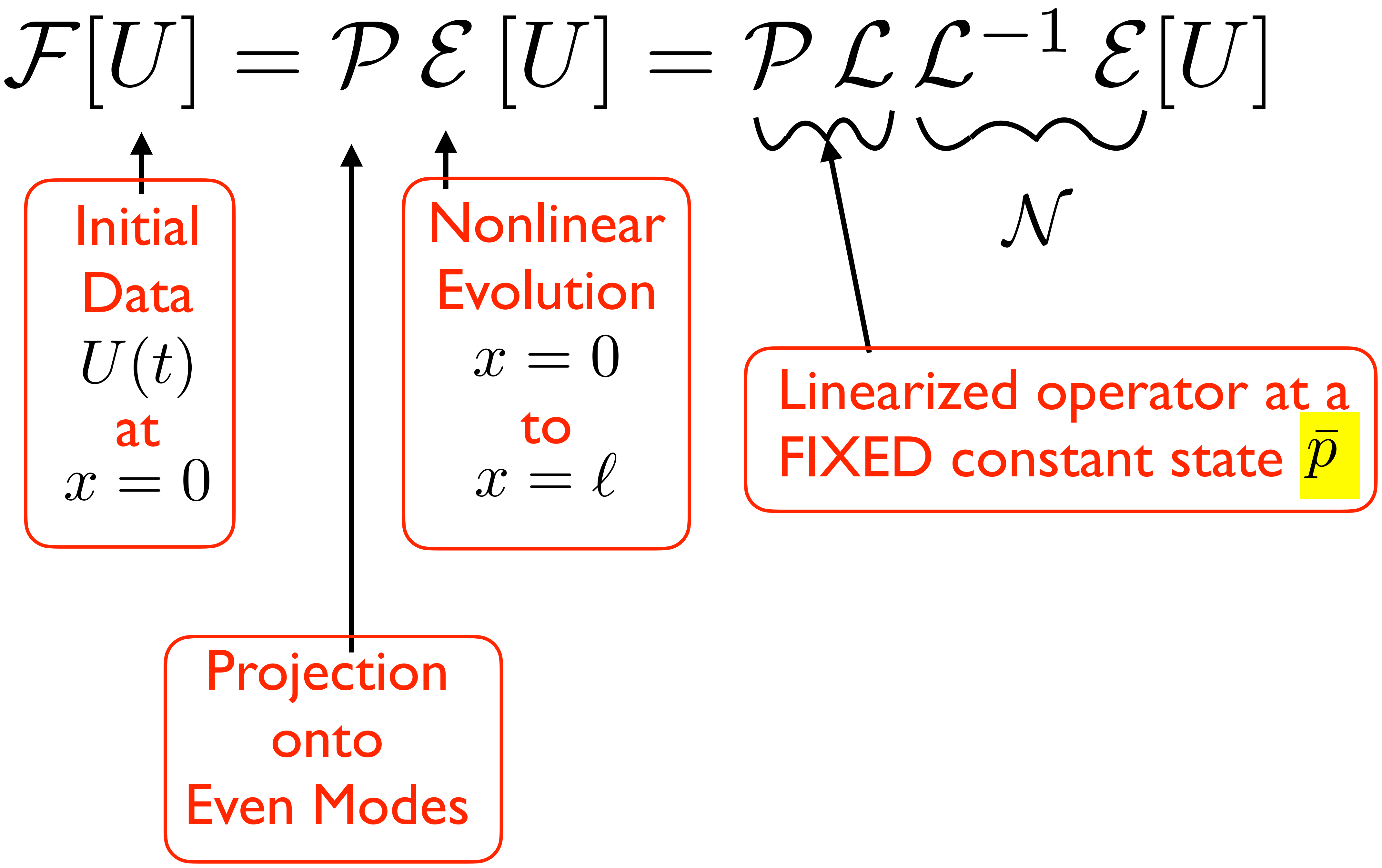

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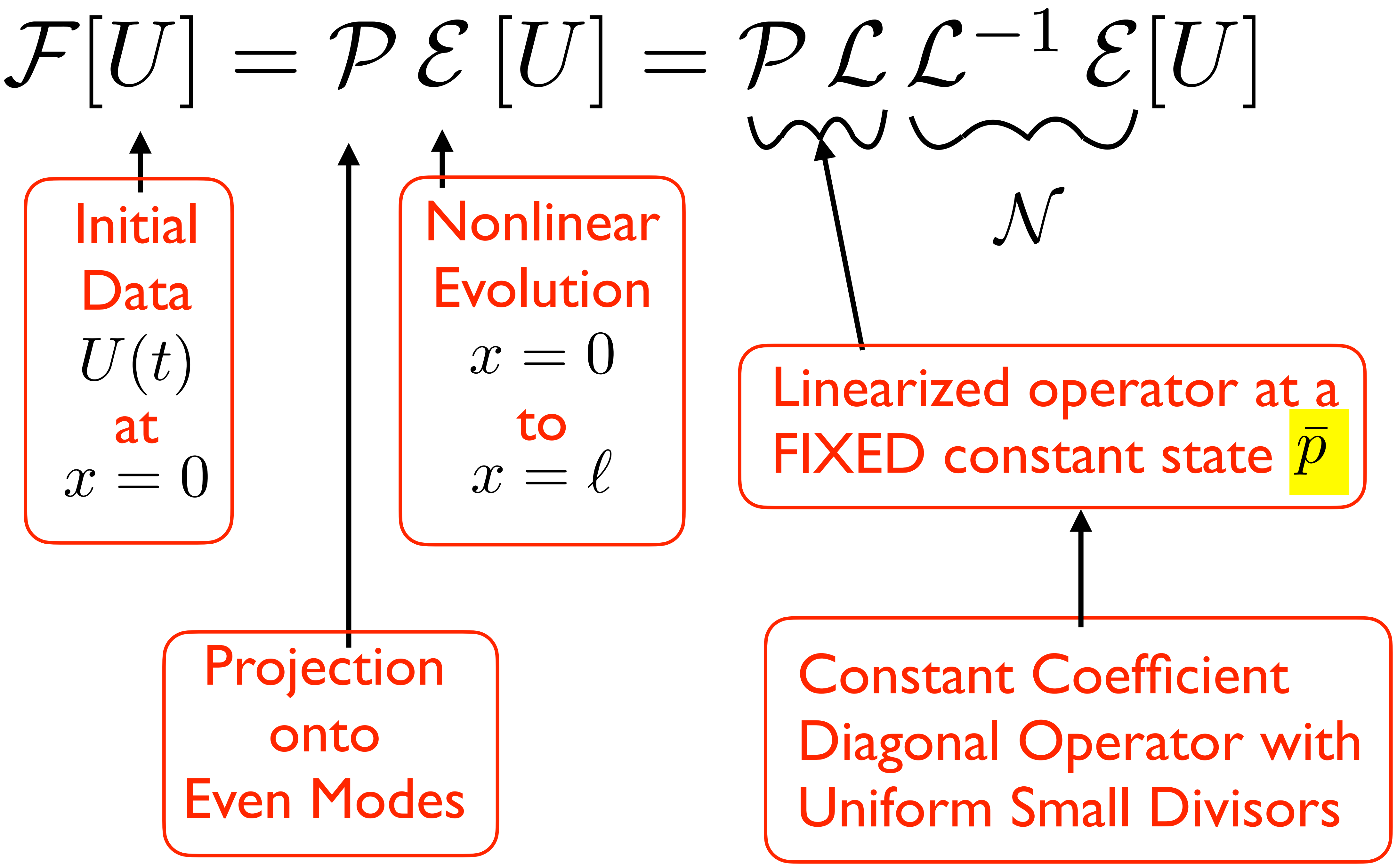

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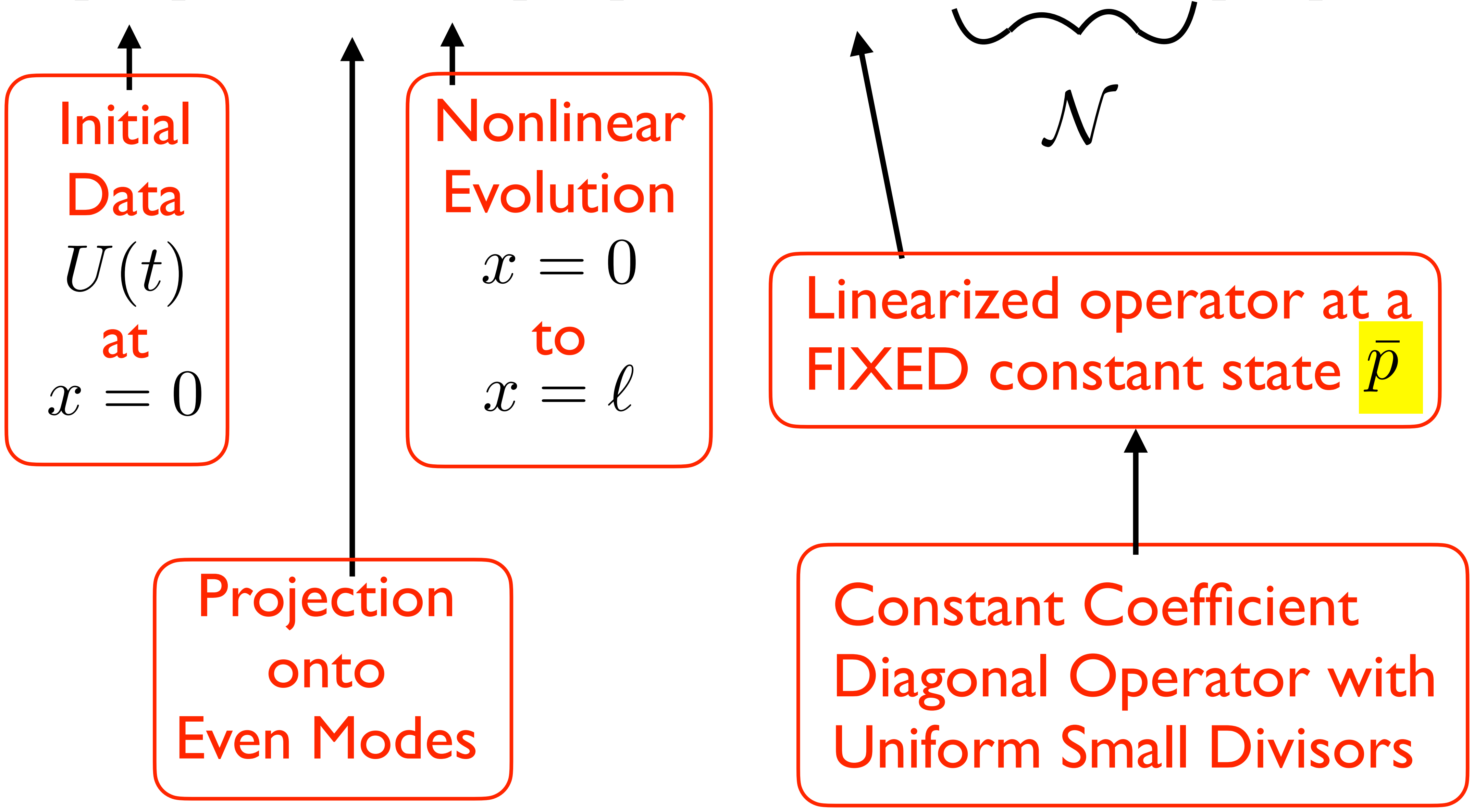

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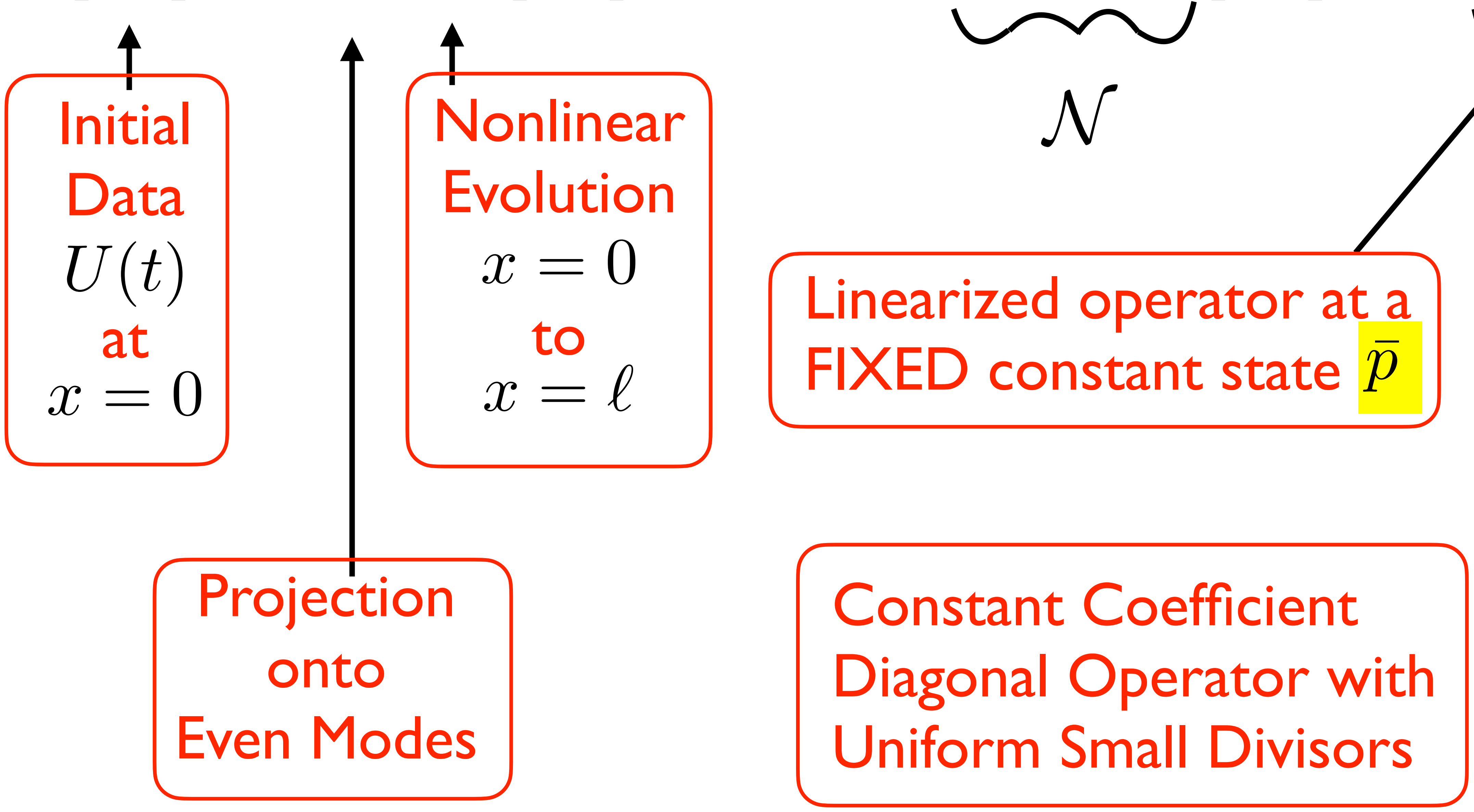

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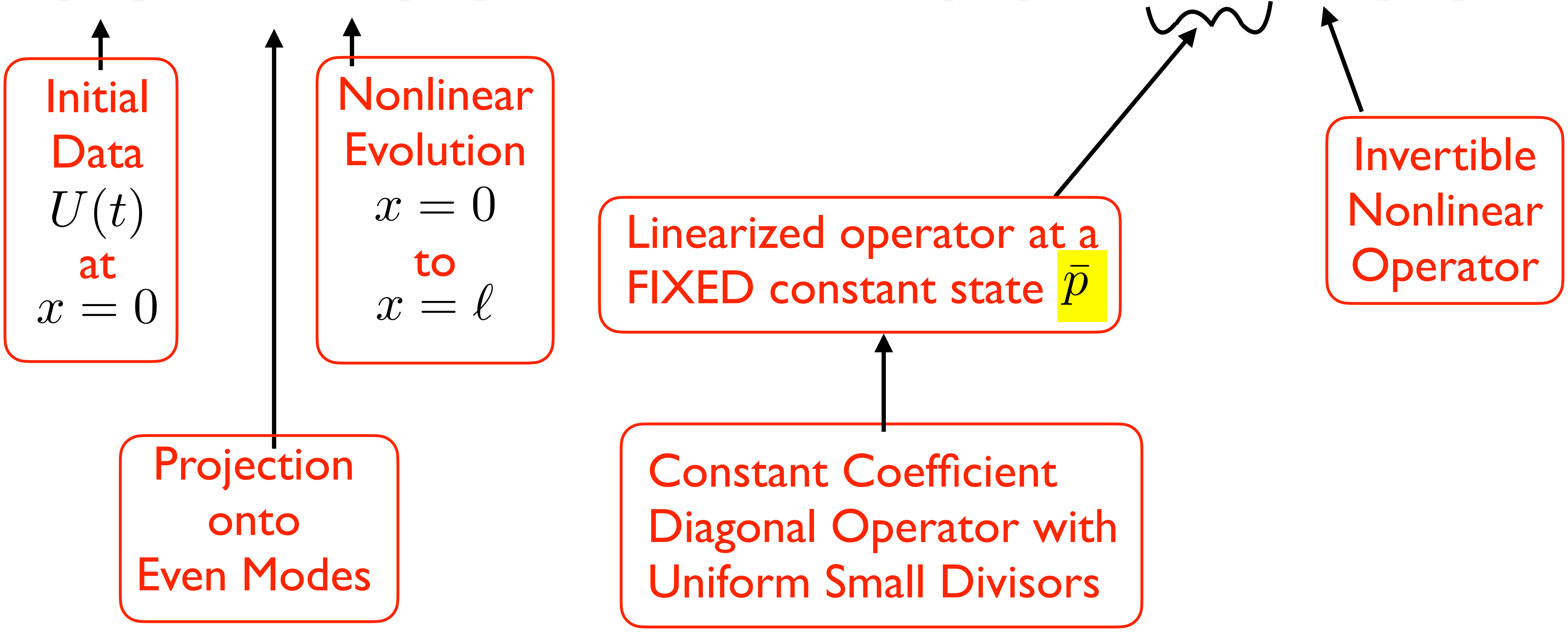

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Linearized operator at a FIXED constant state!

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Essentially: $\mathcal{P}\,\mathcal{L} = \mathrm{Diag}\,\{\lambda_1, \lambda_2, ...\}\,,\;\; \lambda_i \to 0$ (invertible)

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★ We do not need Nash-Moser, and we do not need to expunge!

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$$s(x) = s_0(x), \quad 0 \le x \le \ell.$$

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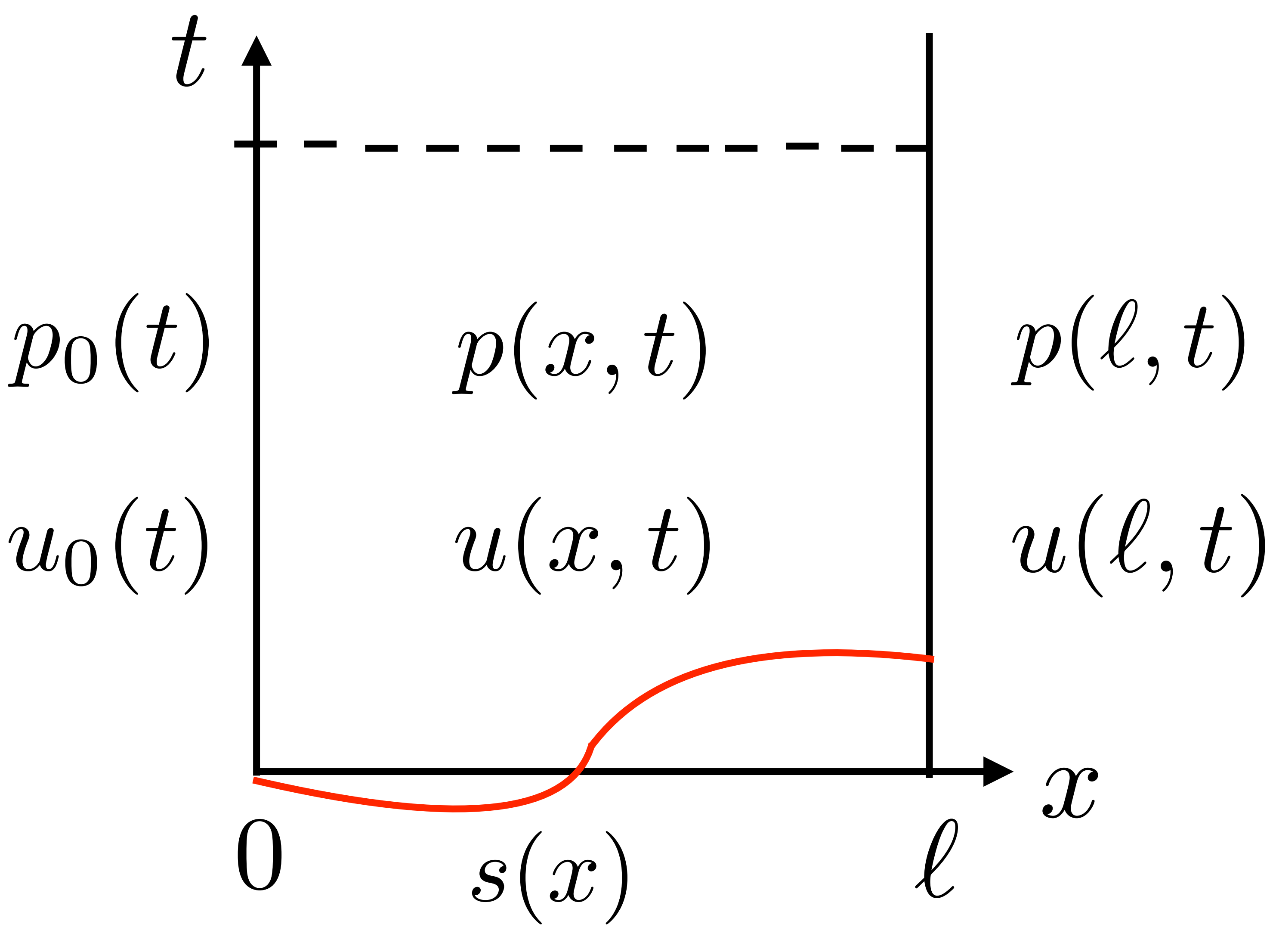

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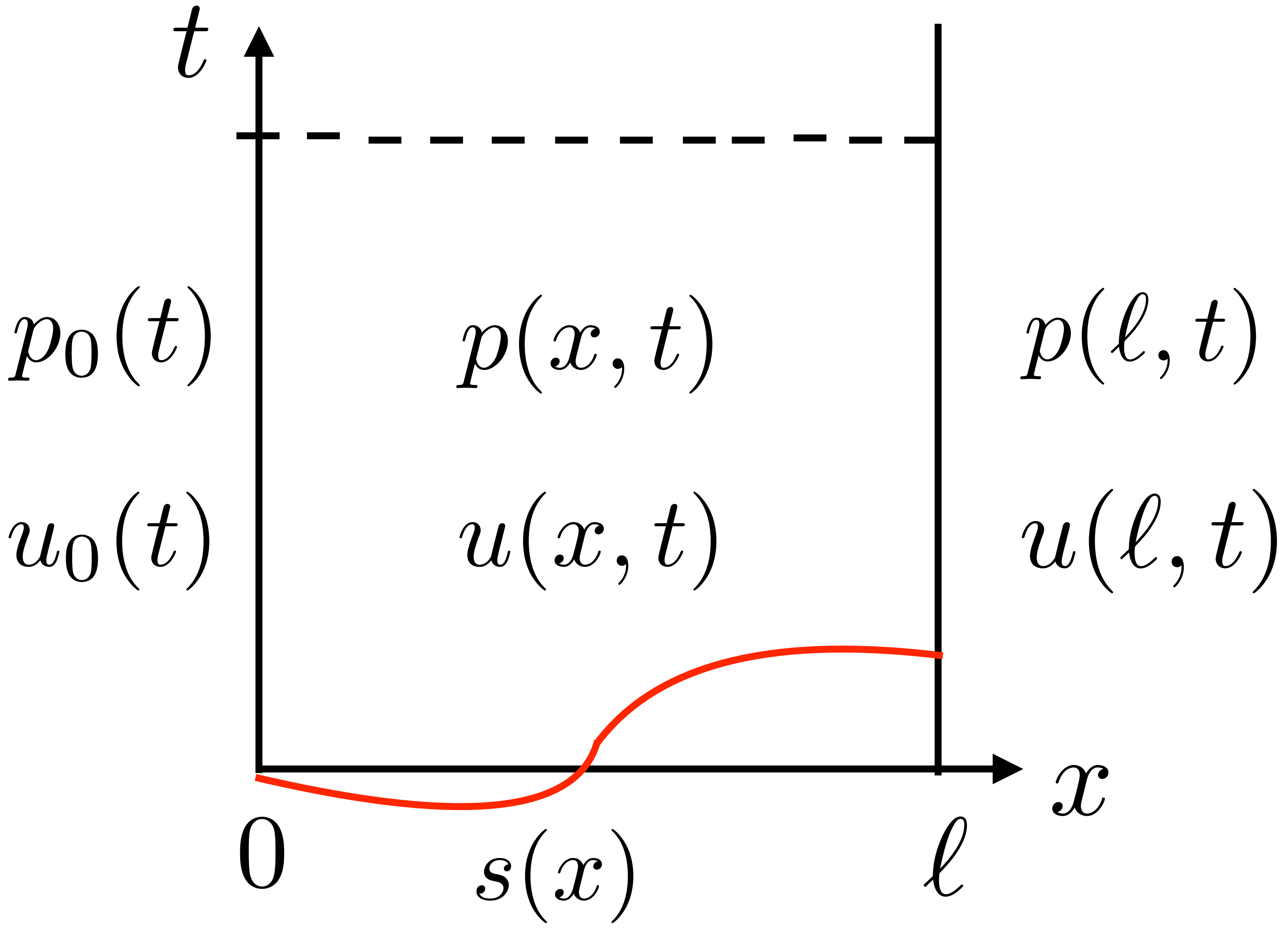
Observation 1: Evolution in x preserves the symmetries



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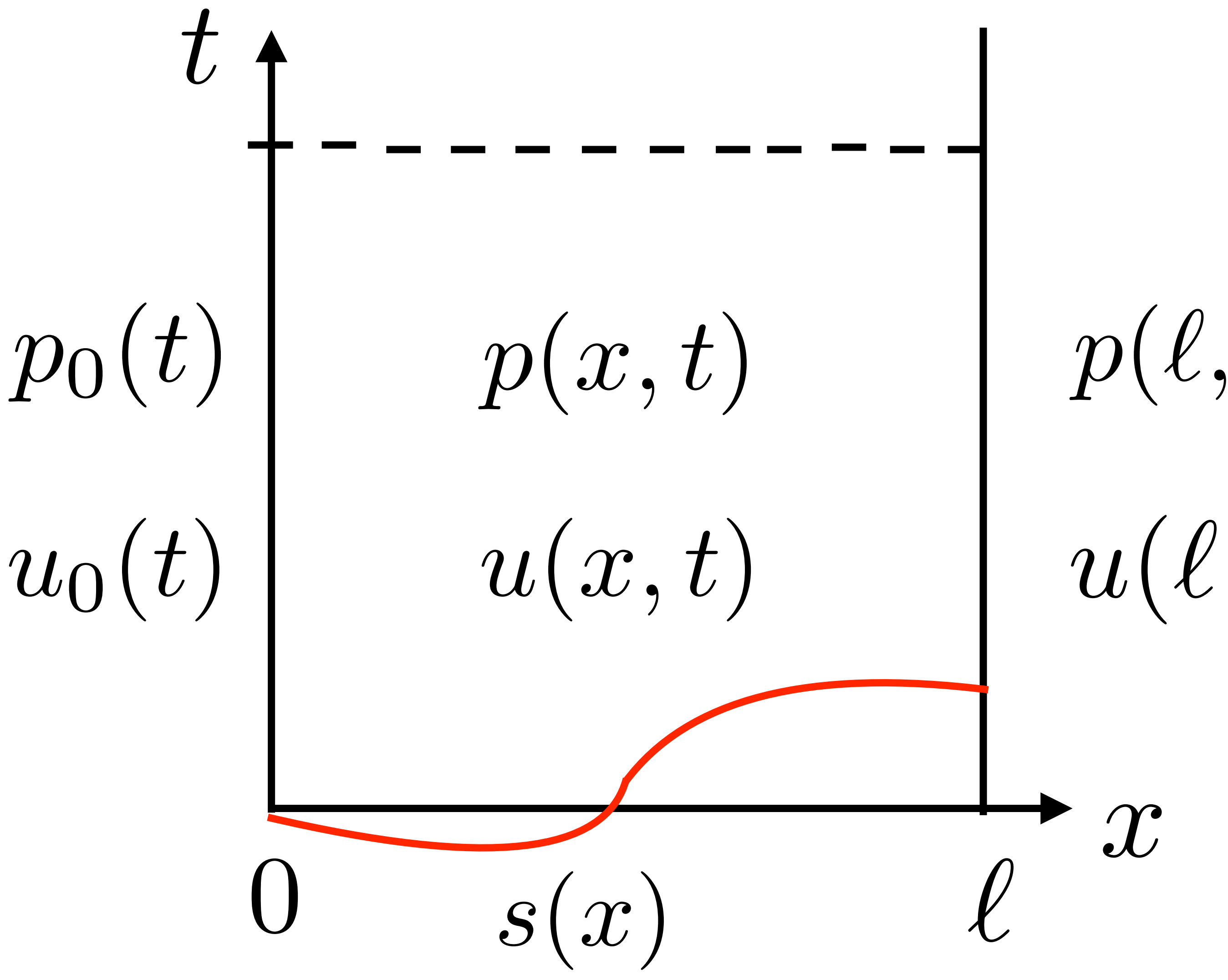
$p(x,t)$ even in t $\quad$ $p(x,-t) = p(x,t)$



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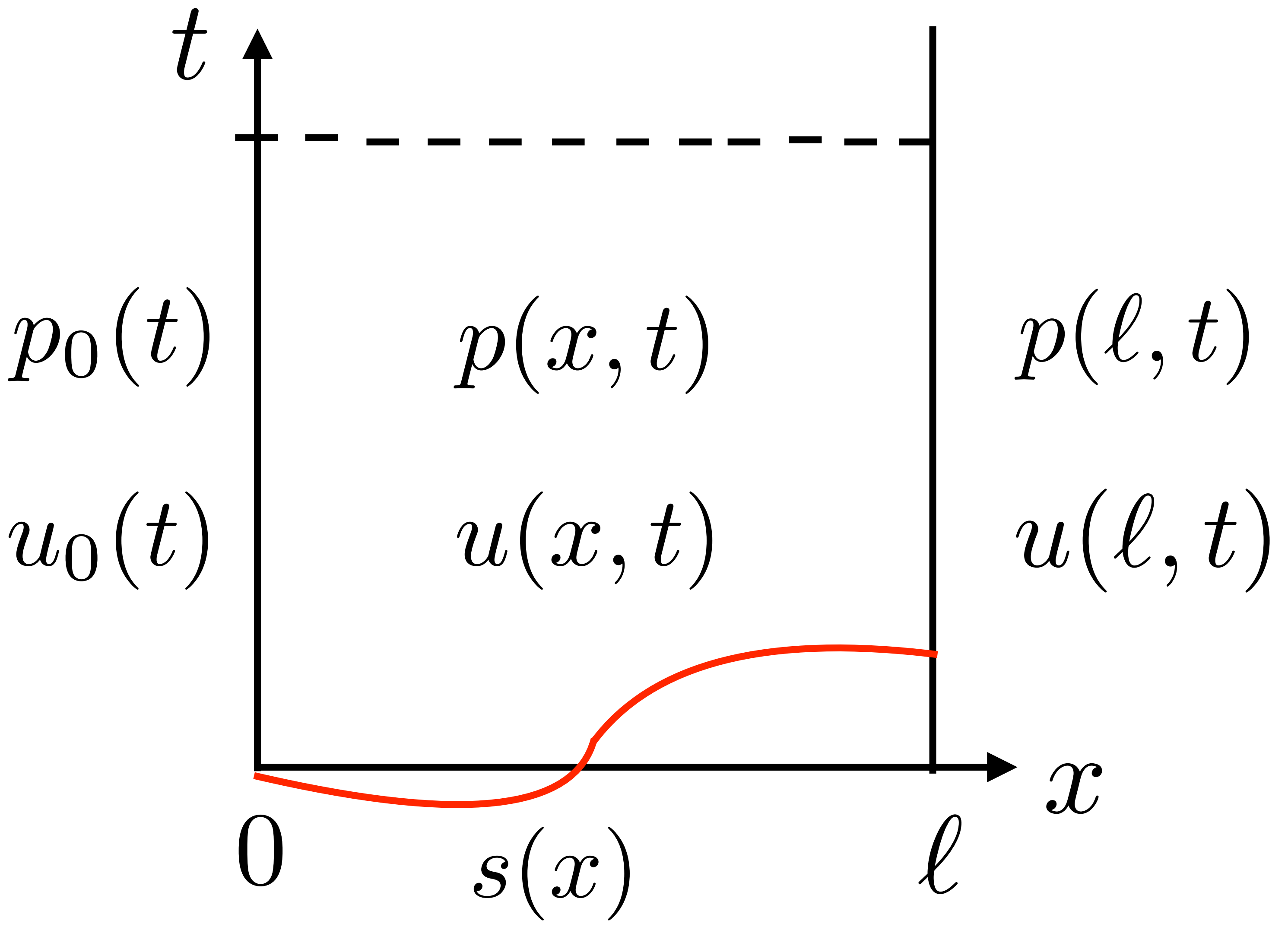
$u(x,t)$ odd in t $\qquad u(x,-t) = -u(x,t)$



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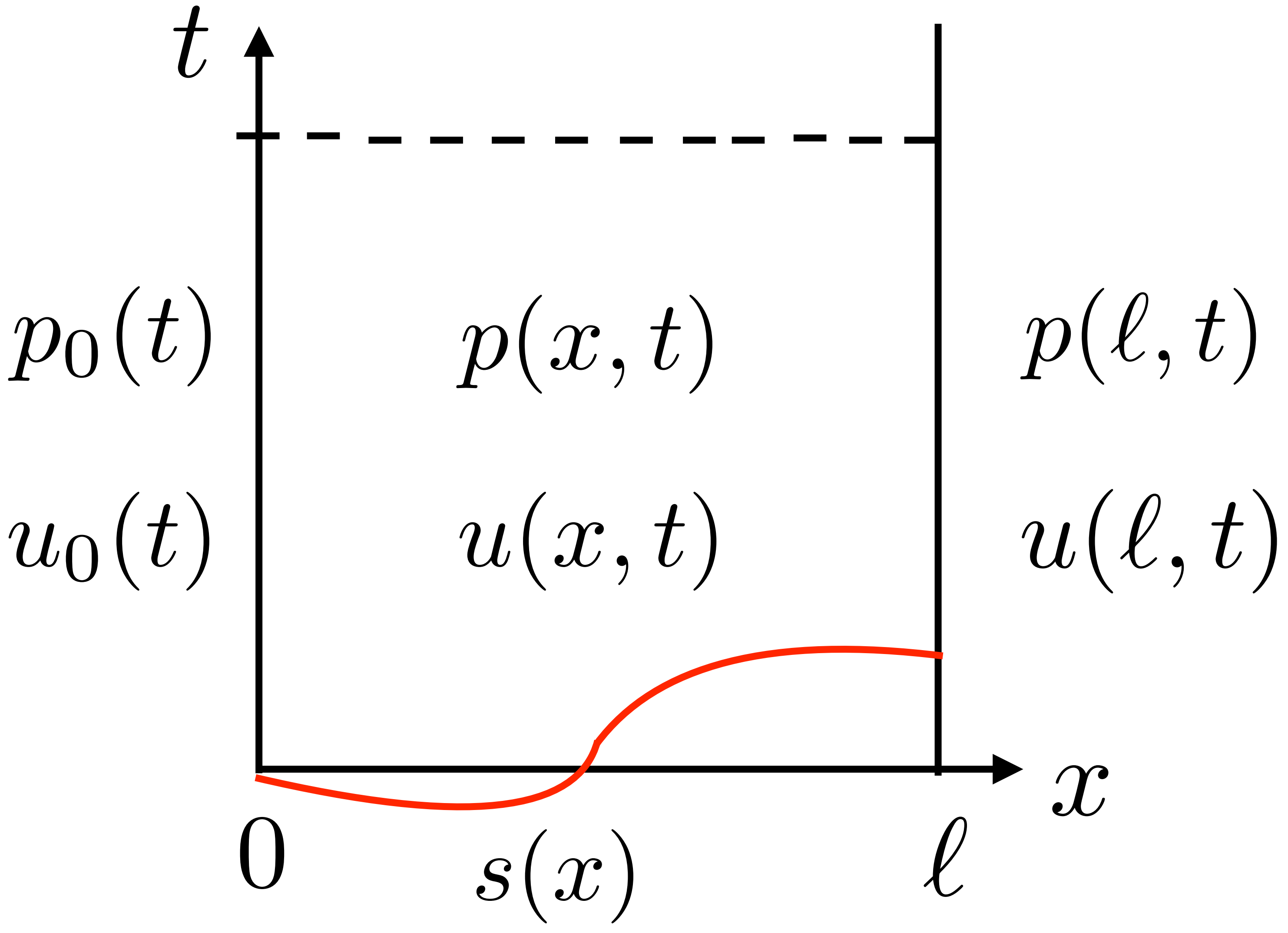

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$$p(-x,t) = p(x,t),$$
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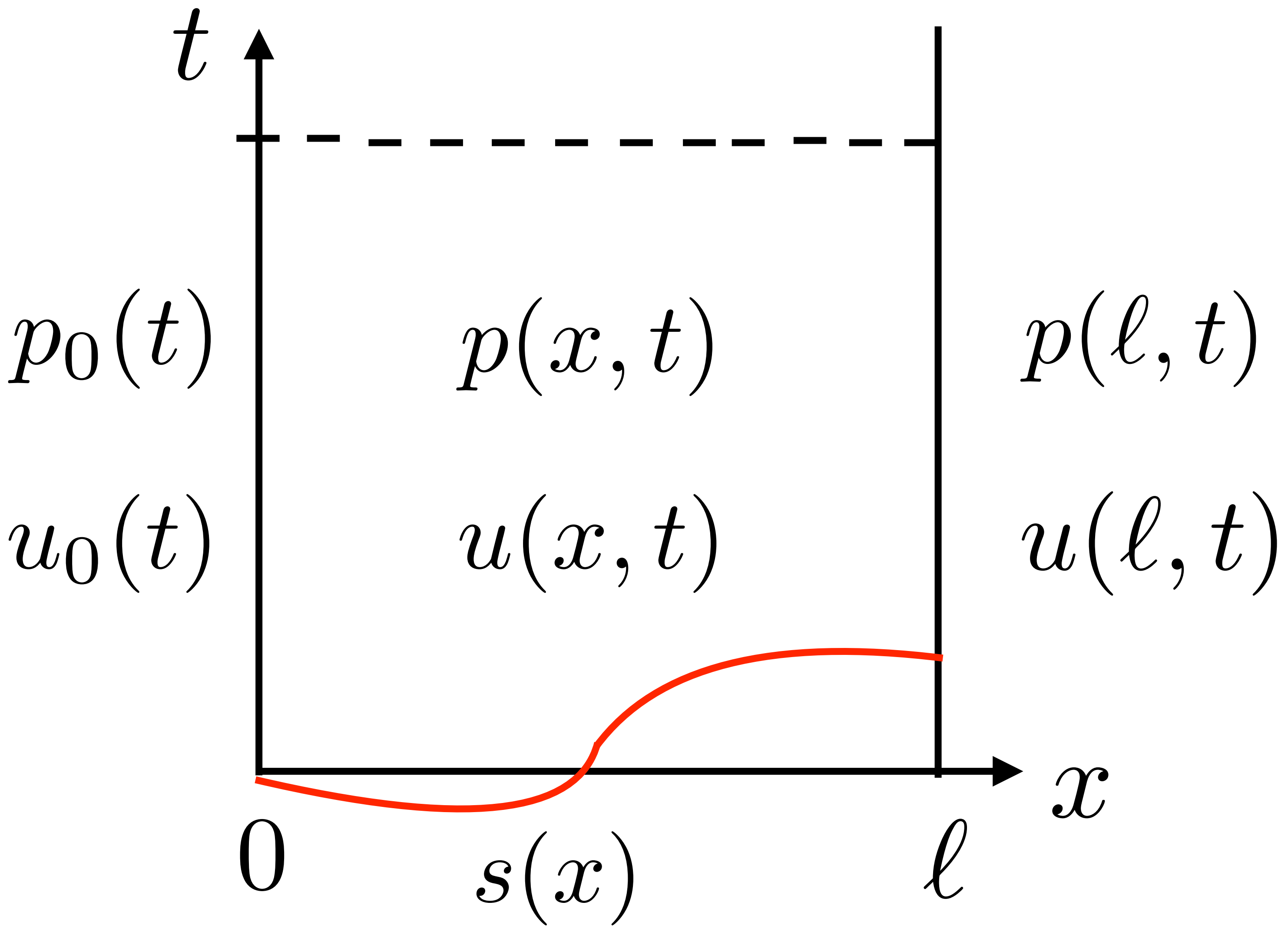

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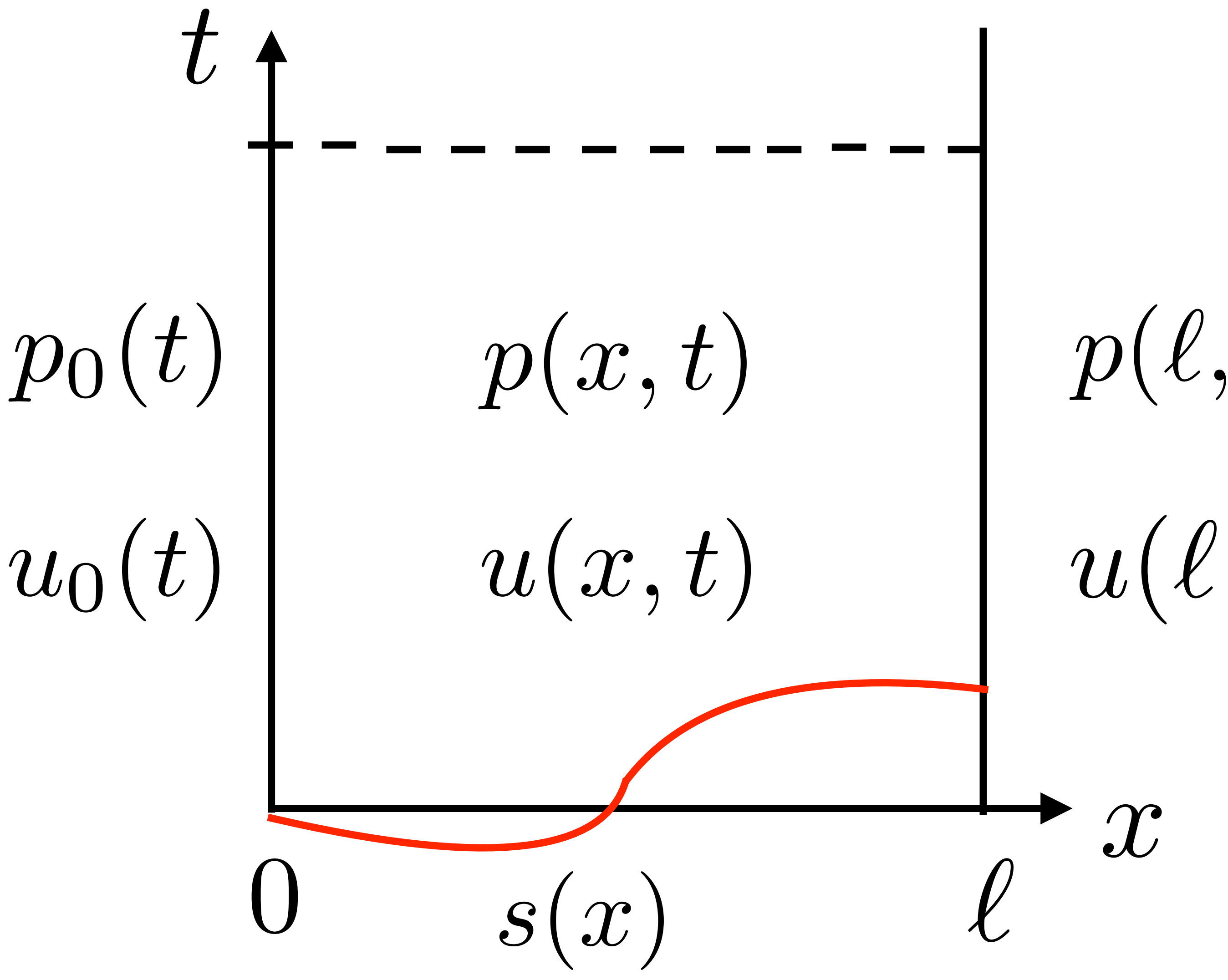

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Condition (2) is analogous to (1) observing that even/odd periodic functions are also even/odd about the half period T/2.

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✸ We now impose boundary conditions at $x = 0$ and $x = \ell$ which guarantee continuity at axes of reflection

$$x = 0 \quad \text{and} \quad x = \ell.$$

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These are self-adjoint boundary conditions!

Theorem: Assume a smooth T-periodic nonlinear solution satisfies (IC) and (BC). Then the solution extends by reflection symmetries to a smooth

4ℓ-periodic solution of compressible Euler.

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Conclude: To construct T-time and 4ℓ-space periodic solutions of compressible Euler, it suffices to prove existence of solutions to the boundary value problem:

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$$x = 0: \qquad u(0,t) = 0 \qquad \text{(IC)}$$

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so $p(x,\cdot)$ even and $u(x,\cdot)$ odd, $0 \le x \le \ell$.

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We prove existence of nonlinear solutions by perturbing off of linearized solutions.

THE RESULT IN A NUTSHELL

Conclude: To construct T-time and 4ℓ-space periodic solutions of compressible Euler, it suffices to prove existence of solutions to the boundary value problem:

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v, s(x))_x &= 0 \end{aligned} \qquad \text{(Euler)}$$

$$x = 0: \qquad u(0,t) = 0 \qquad \text{(IC)}$$

$$x = \ell: \qquad \begin{aligned} p(\ell, t + T/2) &= p(\ell, t) \\ u(\ell, t + T/2) &= -u(\ell, t) \end{aligned} \qquad \text{(BC)}$$

We prove existence of nonlinear solutions by perturbing off of linearized solutions.

Linear solutions can be constructed by Sturm-Liouville Theory because the boundary conditions are self-adjoint!

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to get

$$p_x + u_t = 0, \qquad u_x + \sigma^2(x)\, p_t = 0,$$

which is equivalent to (L).

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Sturm-Liouville theory implies ω_k are isolated, and grow linearly with wave number k.

THE EXISTENCE THEOREM

Solutions of the linearized equations by Sturm-Liouville

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Theorem [TY 2023]: Linearizing (Euler) around a quiet state $p = \bar{p}, u = 0, s(x)$, Sturm-Liouville theory determines a sequence of pure-tone solutions of the linear (BVP):

$$\begin{aligned} p(x,t) &= \bar{p} + \phi_k(x)\cos(\omega_k t) \\ u(x,t) &= \psi_k(x)\sin(\omega_k t) \end{aligned}$$

where ϕ_k and ψ_k are the eigenfunctions of the Sturm-Liouville problem, $k = 1, 2, 3, ...$.

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Our Main Theorem establishes that each non-resonant linearized pure-tone solution perturbs to a one parameter family of pure-tone solutions of the nonlinear compressible Euler equations, with the same frequency and time period

$$T_k = \frac{2\pi}{\omega_k}.$$

STATEMENT OF THEOREMS

DEFN: A k-mode is non-resonant if ω_k is not a rational multiple of any other eigen-frequency,

$$\frac{\omega_j}{\omega_k} \notin Q \quad \text{for all } j \neq k$$

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Theorem:[TY 2023] All non-resonant linearized k-modes perturb to periodic solutions of the nonlinear compressible Euler equations with the same space/time periods.

In Lagrangian coordinates the solutions take the form

$$p(x,t) = \bar{p} + \alpha\,\phi_k(x)\cos(\omega_k t) + O(\alpha^2),$$

$$u(x,t) = \alpha\,\psi_k(x)\sin(\omega_k t) + O(\alpha^2).$$

α = amplitude = perturbation parameter

$\phi_k, \psi_k \equiv$ Sturm-Liouville eigen-solutions

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Corollary: The compressible Euler equations generically admit pure-tone periodic solutions for every wave number $k = 1, 2, 3...$

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$$\mathcal{B} \equiv \left\{ s \in L^1[0,\ell] \;\middle|\; \sigma(x) = \sqrt{-v_p(\bar{p}, s(x))} \in L^1,\ \log \sigma(\cdot) \in BV \right\},$$

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I.e., completely non-resonant entropy profiles are of second Baire category in $\mathcal{B}$ with respect to the L^1-topology on $\mathcal{B}$.

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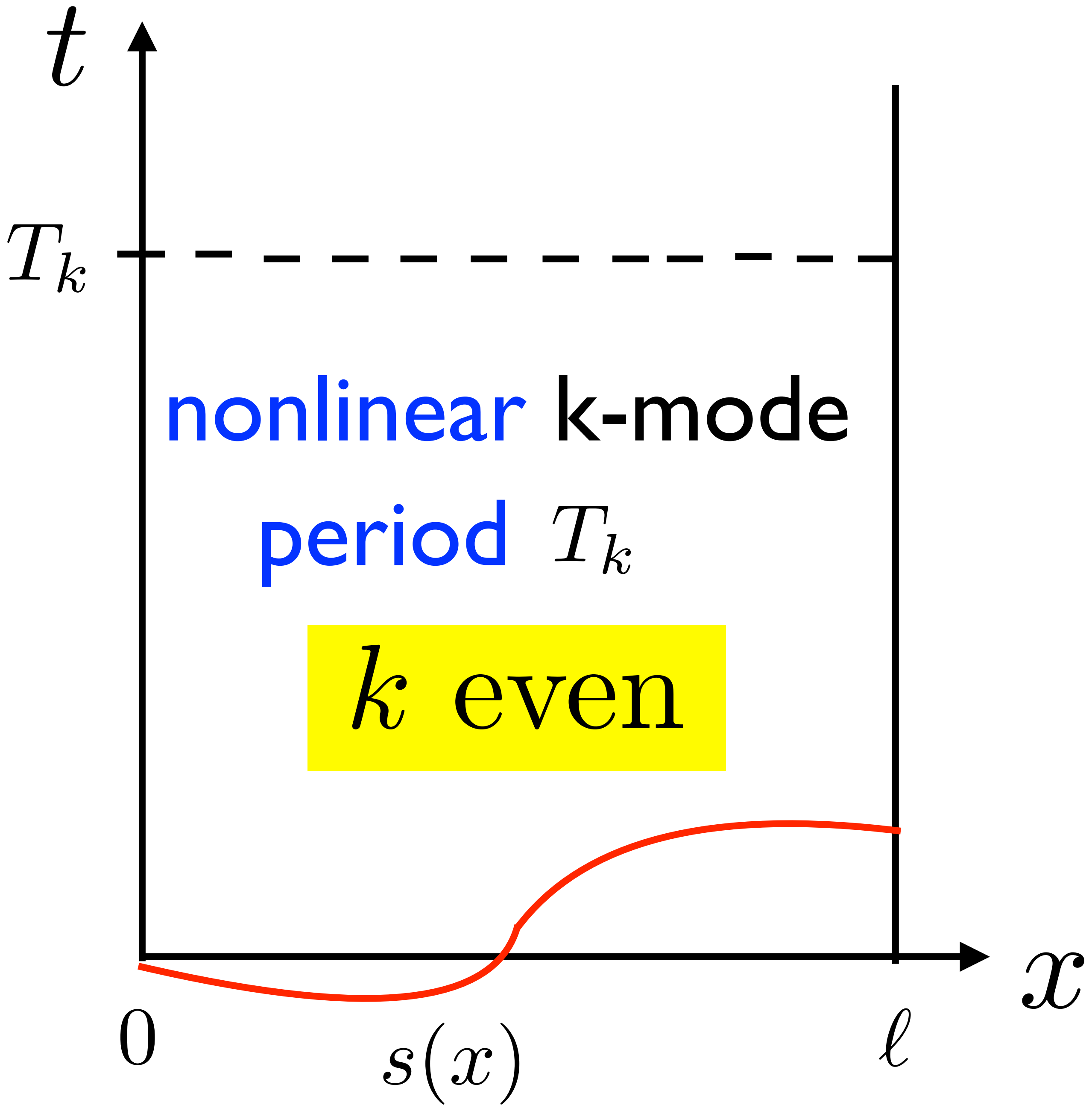
As a corollary we obtain an infinite family of pure tone solutions satisfying the acoustic boundary condition $u = 0$.



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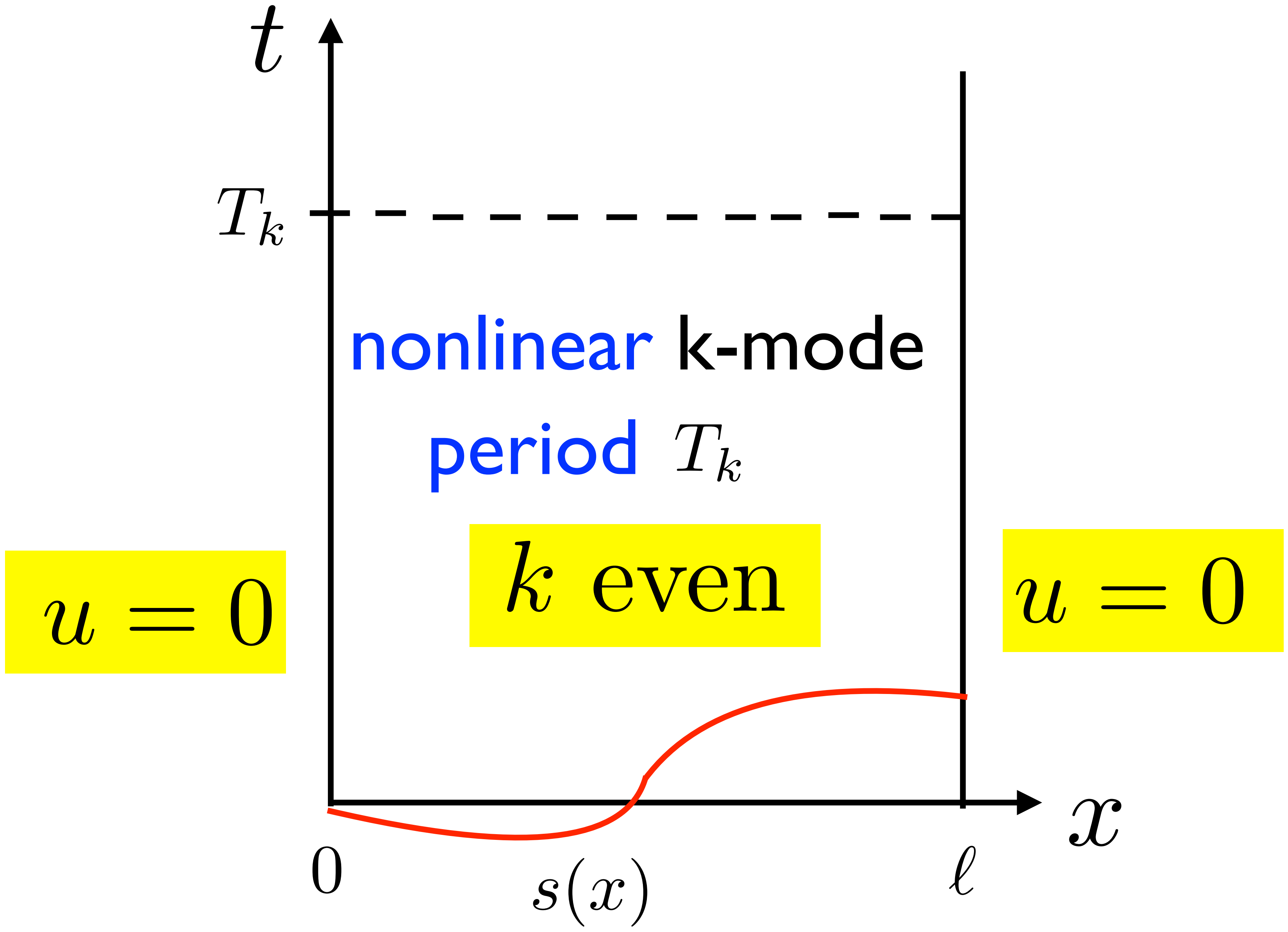

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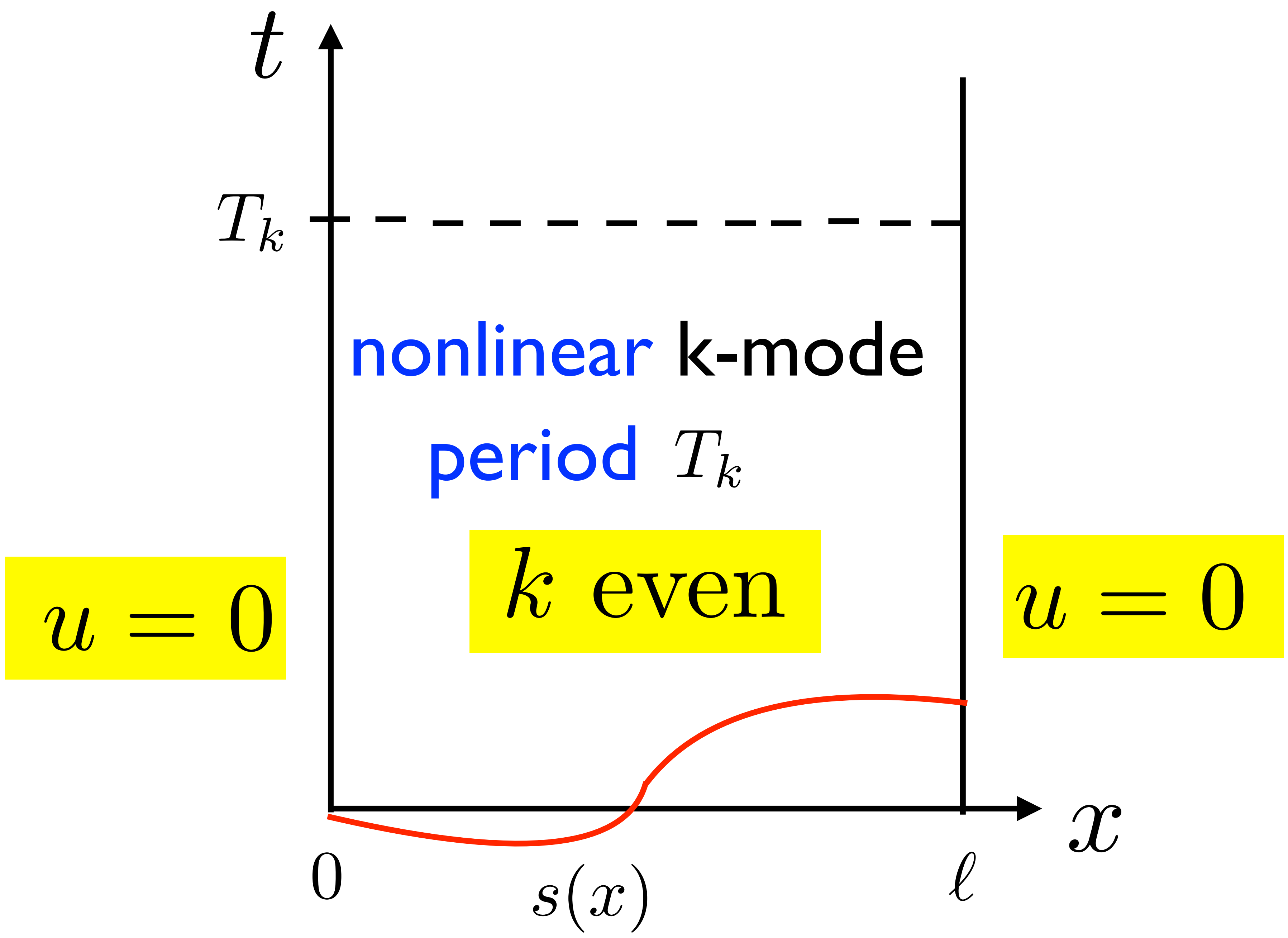

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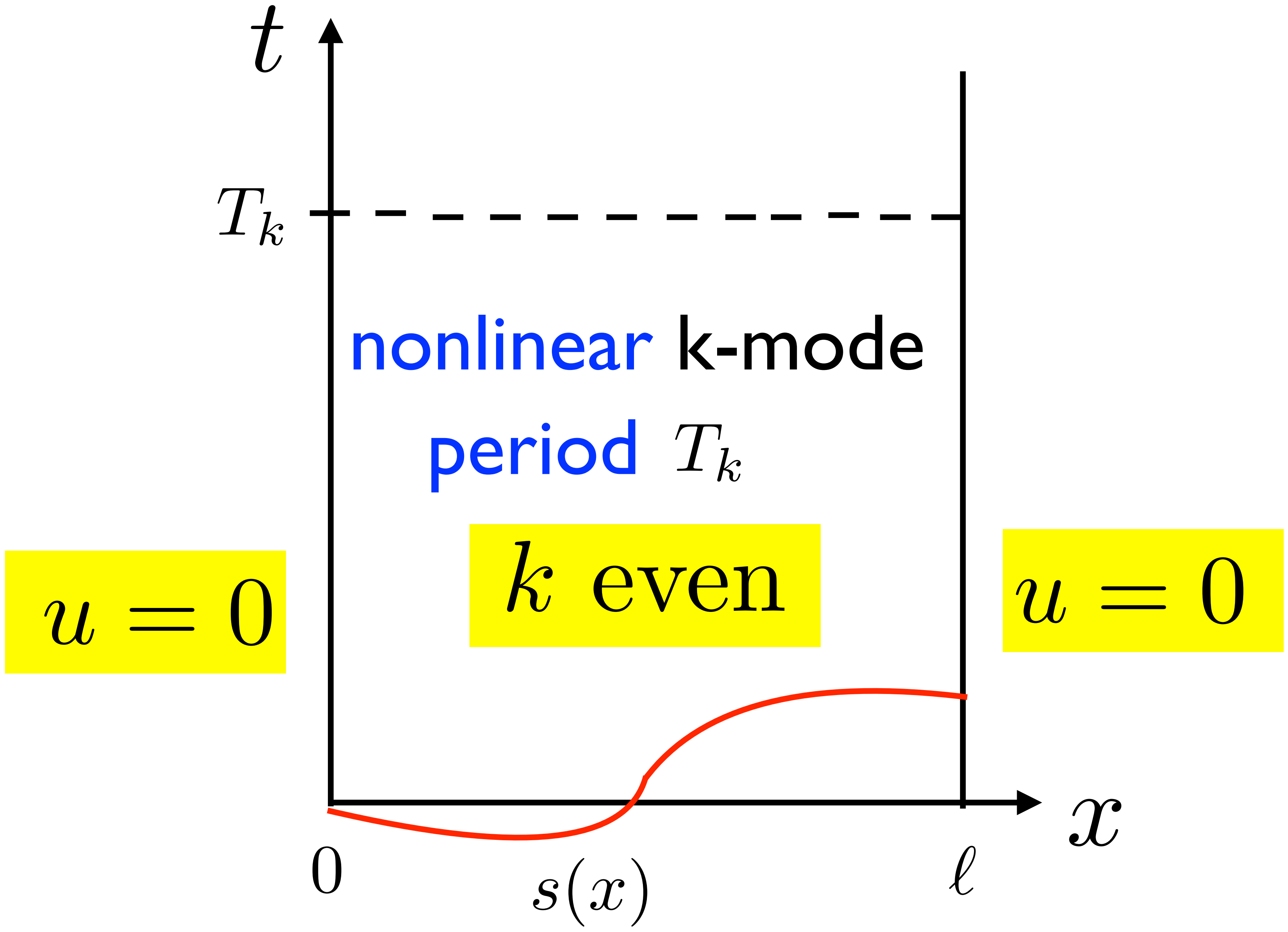

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period T_k

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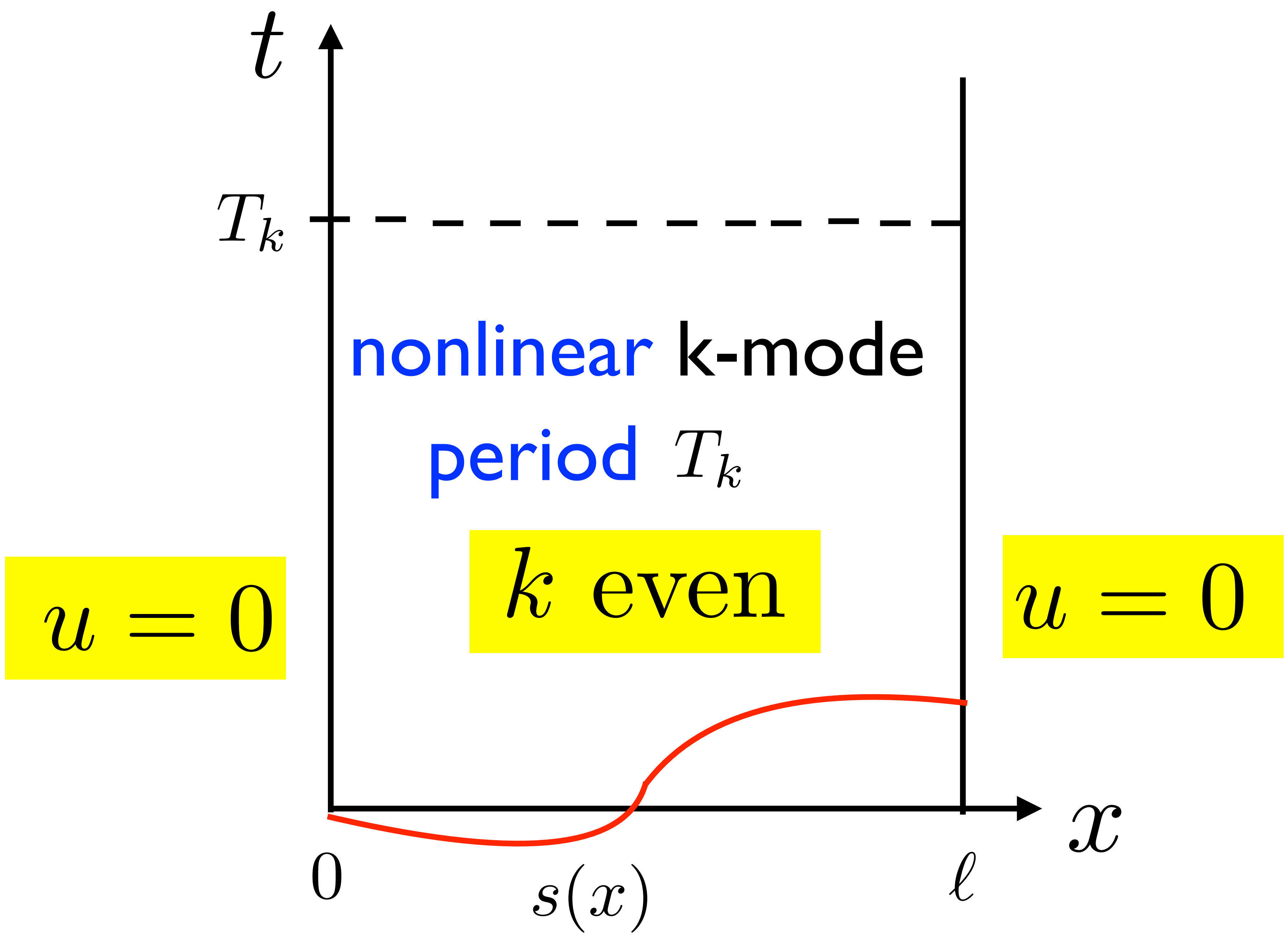

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A Symmetry Reflection Principle extends solutions on $[0, \ell]$ to solutions with time period T_k and space period 4ℓ.

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$$u(0,t) = 0. \qquad \text{(IC)}$$

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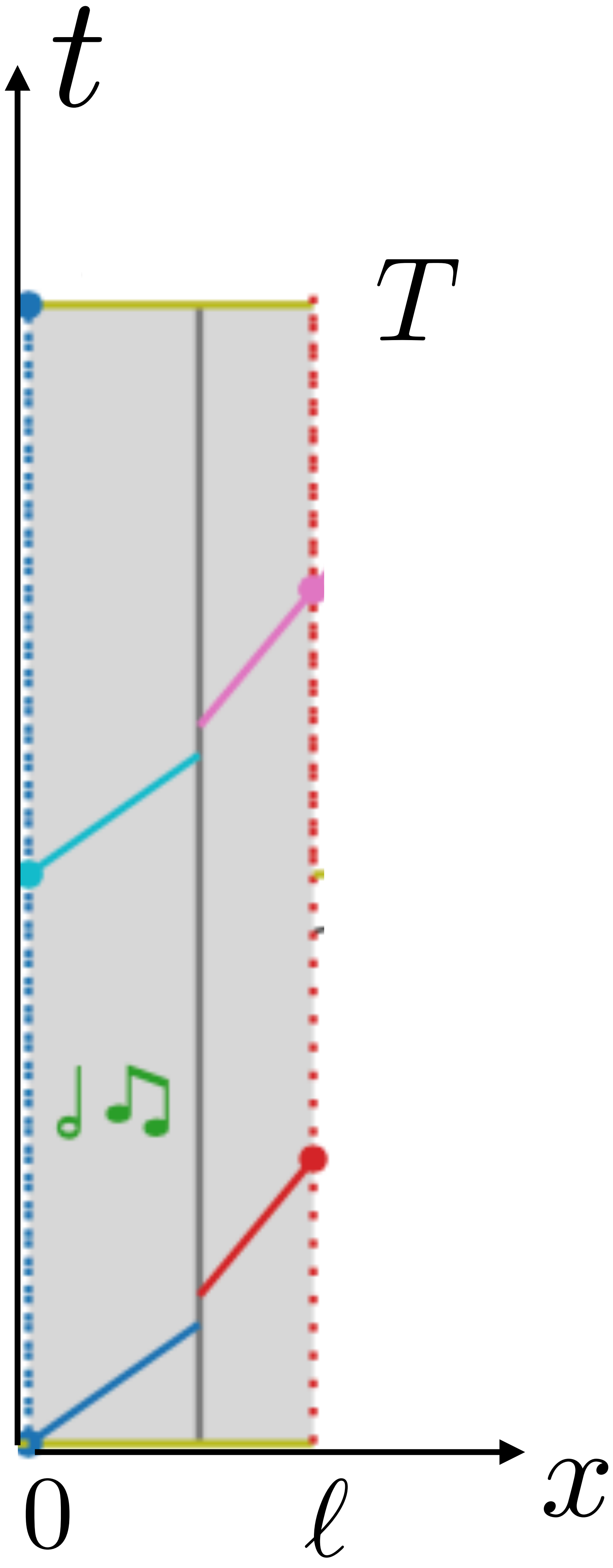
Then the solution extends by reflection to a 4ℓ-periodic solution of compressible Euler.

THE PERIODIC TILE

T-periodic solution $0 \leq x \leq \ell$



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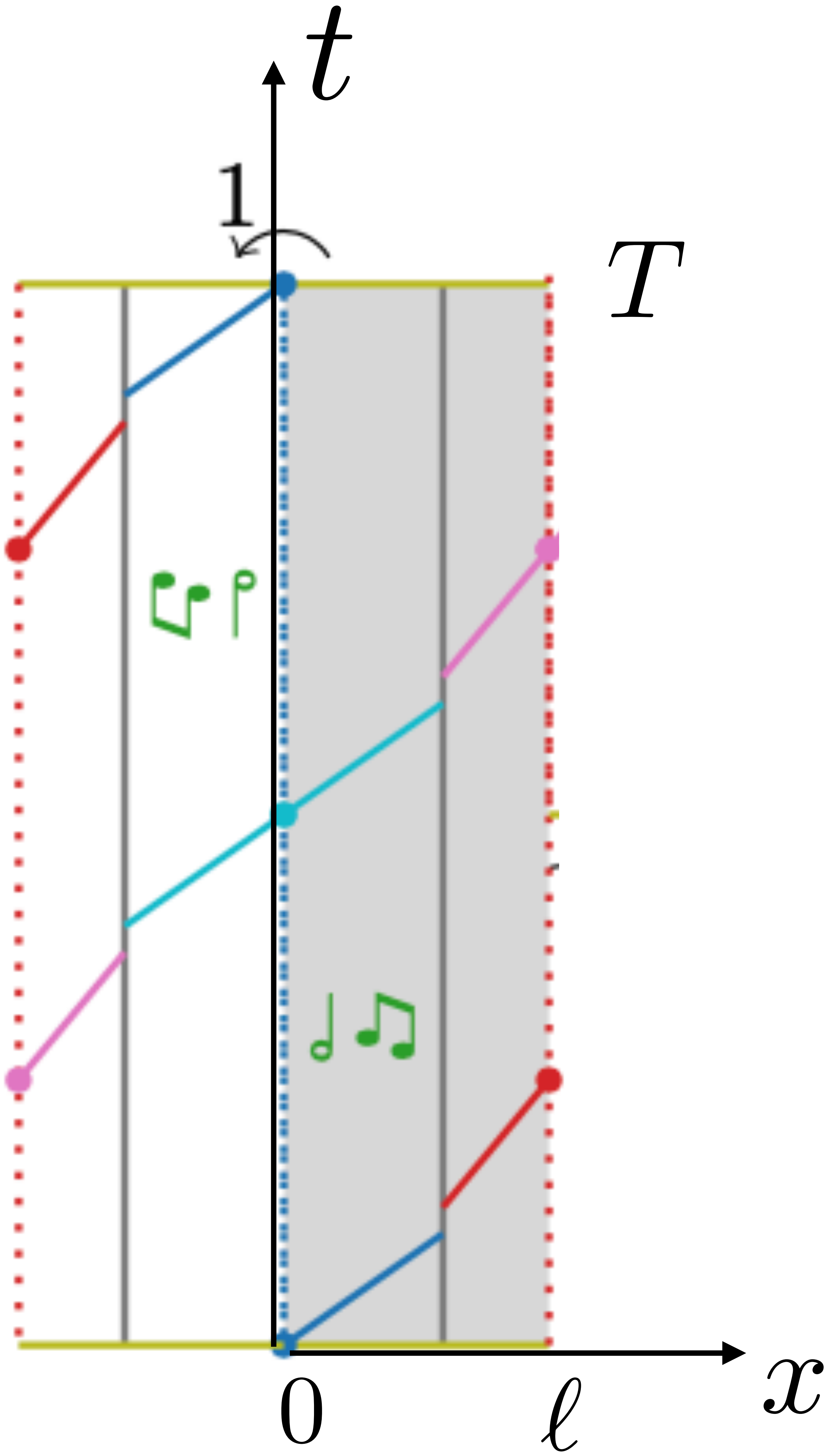

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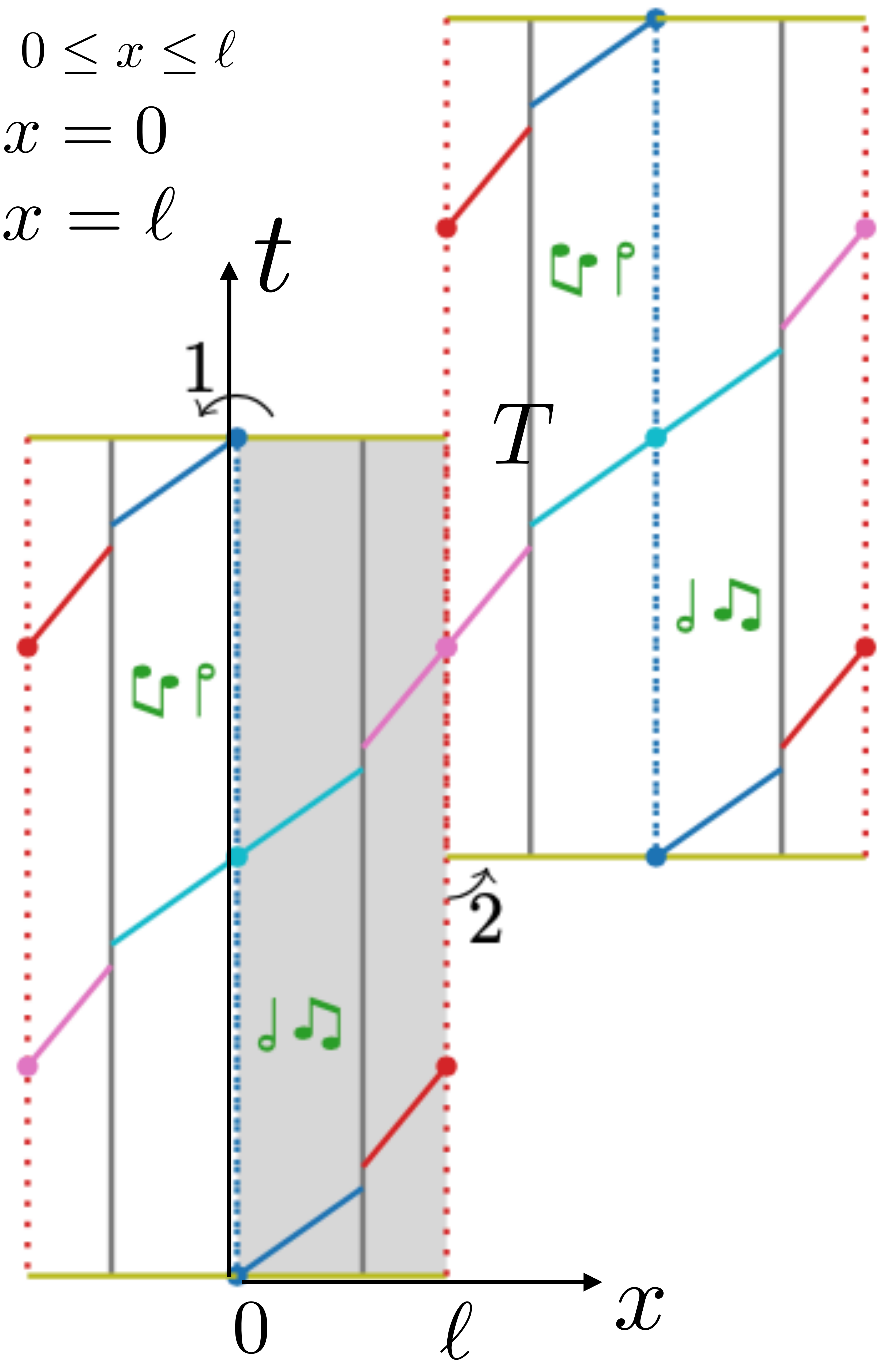
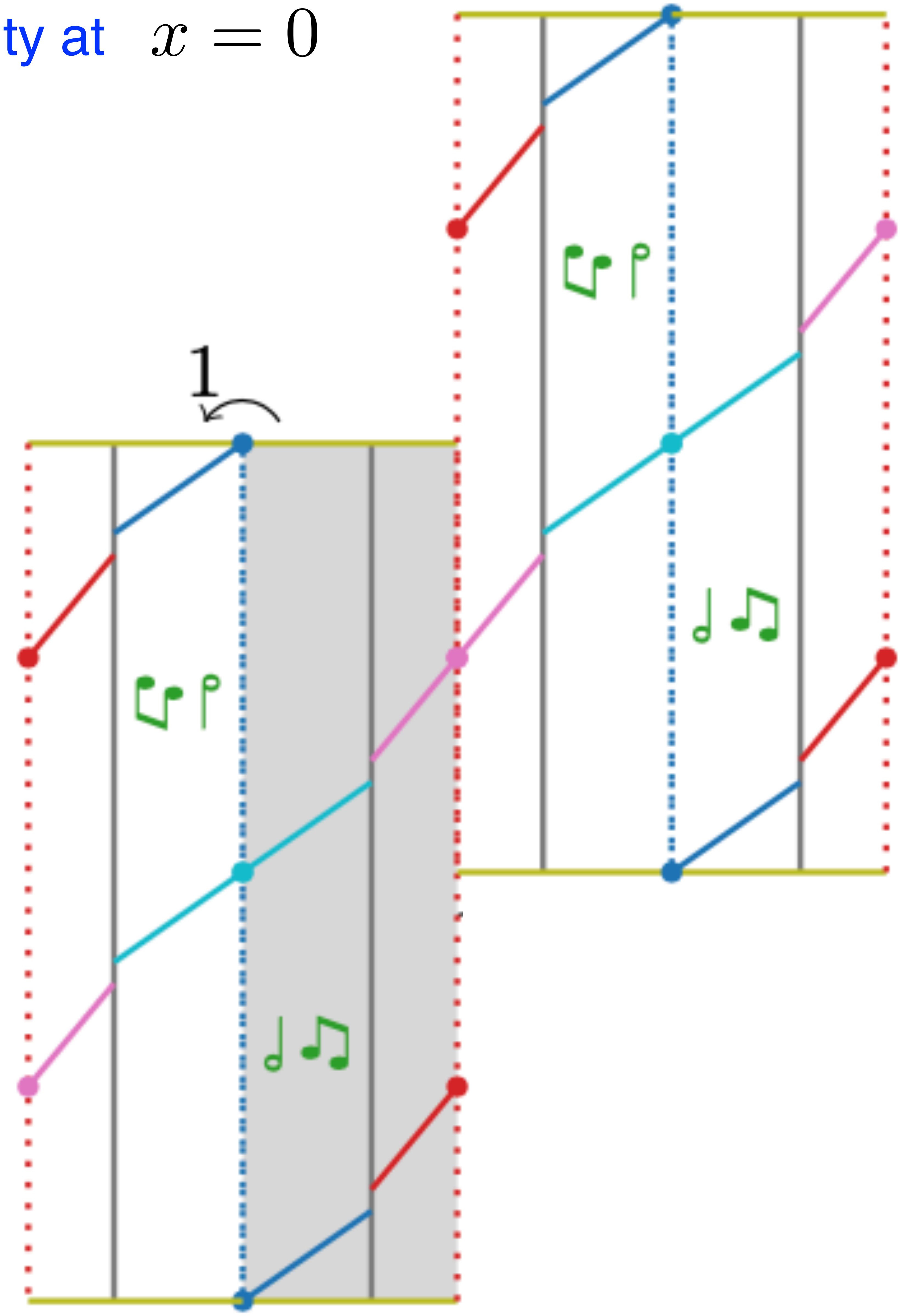


THE PERIODIC TILE

(IC) gives continuity at $x = 0$



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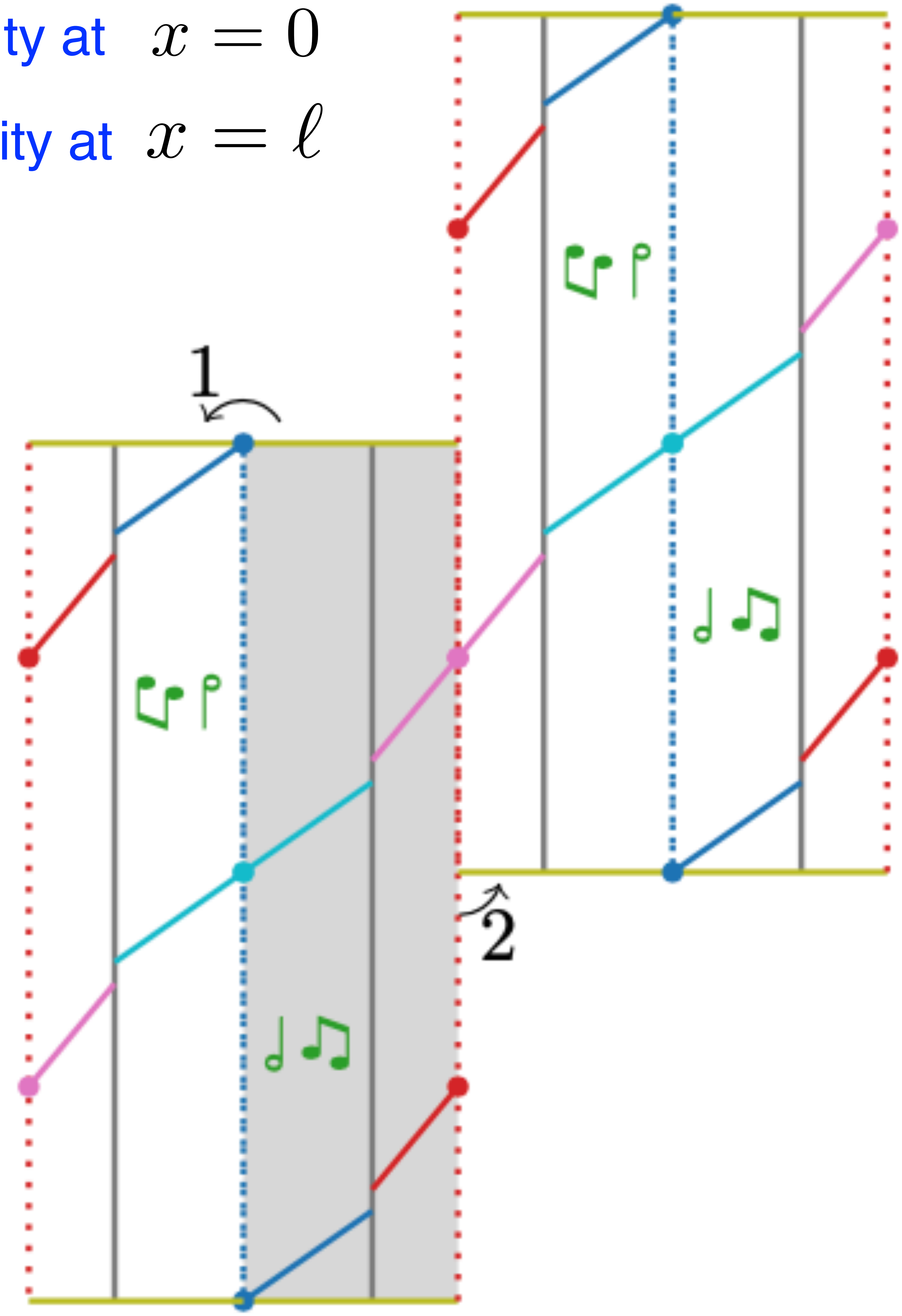

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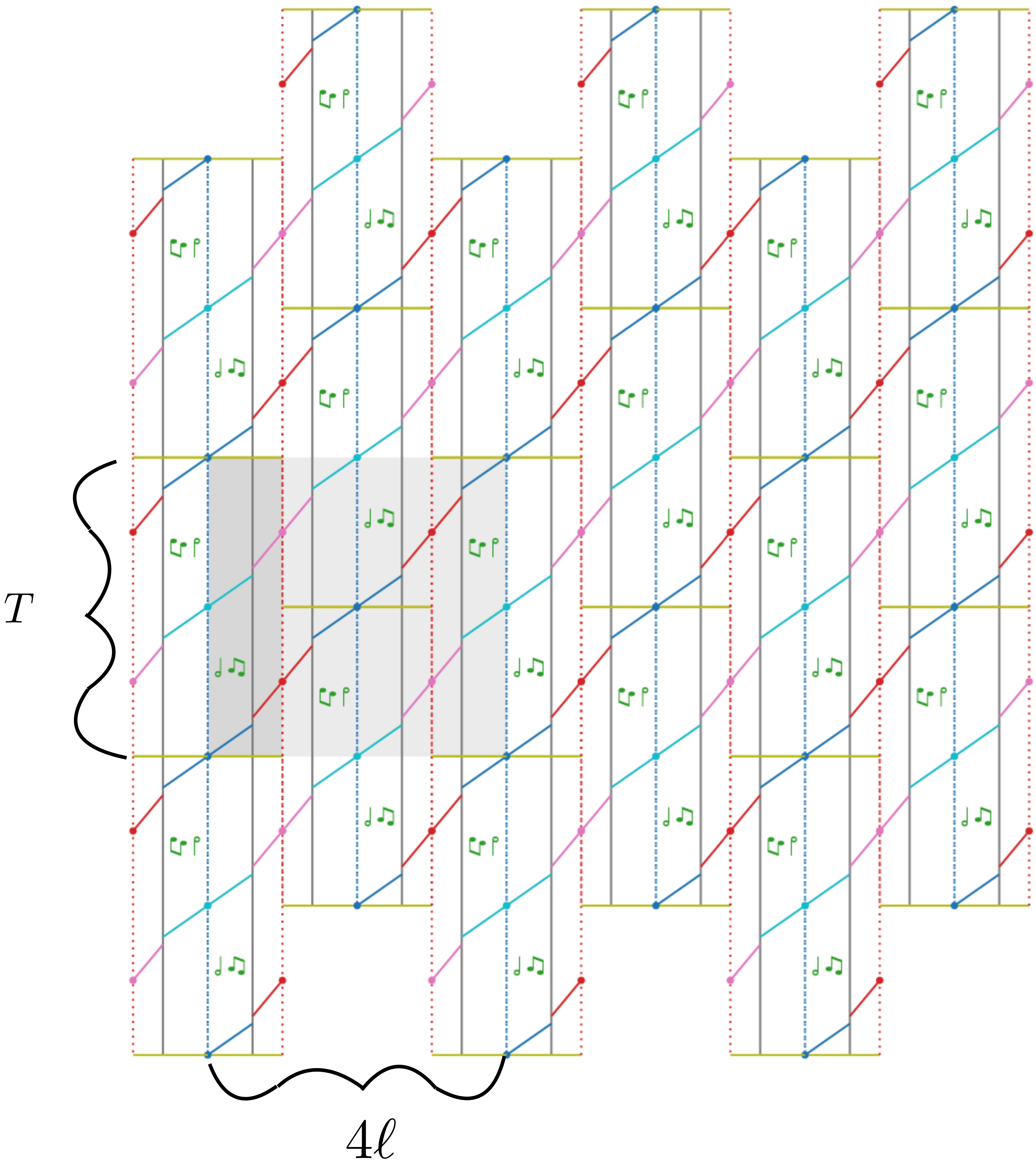
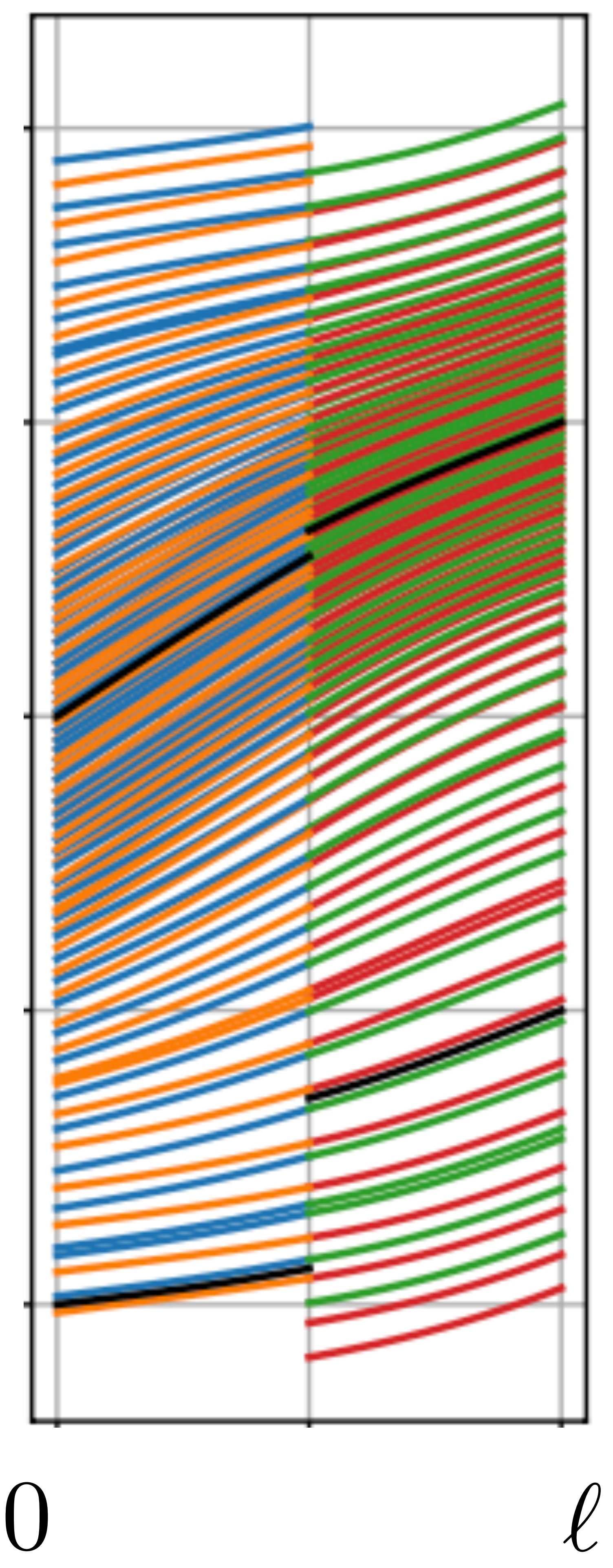


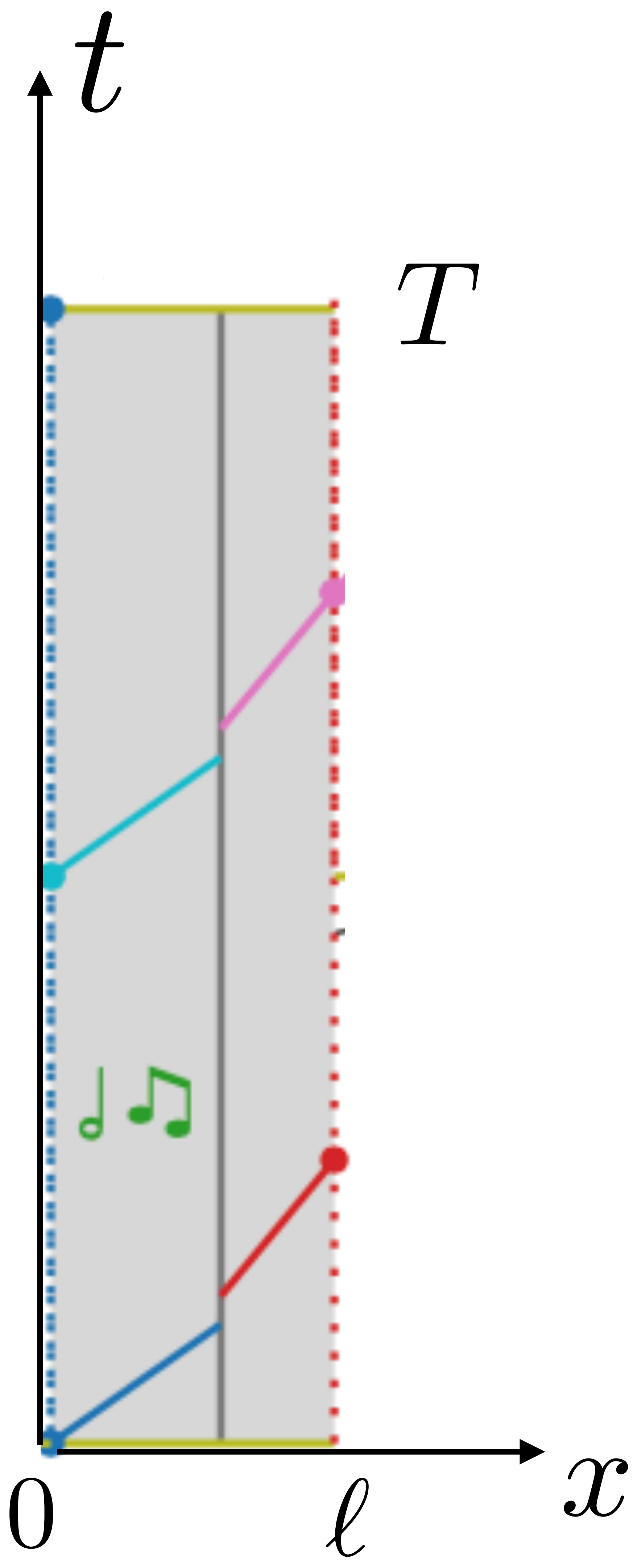
THE PERIODIC TILE

Characteristics in nonlinear solution—one entropy jump



THE PERIODIC TILE

Turns Out: Even modes meet acoustic boundary condition



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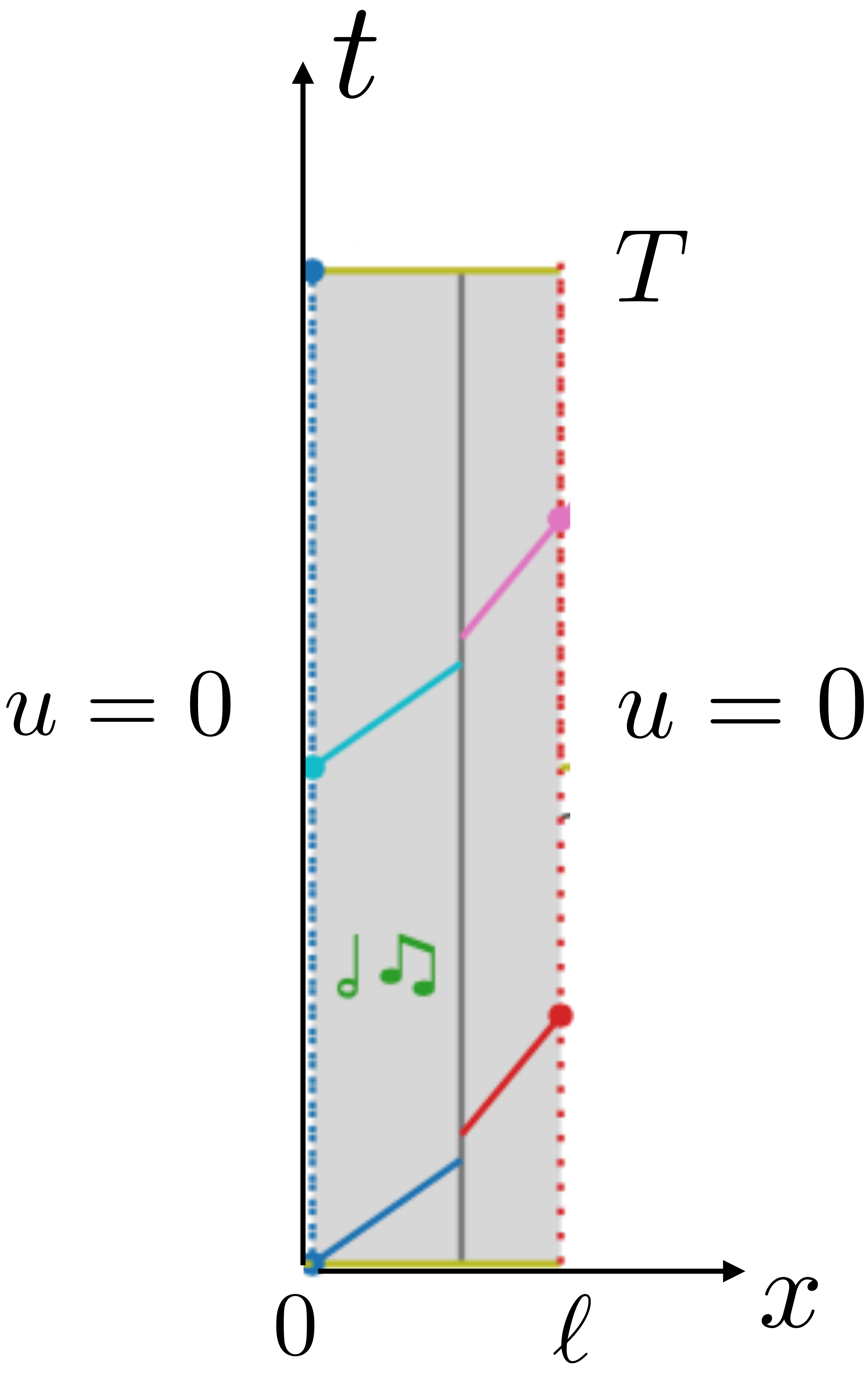

THE PERIODIC TILE

$s(x)$

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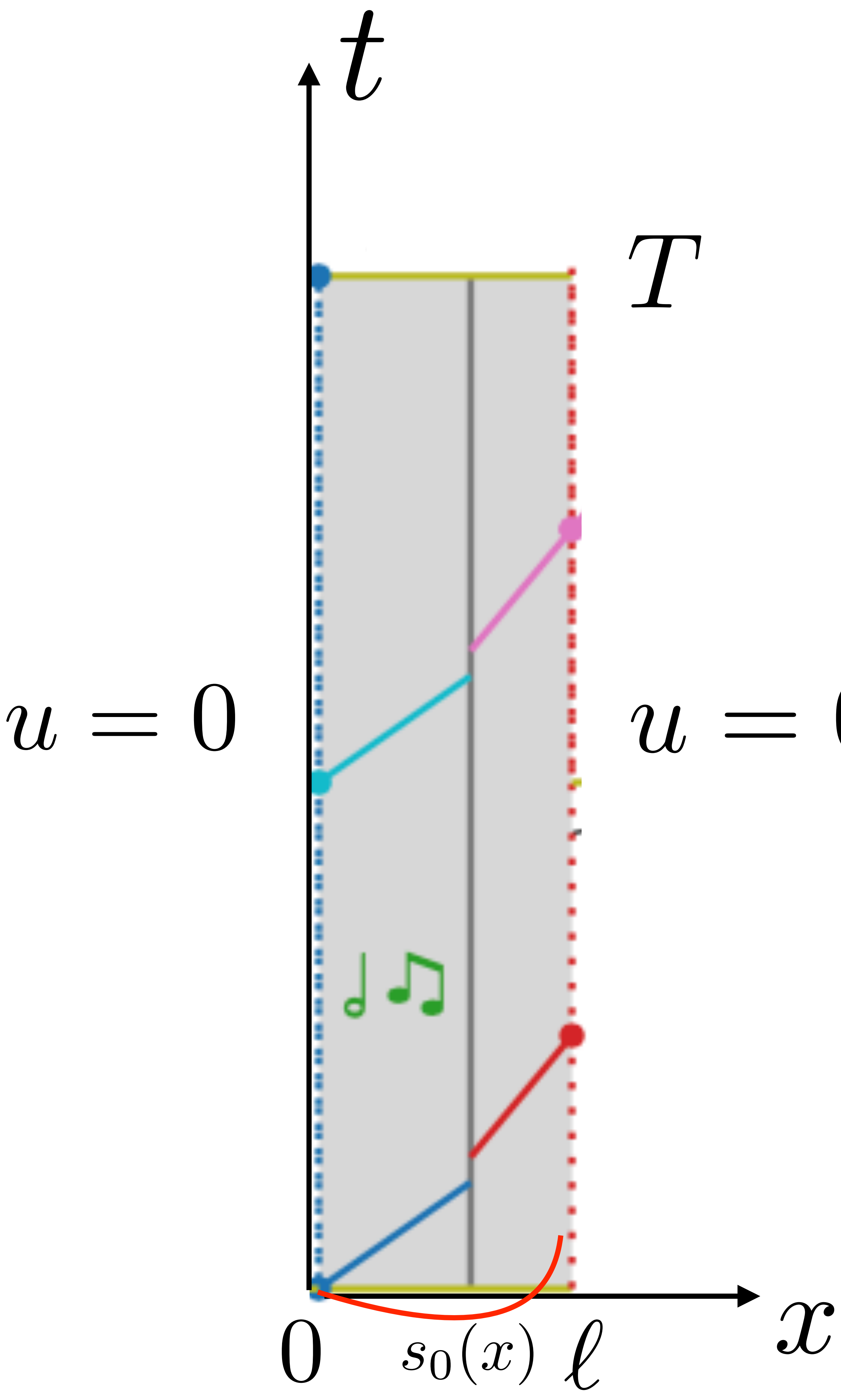

Arbitrary entropy profile $s_0(x),\ 0 \leq x \leq \ell$

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Compressible Euler: $s = s(x), \quad 0 \leq x \leq \ell.$

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Nonlinear solutions of (Euler),(IC),(BC) solve:

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Linearized operator with small divisors at a FIXED constant state $\bar{p}$!

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We can change constant state to $\bar{p}+z$ while keeping $\mathcal{L}$ FIXED at constant state $\bar{p}$!

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Choose T_k in kernel and assume non-resonance…

THE EXISTENCE THEOREM

$$\mathcal{F}[U] = D_{\bar{p}}\mathcal{F} \circ \mathcal{N}[U] = 0$$

Linearized operator at a FIXED constant state p_0

Invertible Nonlinear Operator

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By construction, only the k-mode is in the kernel of $d\mathcal{F}_{p_0}$!

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Conclude:

THE EXISTENCE THEOREM

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Invertible Nonlinear Operator

Conclude: $D\mathcal{F}_{\bar{p}} = \mathrm{Diag}\left\{\lambda_1, \lambda_2, ...\right\}, \ \ \lambda_i \to 0$ (invertible)

$\mathcal{N} = \mathcal{I} + O(\alpha)$ (bounded invertible)

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Essentially: $D\mathcal{F}_{\bar{p}} = \mathrm{Diag}\left\{\lambda_1, \lambda_2, \ldots\right\}, \ \lambda_i \to 0$ (invertible)

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Theorem: Solutions of $\mathcal{F}[U] = 0$ exist by the Implicit Function Theorem in Banach Spaces.

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(Auxiliary and Bifurcation equations of Lyapunov-Schmidt).

Our new proof arose from ideas in the completion of Steps 1 and 2.

Review of Prior Results.

PRIOR WORK

In prior work: YoTe proposed a simplest possible wave pattern for periodic propagation…

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The pattern formally balanced compression and rarefaction along every characteristic:

The Mechanism requires at least three coupled equations…

SIMPLEST PERIODIC PATTERN

We looked to construct the simplest solution of

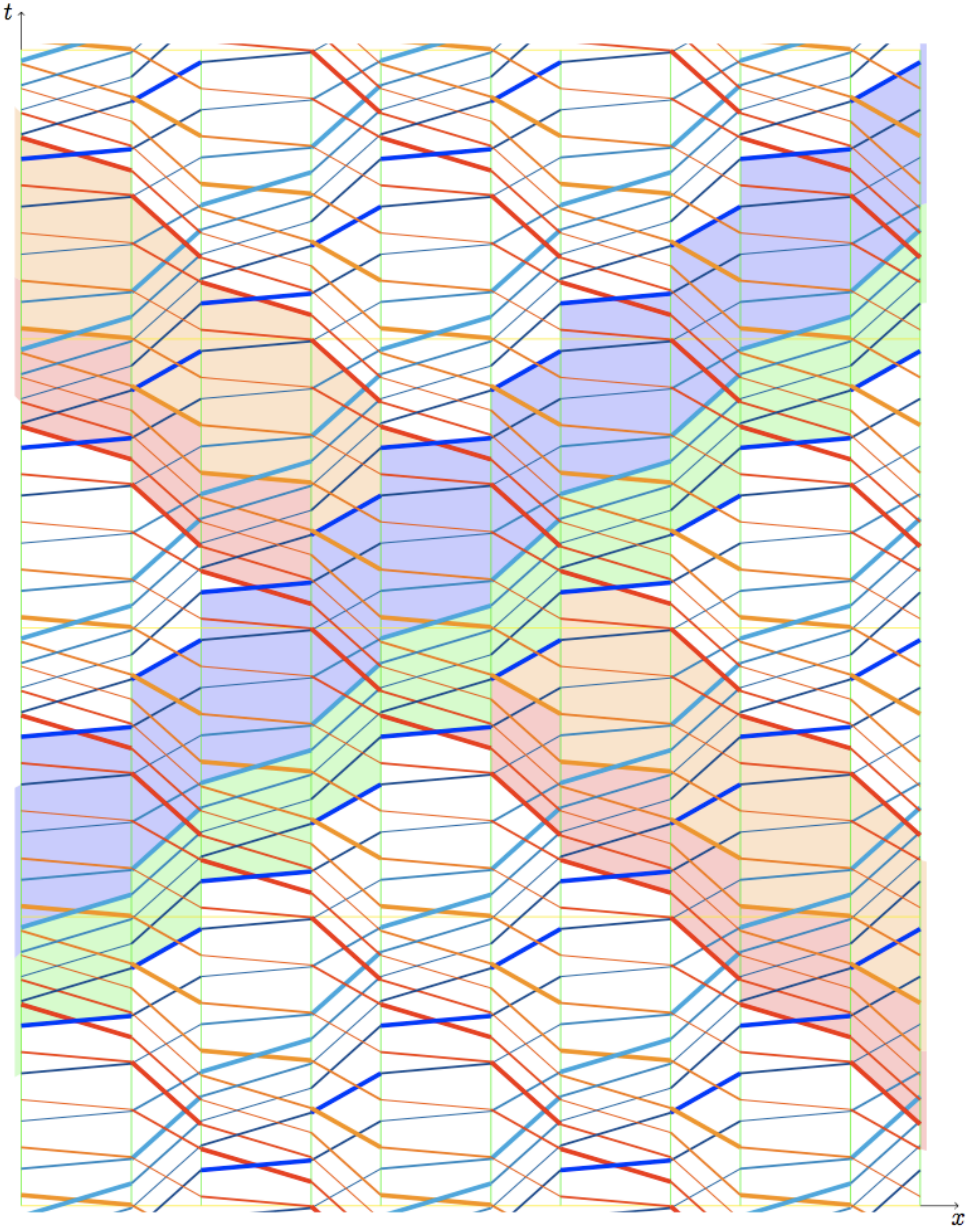
$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v, s(x))_x &= 0 \\ s_t &= 0 \end{aligned} \qquad \text{(L)}$$

such that Rarefaction and Compression is in balance along every characteristic…

We perturb off "quiet state" solutions:

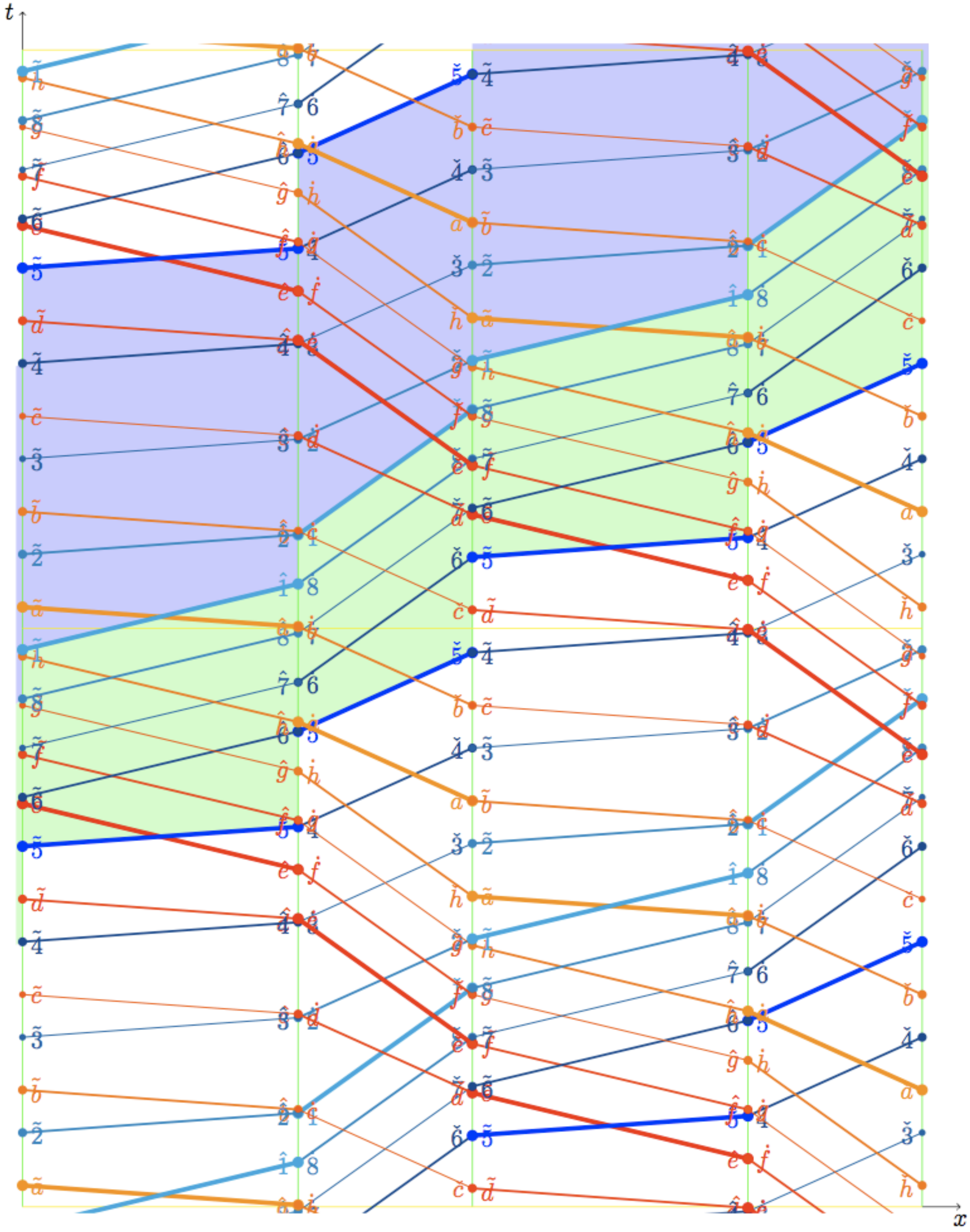
$$p = p_0, \quad u = 0, \quad s = s(x)$$

Simplest case: $s(x)$ jumps between two constant states.



---Our Proposal—

The simplest global periodic structure in the xt-plane



Our Proposal

RAREFACTION
and
COMPRESSION

Compressible Euler Equations:
Lagrangian Coordinates:

Assume given entropy profile: $s_0(x)$

$$\begin{aligned} p_x + u_t &= 0, \\ u_x - v_p(p, s_0(x))p_t &= 0 \end{aligned} \qquad \text{(L)}$$

Compressible Euler Equations:
Lagrangian Coordinates:

The system supports three wave families:

1-waves

$\lambda_1 = -c$

2-waves

$\lambda_2 = 0$

3-waves

$\lambda_3 = c$

- 3 characteristic families associated with (λ_i, R_i) :

1-waves
(back)

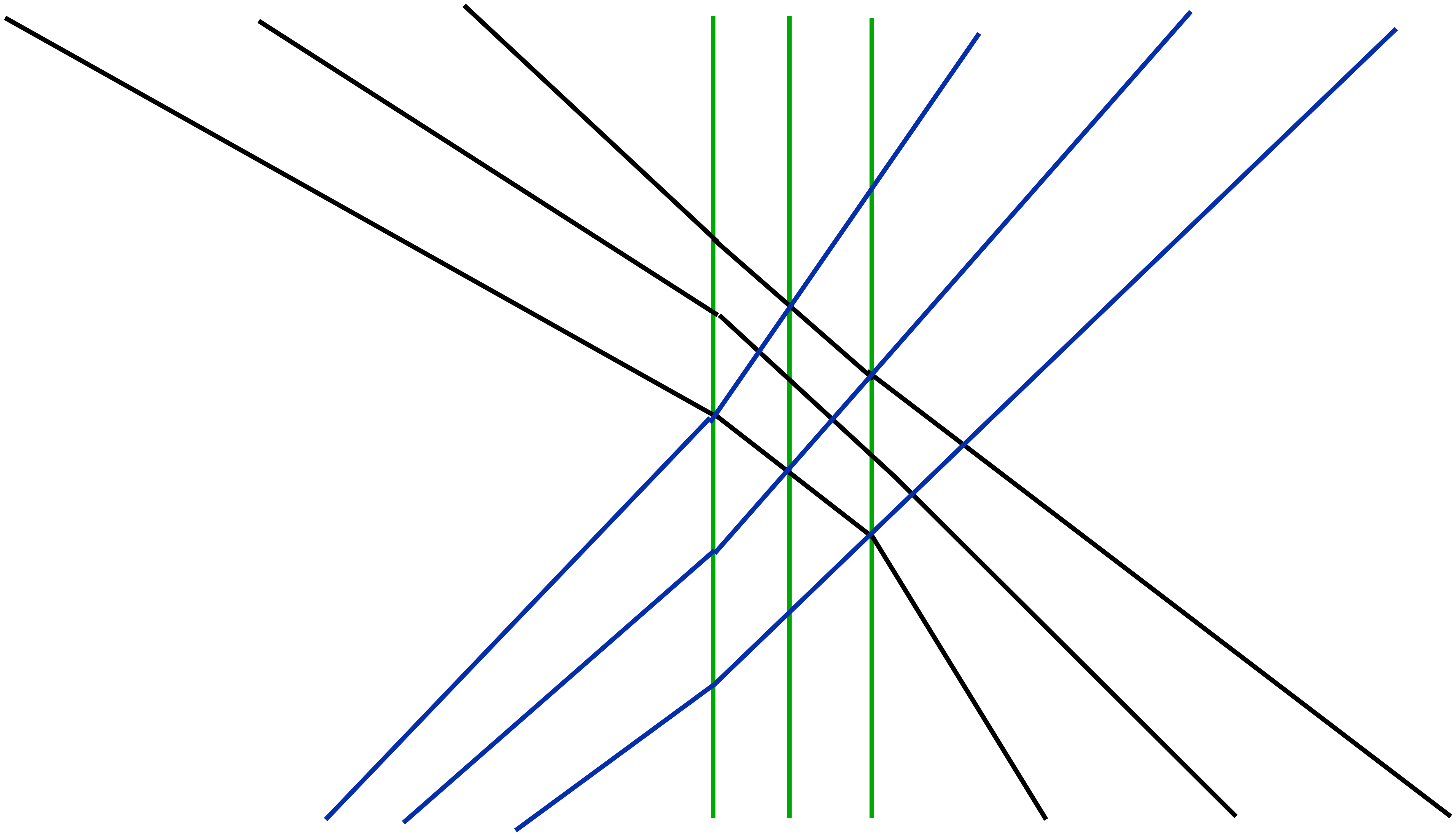
$$\frac{dx}{dt} = \lambda_1 = -c$$

2-waves

$$\frac{dx}{dt} = \lambda_2 = 0$$

3-waves
(forward)

$$\frac{dx}{dt} = \lambda_3 = c$$



3-waves

Genuinely
Nonlinear

$$\nabla_{R_1} \lambda_1 > 0$$

2-waves

Linearly
Degenerate

$$\nabla_{R_2} \lambda_2 \equiv 0$$

1-waves

Genuinely
Nonlinear

$$\nabla_{R_3} \lambda_3 > 0$$

- Three eigen-families of dF ...

1-waves

$$\boxed{\begin{array}{c}\lambda_1 = -c \\ R_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}\end{array}}$$

2-waves

$$\boxed{\begin{array}{c}\lambda_2 = 0 \\ R_2 = \begin{pmatrix} -p_S/p_\tau \\ 0 \\ 1 \end{pmatrix}\end{array}}$$

3-waves

$$\boxed{\begin{array}{c}\lambda_3 = c \\ R_3 = \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix}\end{array}}$$

Conclude:

$$\boxed{S \text{ is constant through 1,3-waves}}$$

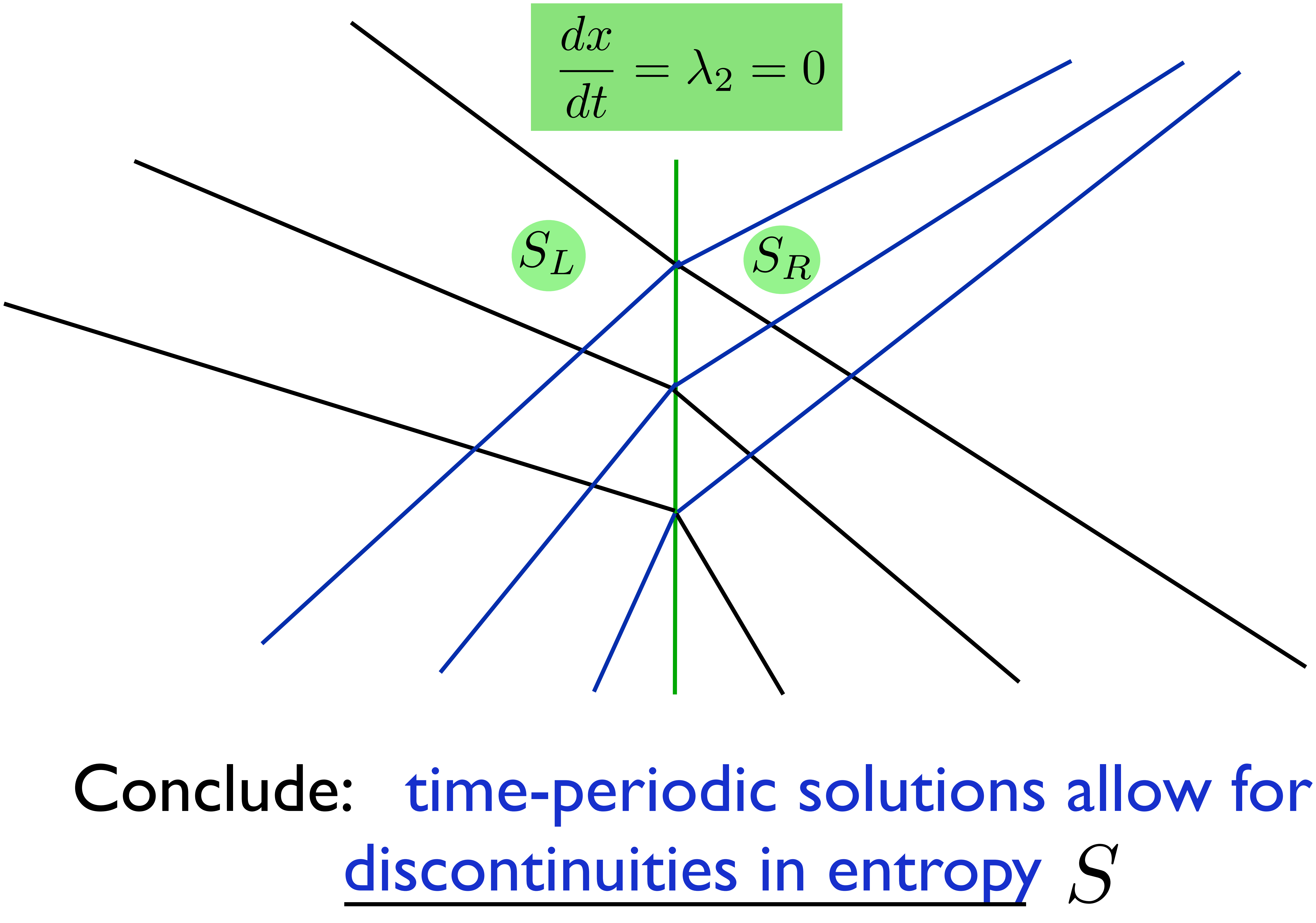
$$\boxed{u, p \text{ are constant through 2-waves}}$$

- The 2-field (λ_2, R_2) is Linearly Degenerate:

$$\nabla_{R_2}\lambda_2 \equiv 0$$

2-waves can be rescaled into time-reversible contact discontinuities

2-contact discontinuity



Conclude: time-periodic solutions allow for discontinuities in entropy S

Compressive and Rarefactive Waves (R/C)

Consider 1,3-waves at constant entropy S:

$$\text{1-wave} \equiv \text{“}backward\text{”-wave}$$

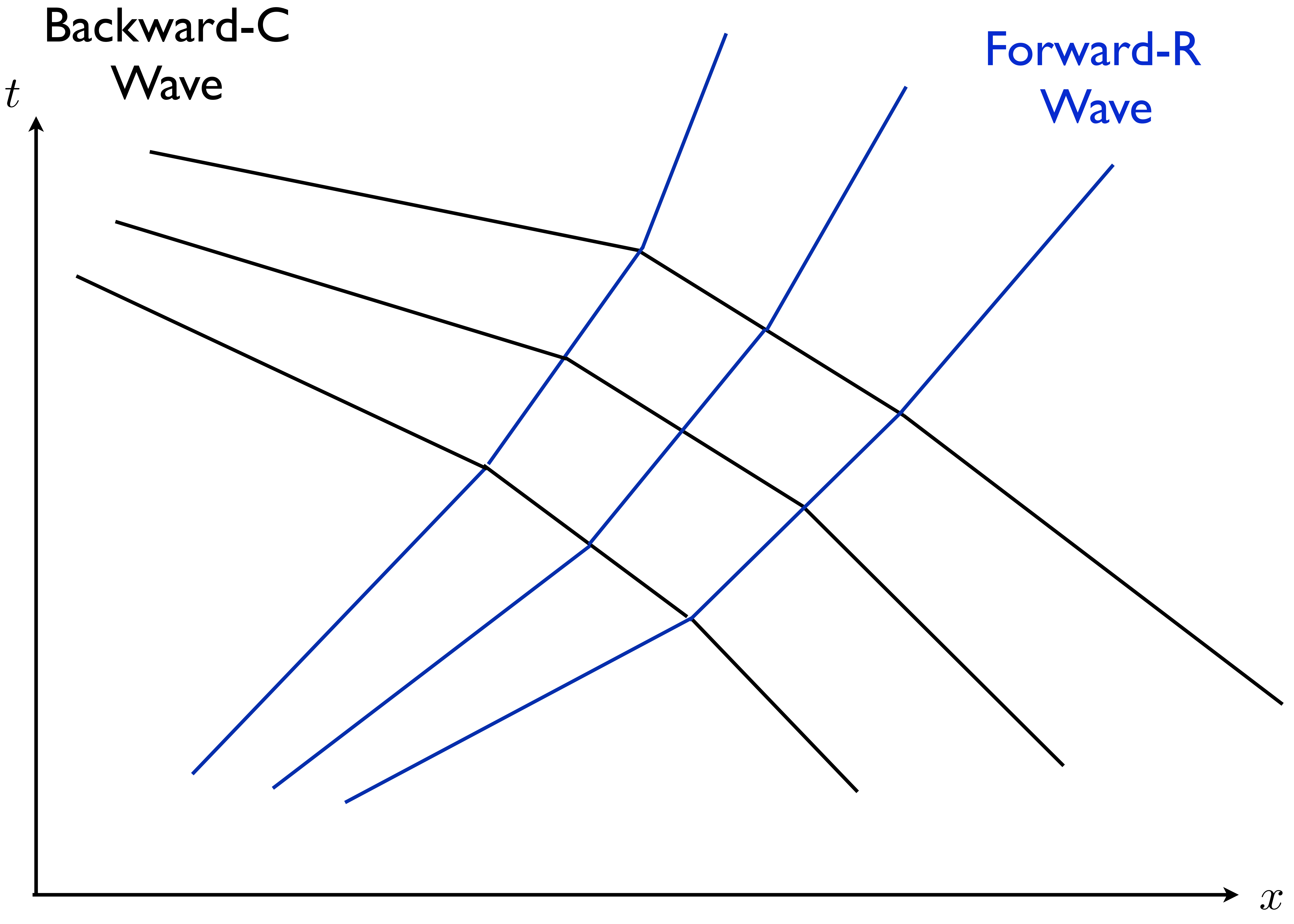
$$\text{3-wave} \equiv \text{“}forward\text{”-wave}$$

Definition: The R/C character of a wave in a general smooth solution is defined (pointwise) by:

$$\begin{array}{rcl} \text{Forward } R & \text{iff} & s_t \leq 0, \\ \text{Forward } C & \text{iff} & s_t \geq 0, \\ \text{Backward } R & \text{iff} & r_t \geq 0, \\ \text{Backward } C & \text{iff} & r_t \leq 0. \end{array}$$

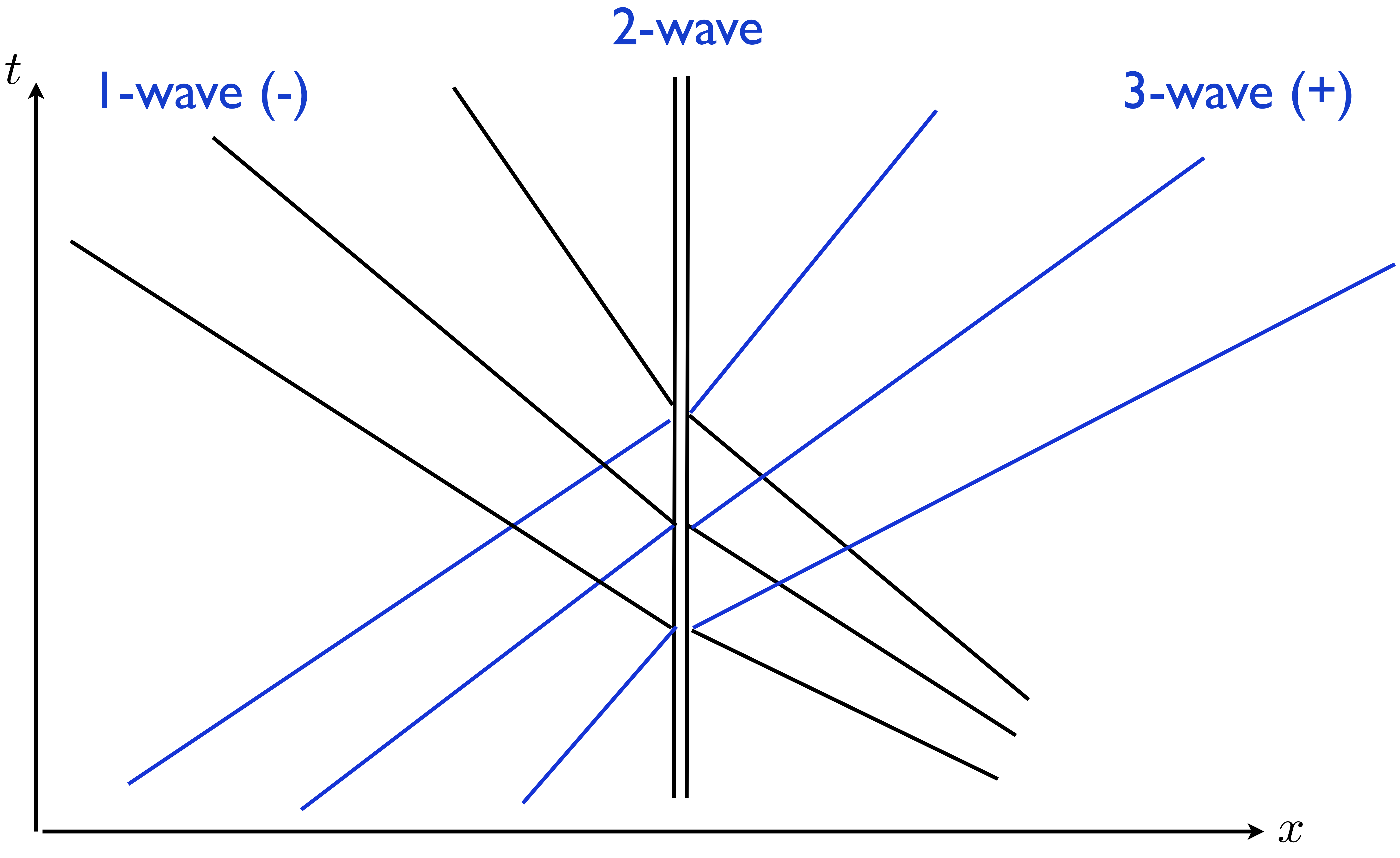
When the ENTROPY is CONSTANT...

Theorem: R/C character is preserved along backward and forward characteristics



The R/C character of a wave
CAN CHANGE
at an entropy jump...

For Example:



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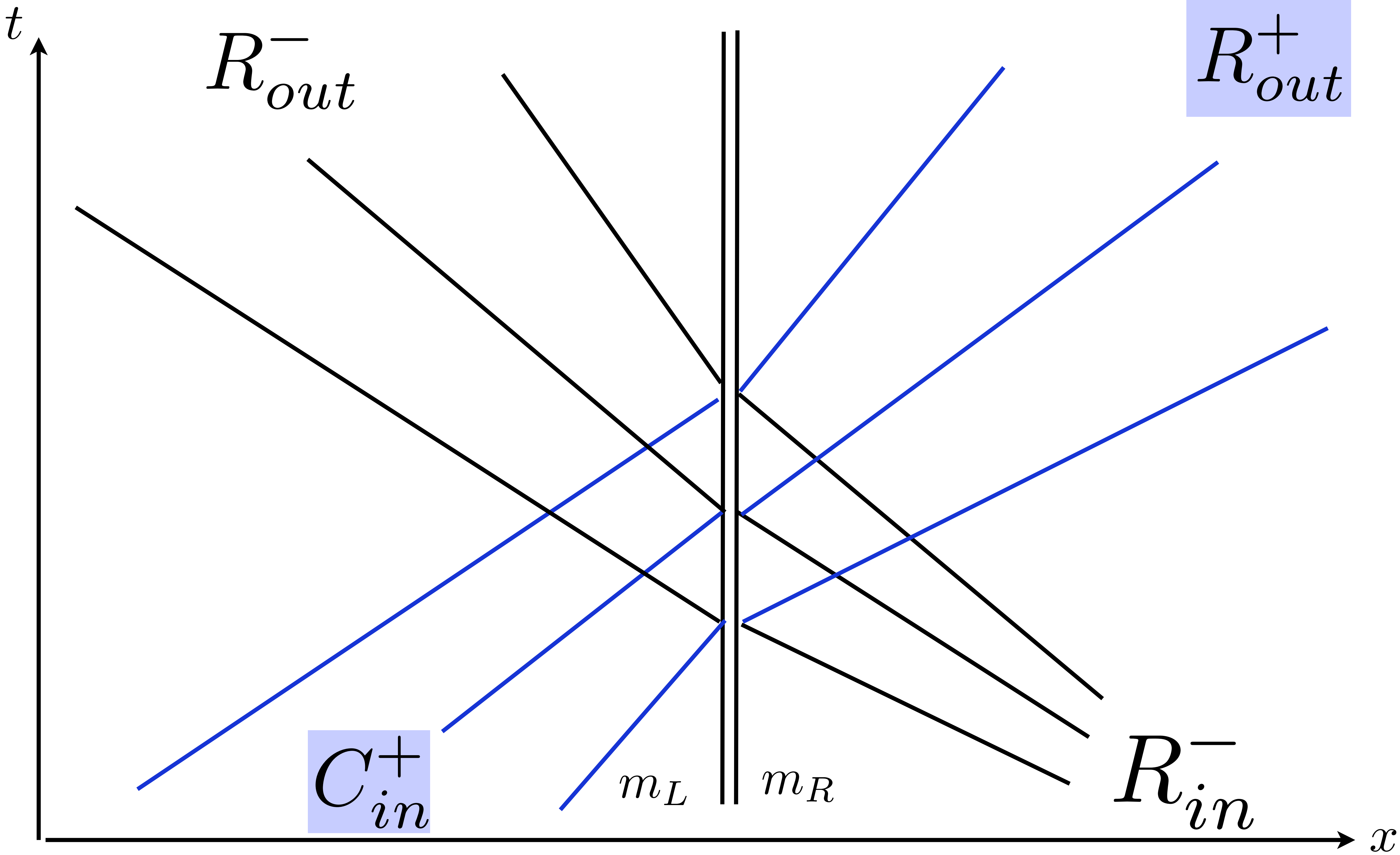

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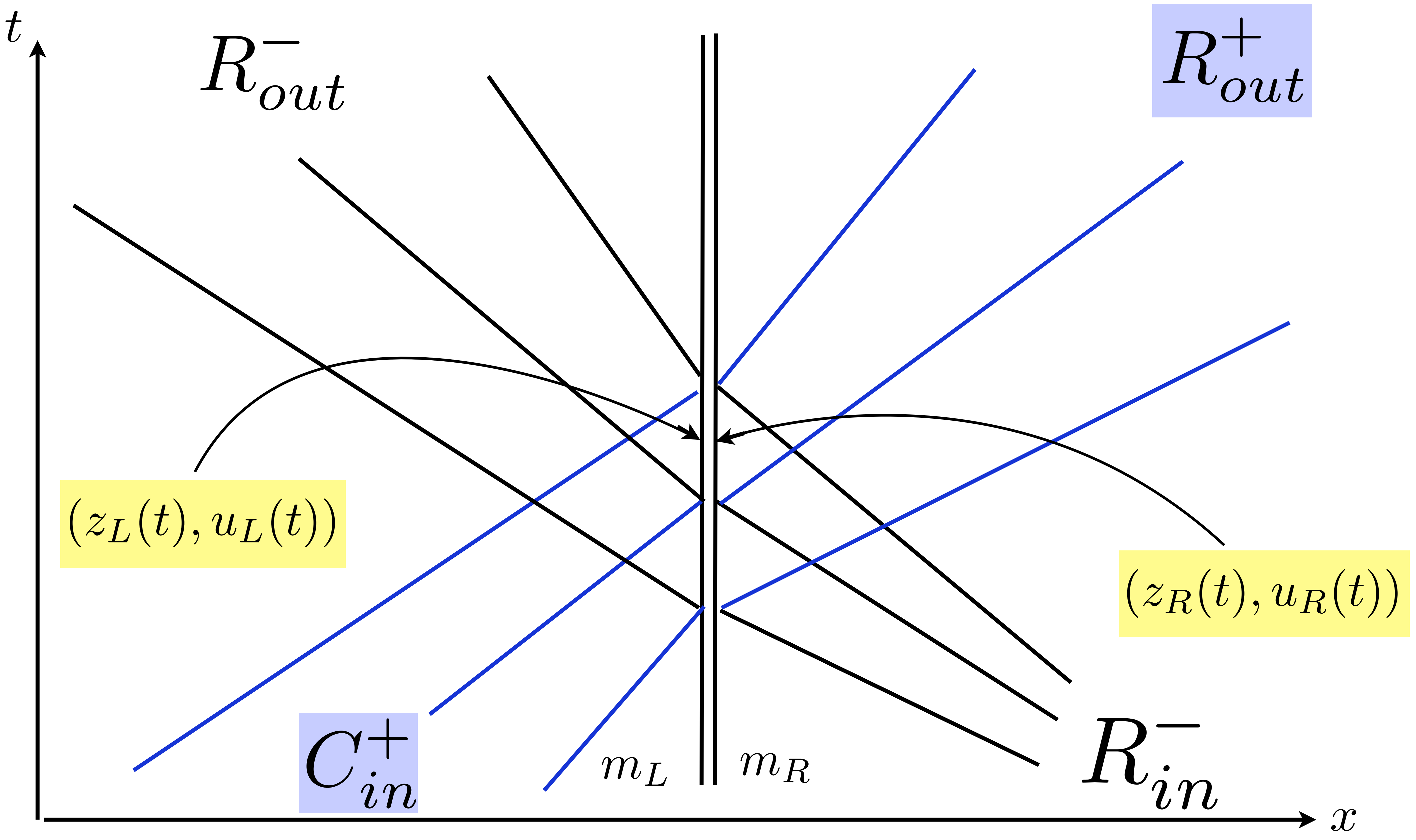


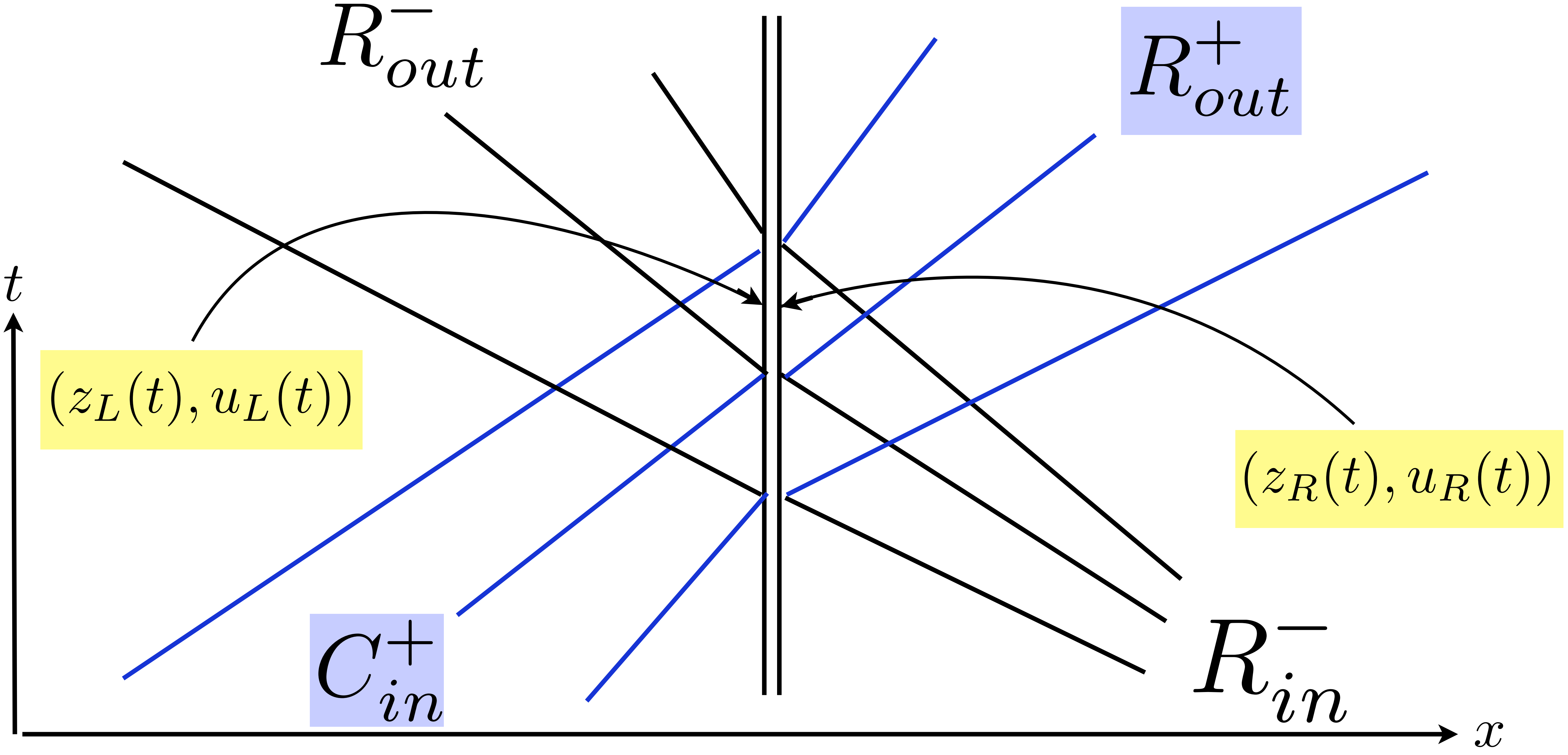
The Rankine-Hugoniot jump conditions characterize how R/C changes at an entropy jump...

Theorem 5. *The following inequalities characterize when a nonlinear wave changes its R/C value at an entropy jump:*

$$\begin{array}{lcr} R^-_{in} \to C^-_{out} & \text{iff} & q^R_L m_L \dot{z}_L < \dot{u}_L < m_L \dot{z}_L, \\ C^-_{in} \to R^-_{out} & \text{iff} & m_L \dot{z}_L < \dot{u}_L < q^R_L m_L \dot{z}_L, \\ R^+_{in} \to C^+_{out} & \text{iff} & -q^R_L m_L \dot{z}_L < \dot{u}_L < -m_L \dot{z}_L, \\ C^+_{in} \to R^+_{out} & \text{iff} & -m_L \dot{z}_L < \dot{u}_L < -q^R_L m_L \dot{z}_L. \end{array}$$

$$\boxed{q^R_L = \left(\frac{m_R}{m_L}\right)^{\frac{1}{\gamma}}}$$

For Example:



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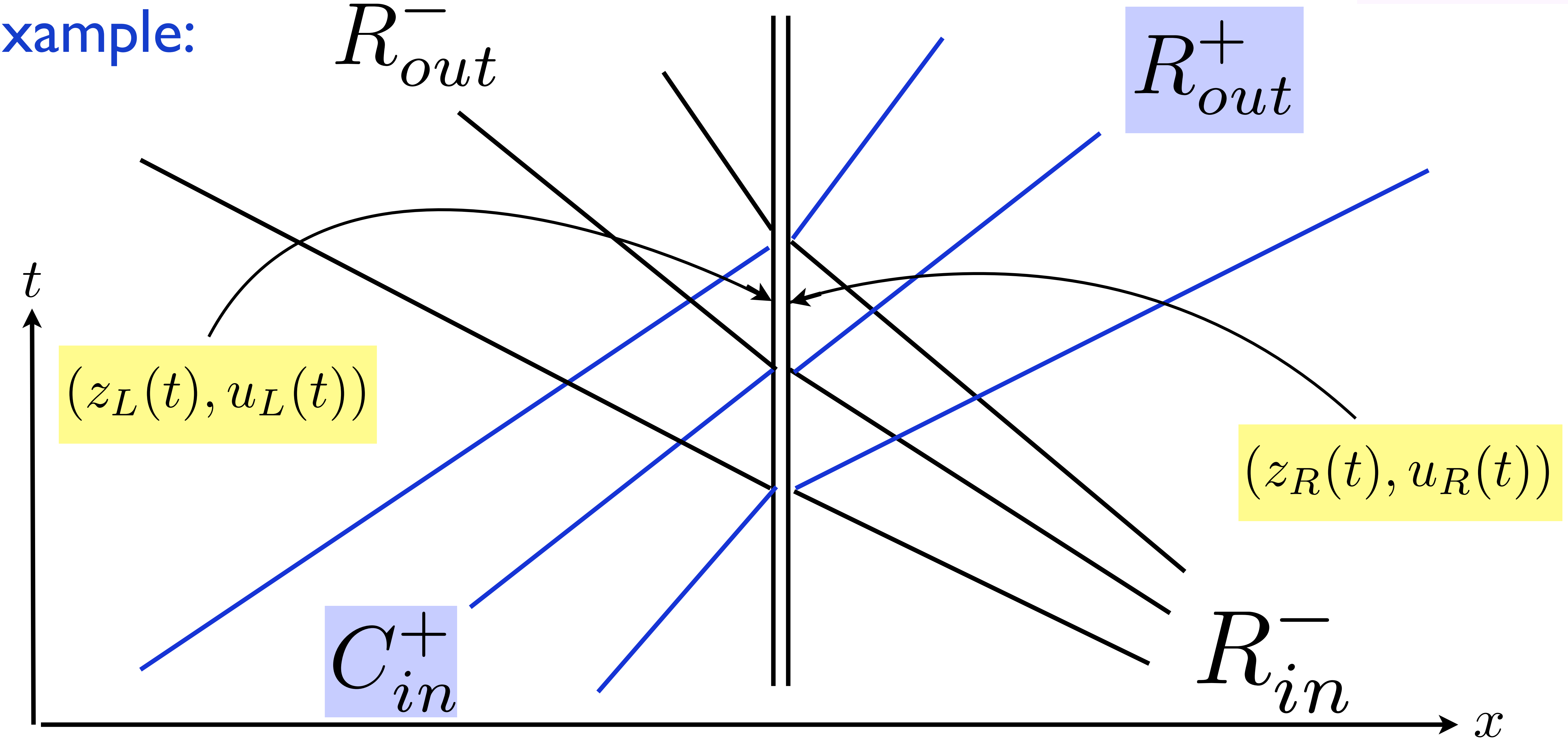
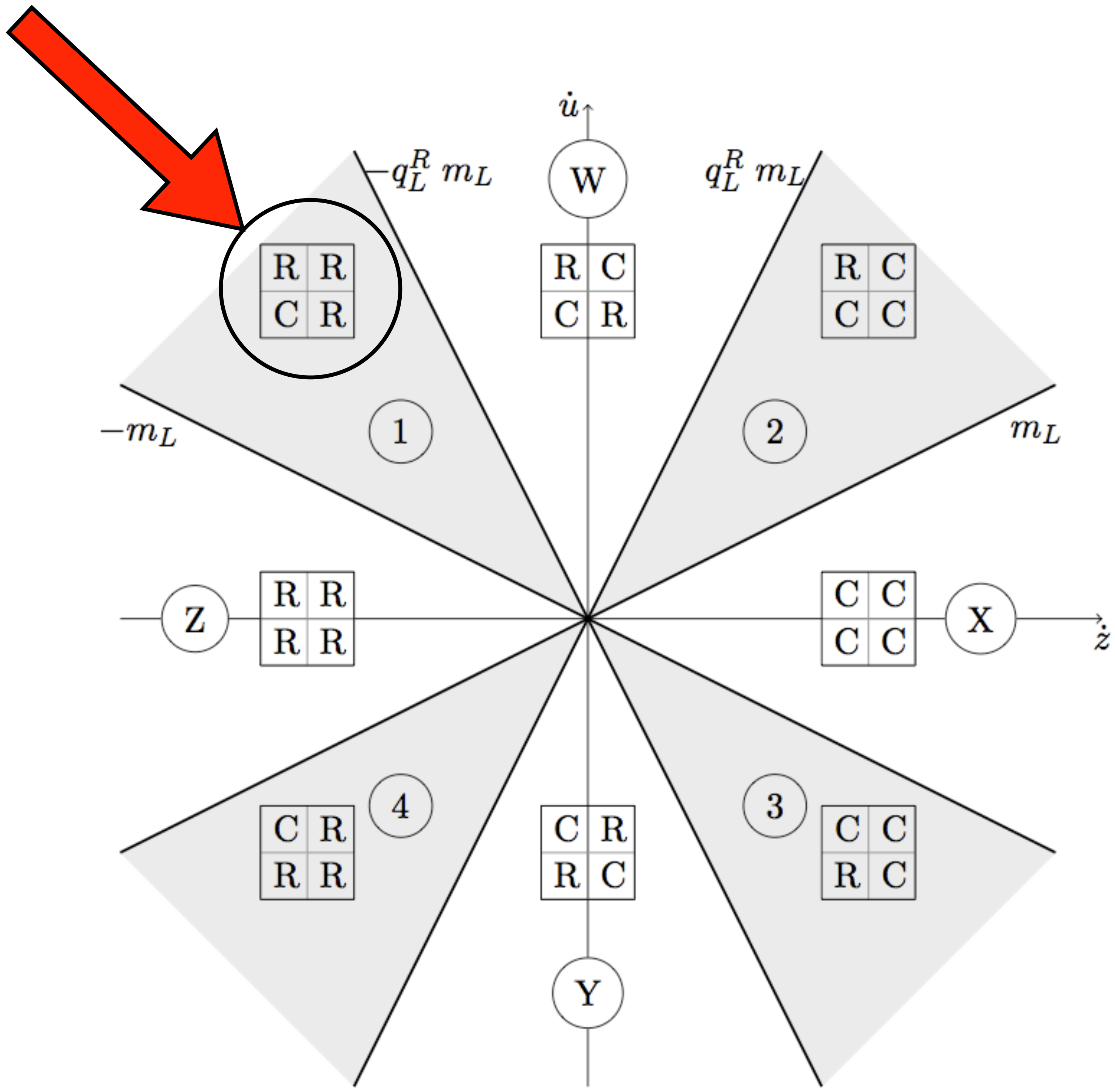
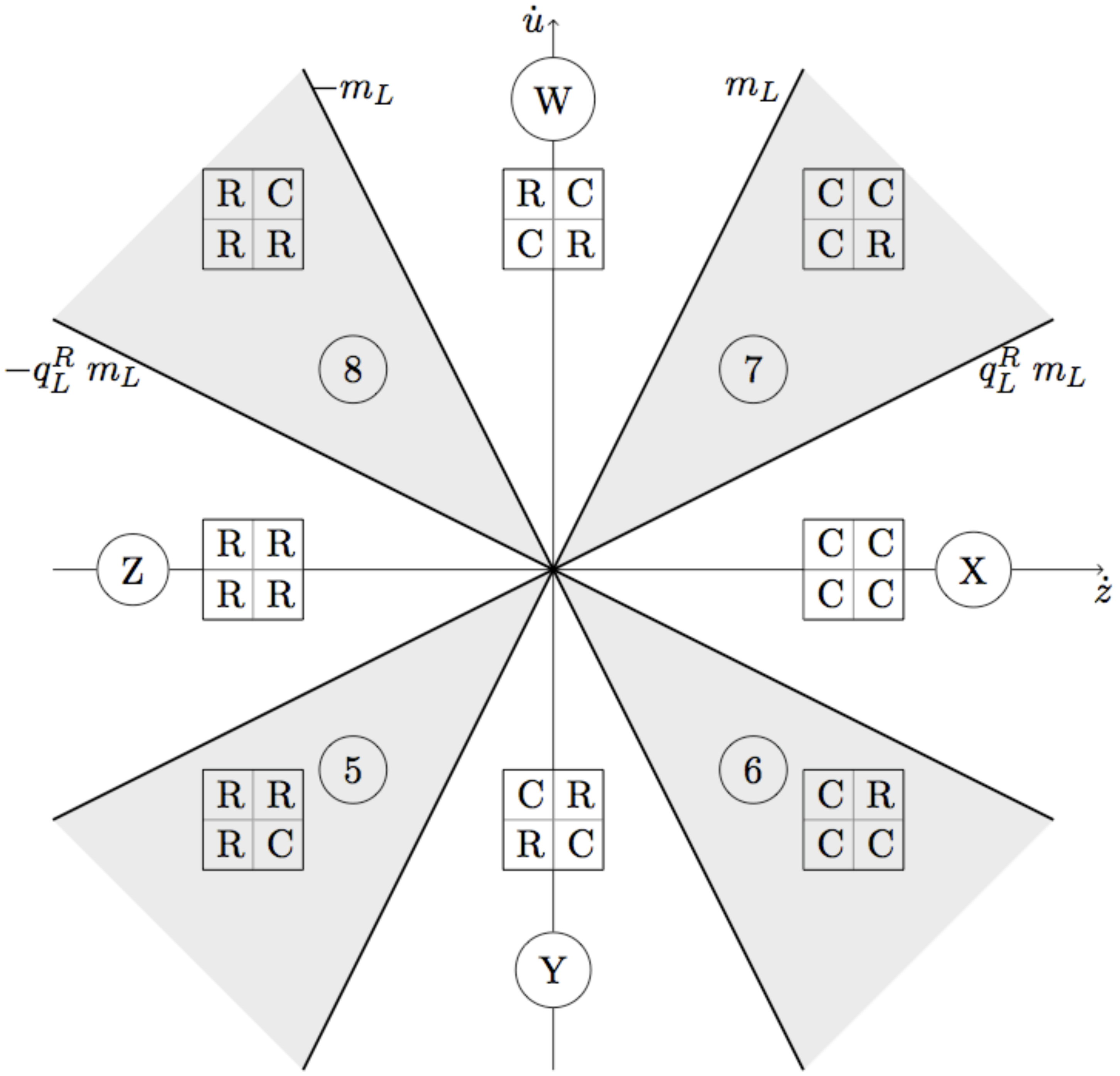


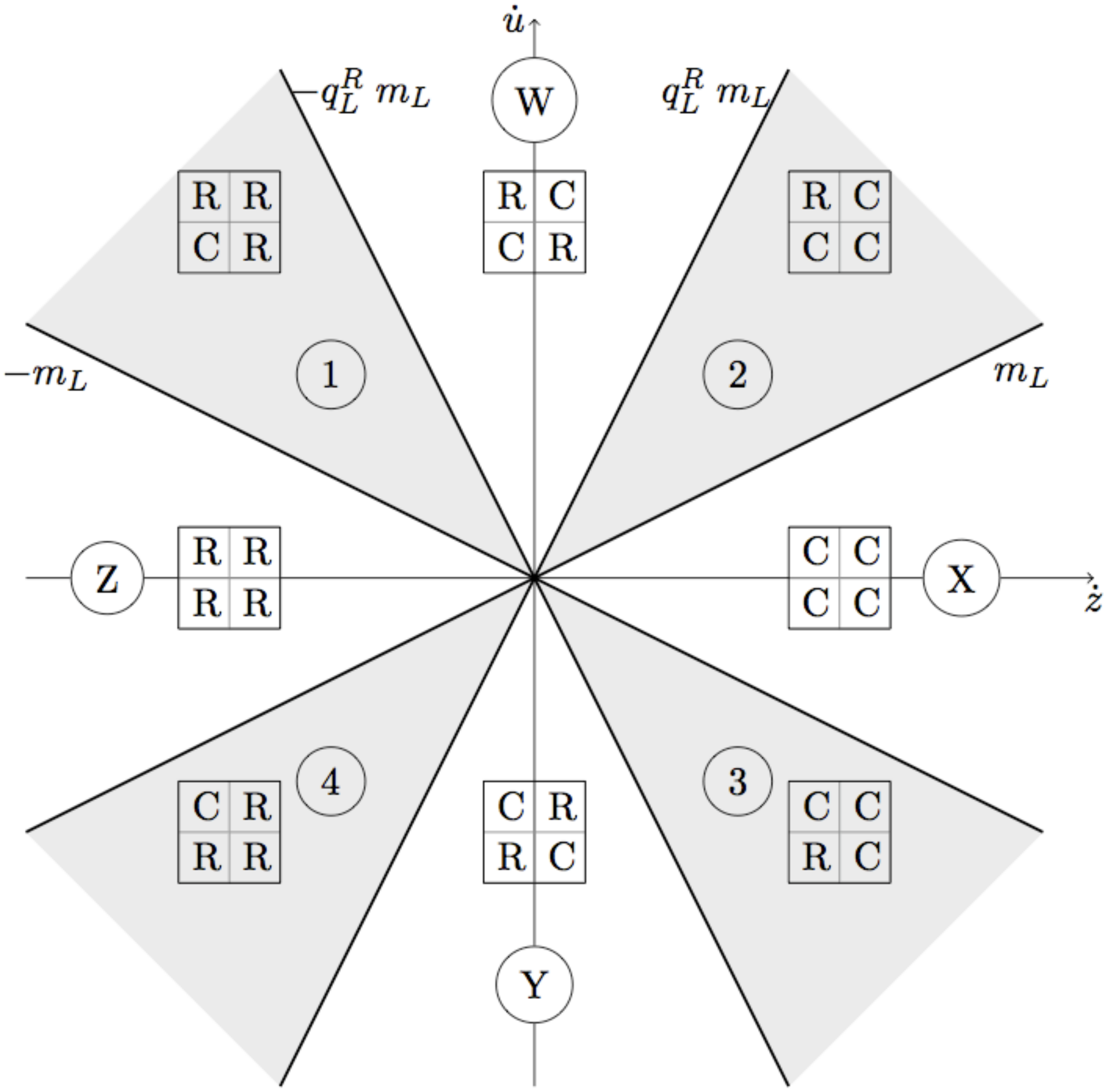


$m_L < m_R$



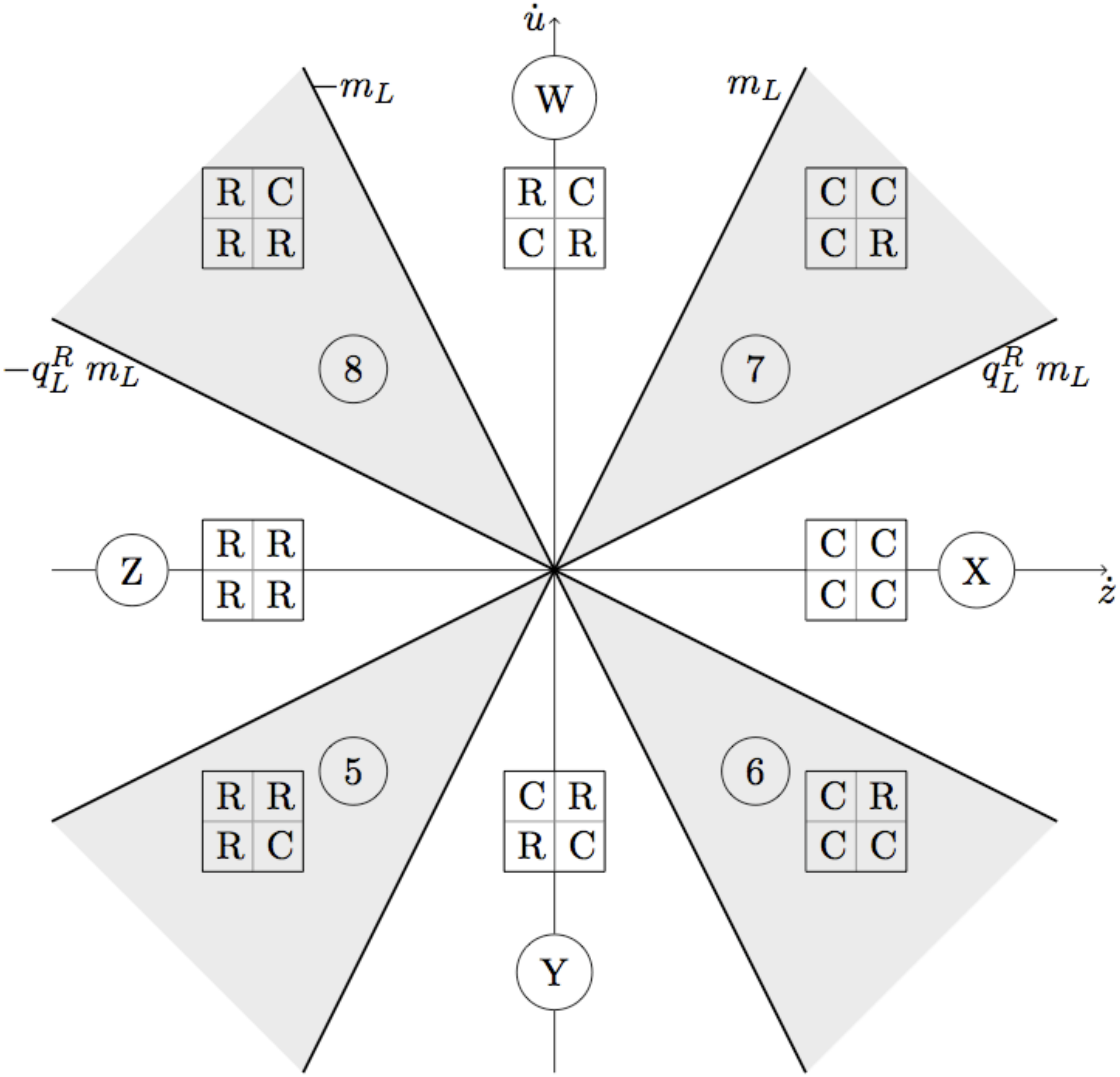
$m_L > m_R$

CONCLUDE: we can determine the R/C changes across the entropy jump from inequalities on the time derivative of the solution at the left hand side of the entropy jump alone. Doing this in all cases yields the following theorem.



Tangent space showing all possible R/C wave structures when

$m_L < m_R$



Tangent space showing all possible R/C wave structures when

$m_L > m_R$

- Note: All 16 possible interaction squares appear EXCEPT ones where R/C value of both waves change simultaneously:

Not possible:

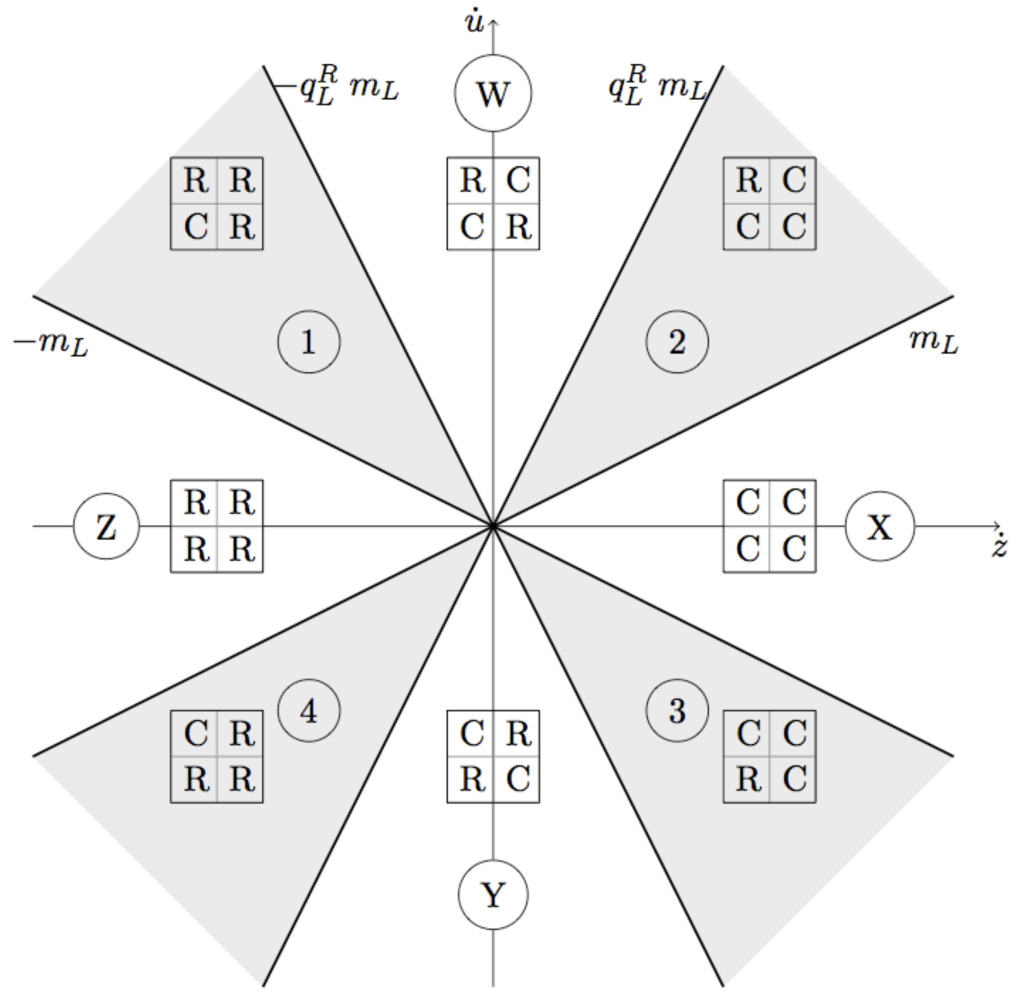
R	R
C	C

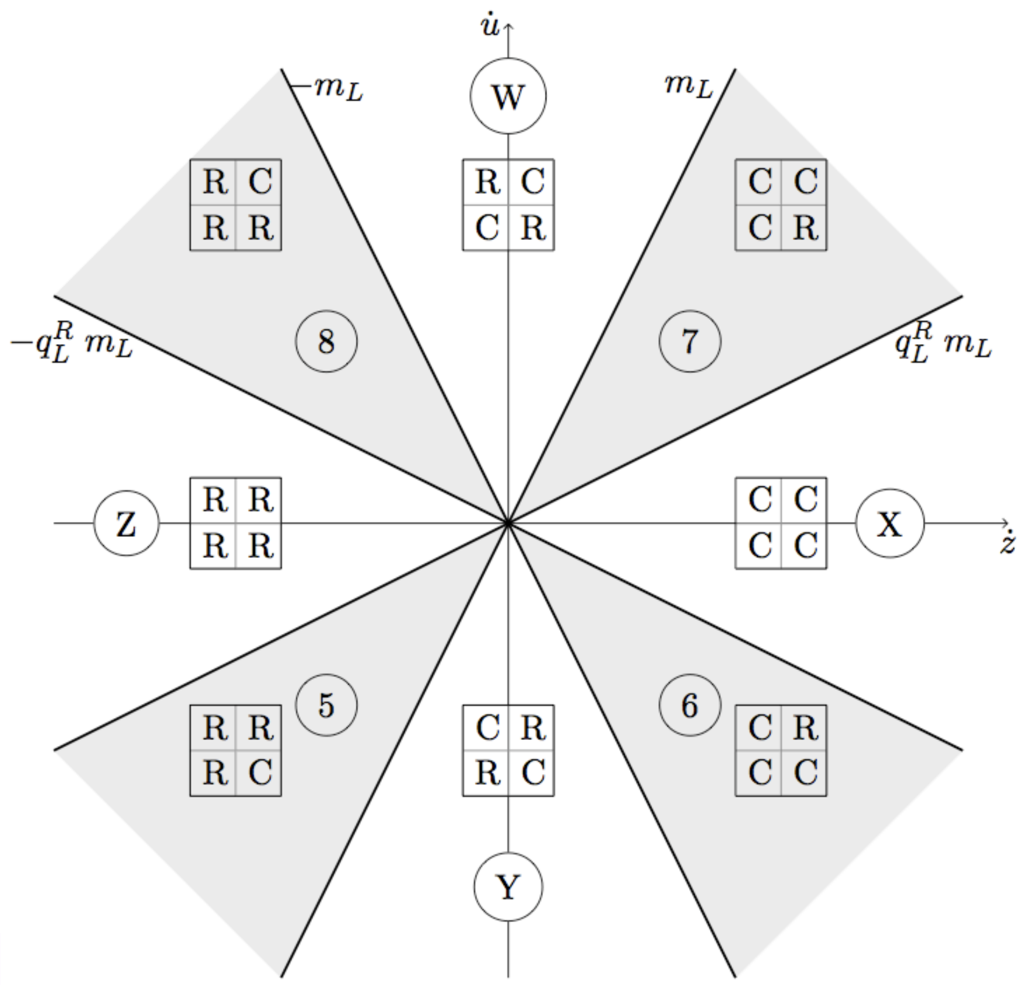
C	C
R	R

R	C
R	C

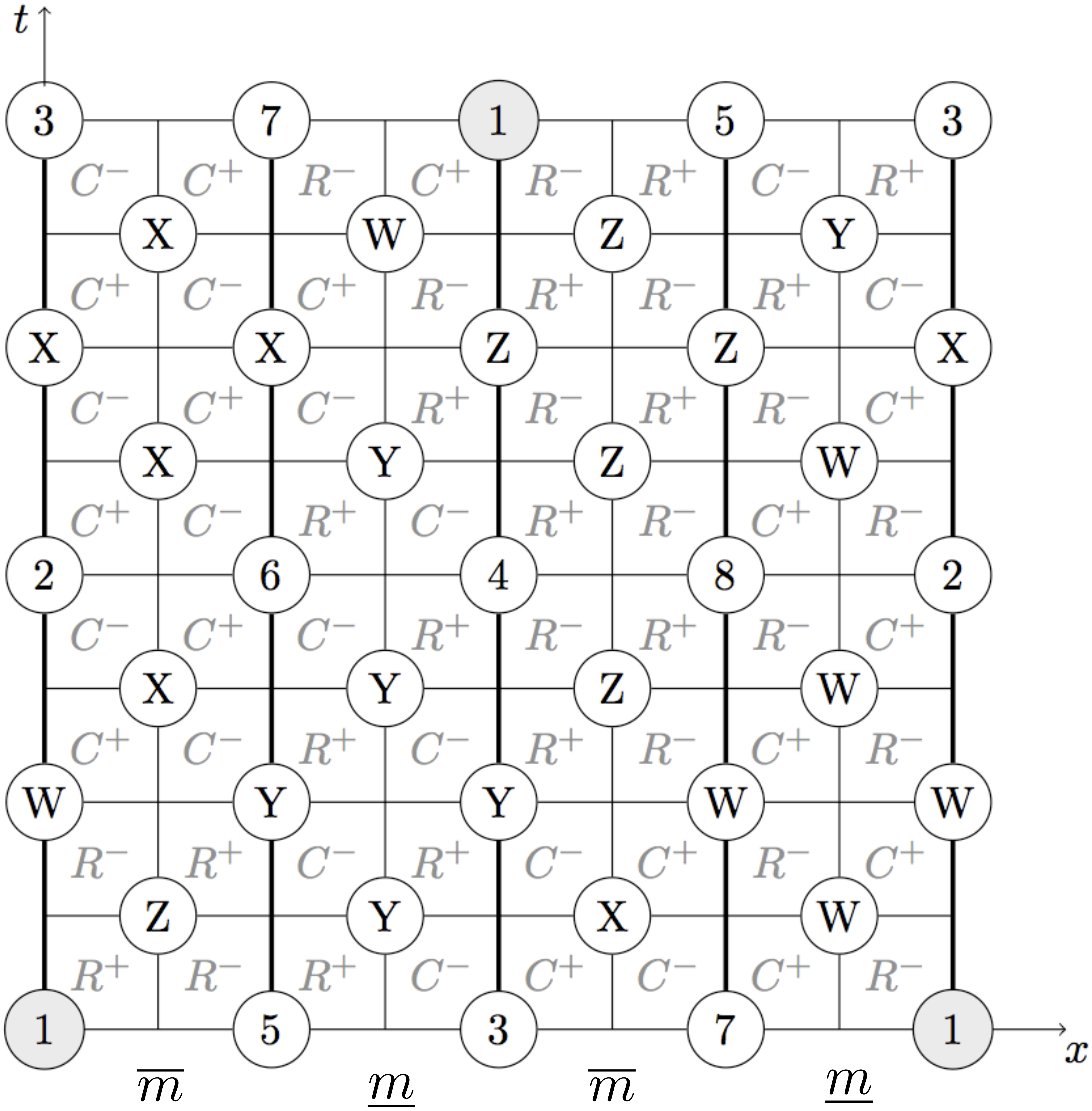
C	R
C	R

- CONCLUDE: A wave in one family can change its R/C value only in the presence of a wave of the opposite family that transmits its R/C value





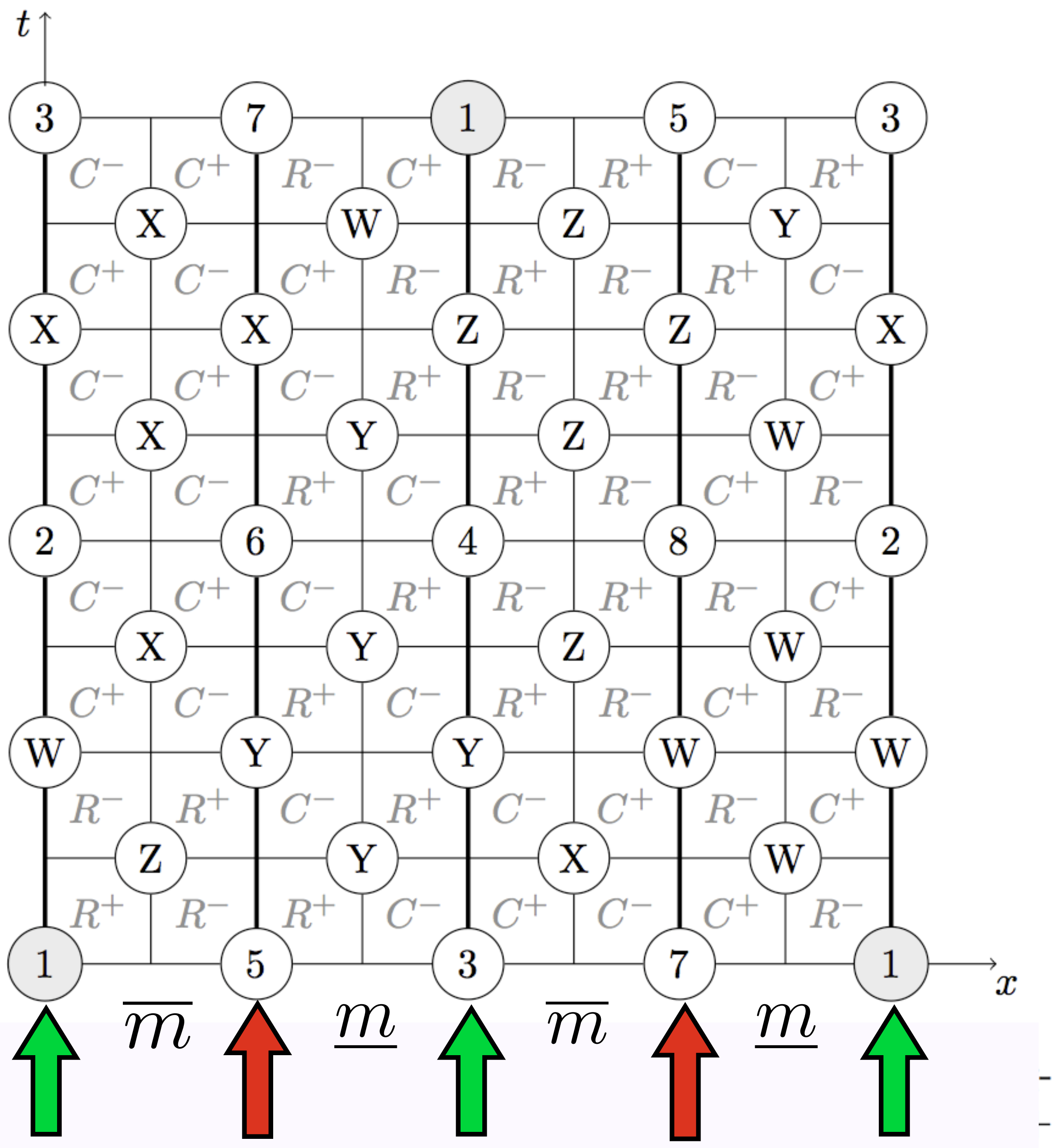
The simpest consistent R/C pattern



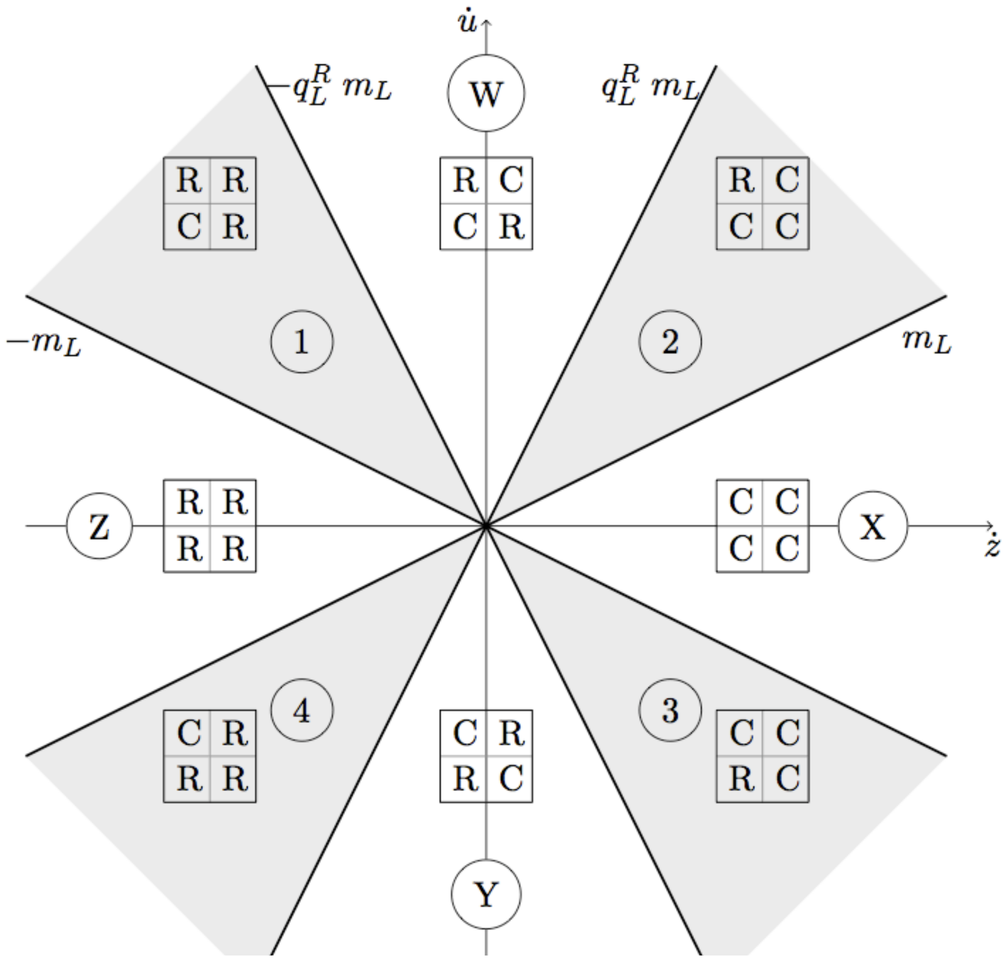
"Extend periodically"

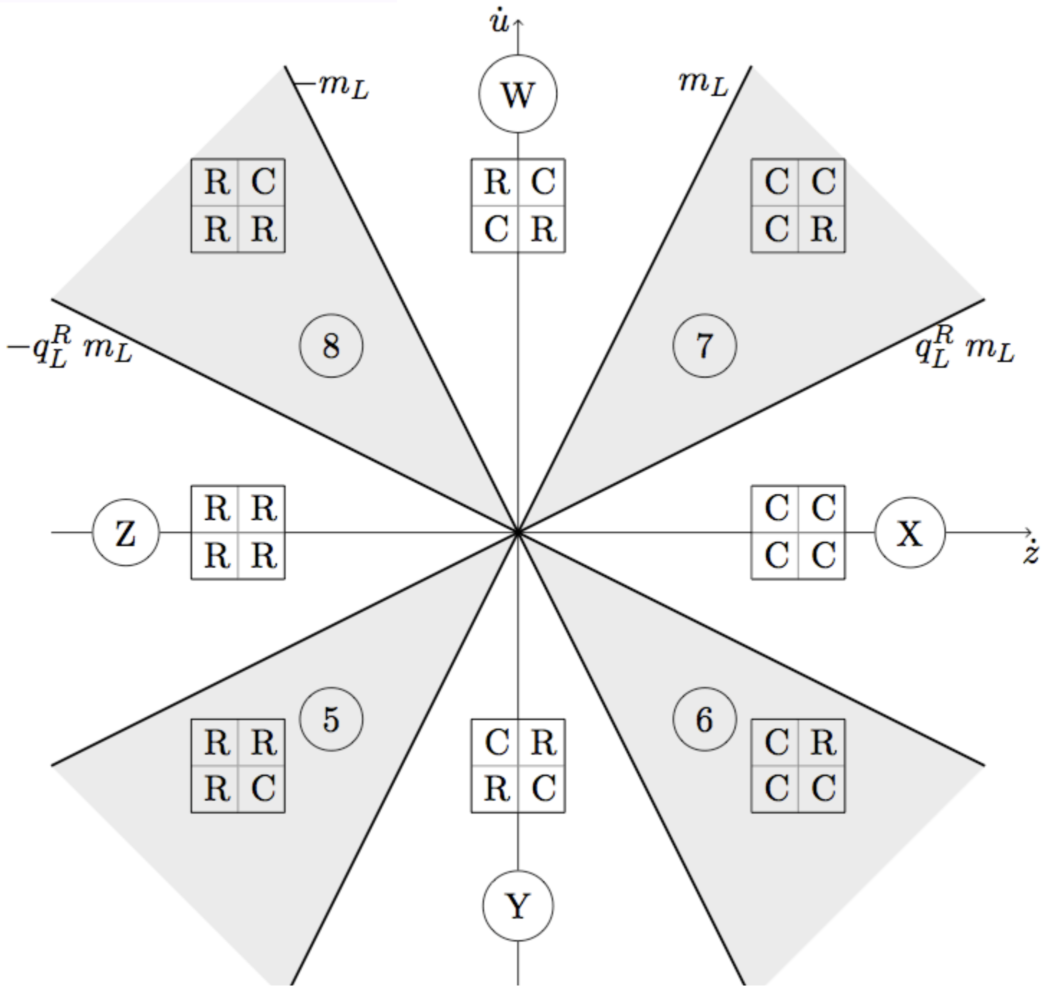
$$\overline{m} > \underline{m}$$

Each number above is consistent with the numbered interaction below



$\overline{m} > \underline{m}$

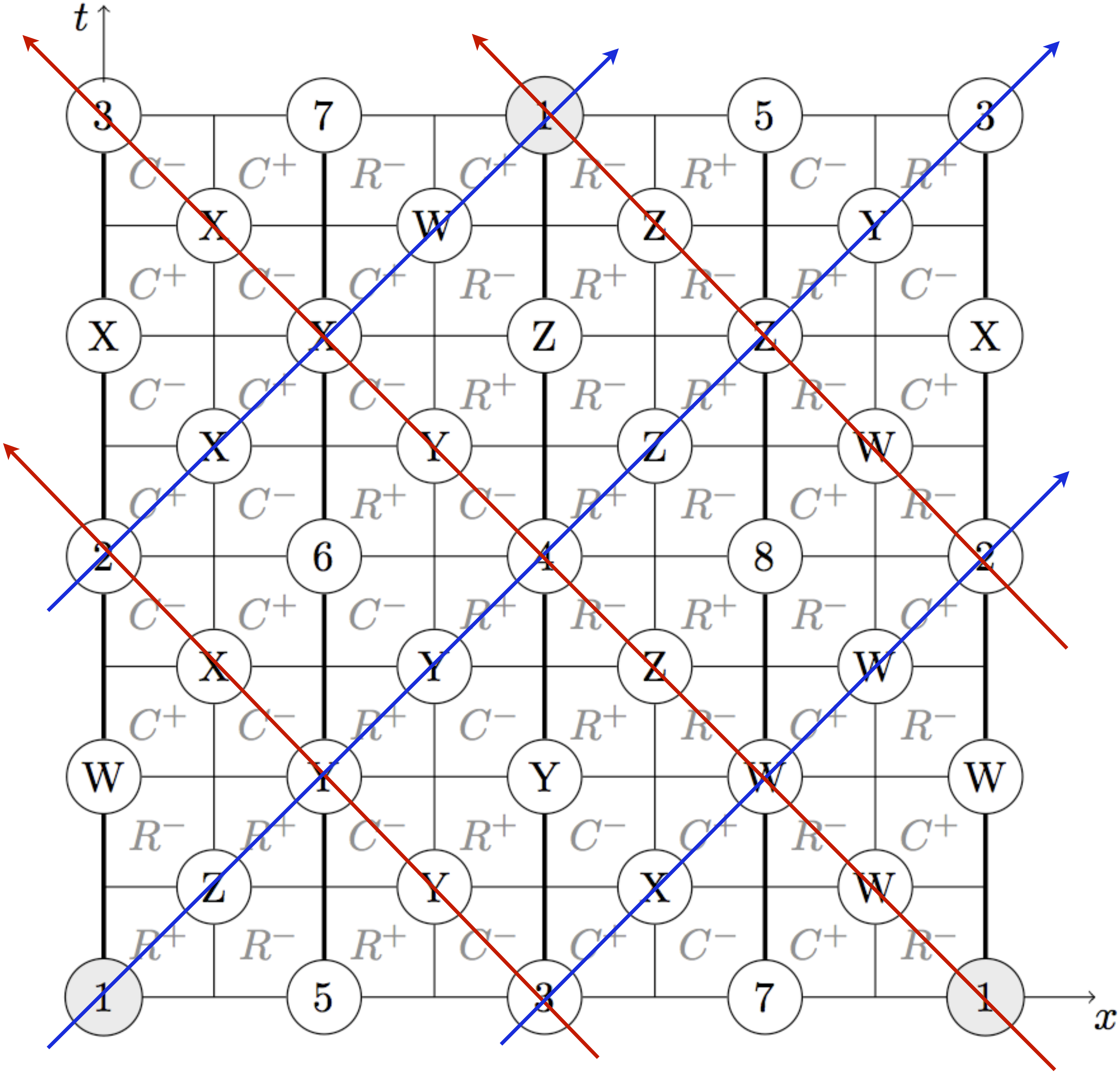




$\underline{m} = m_L < m_R = \overline{m}$

$\overline{m} = m_L > m_R = \underline{m}$

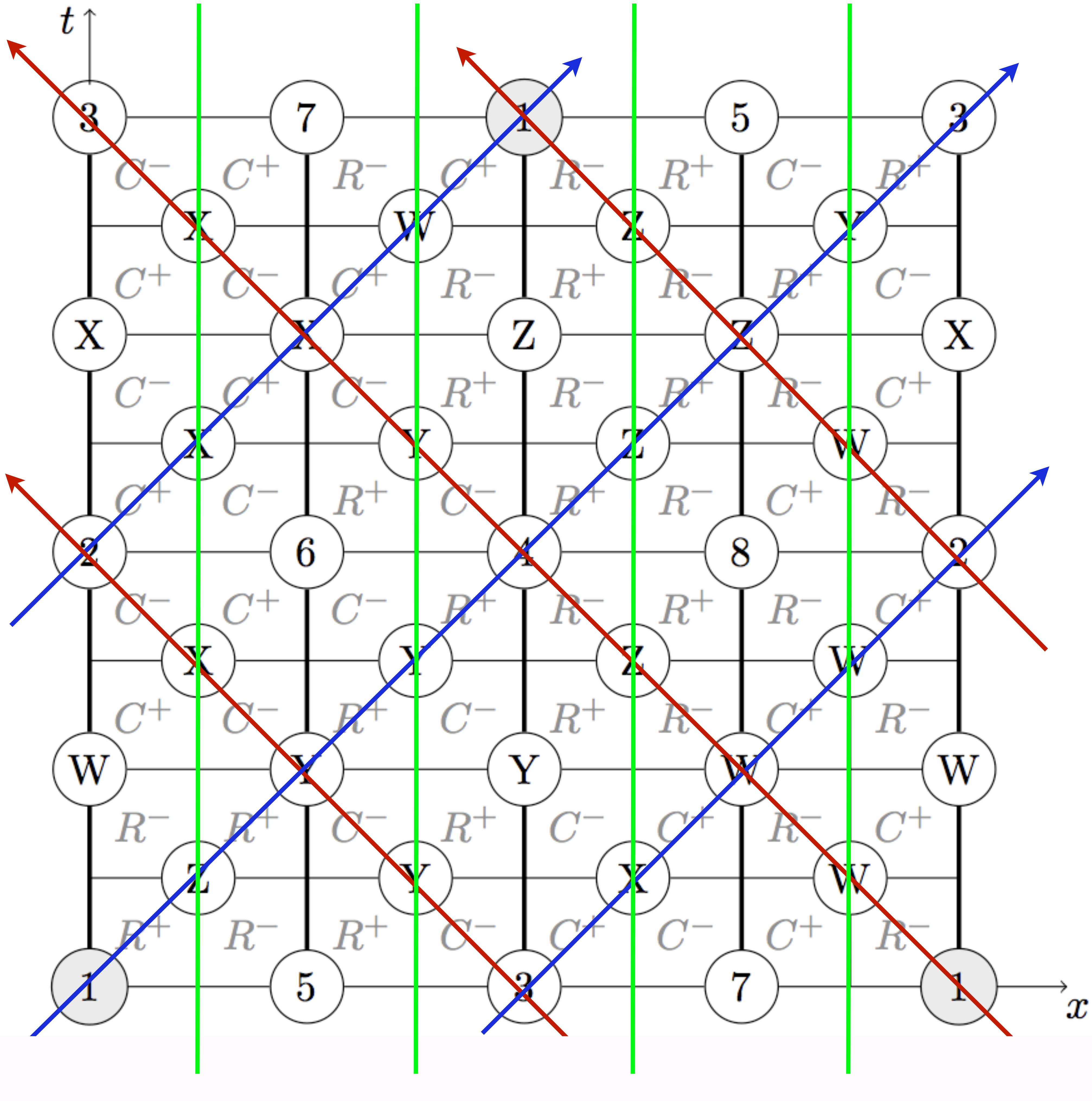
Each 1,3-characteristic traverses 8-C's and 8-R's before returning

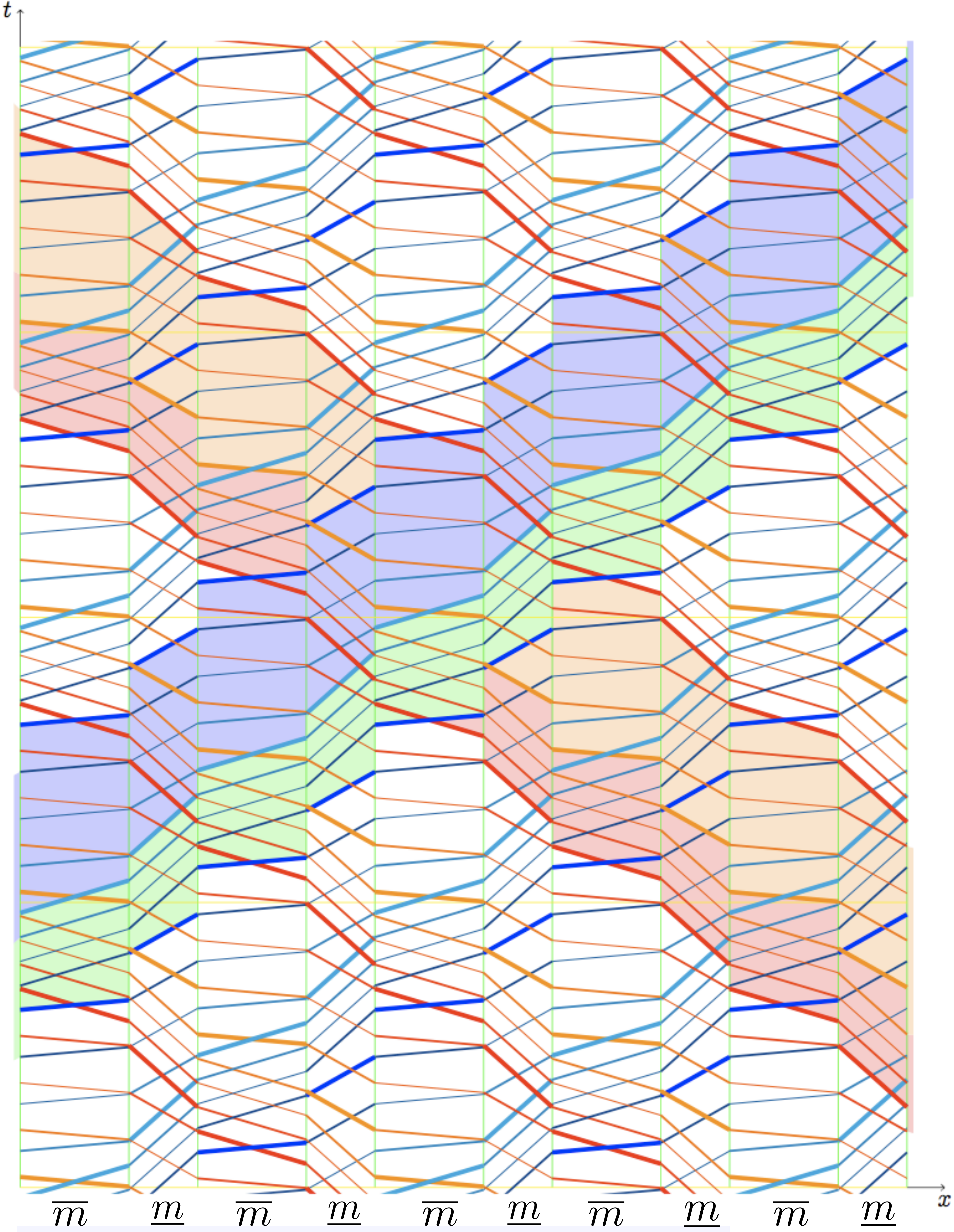


The lettered interactions at constant entropy jump transmit R/C

Identifying these

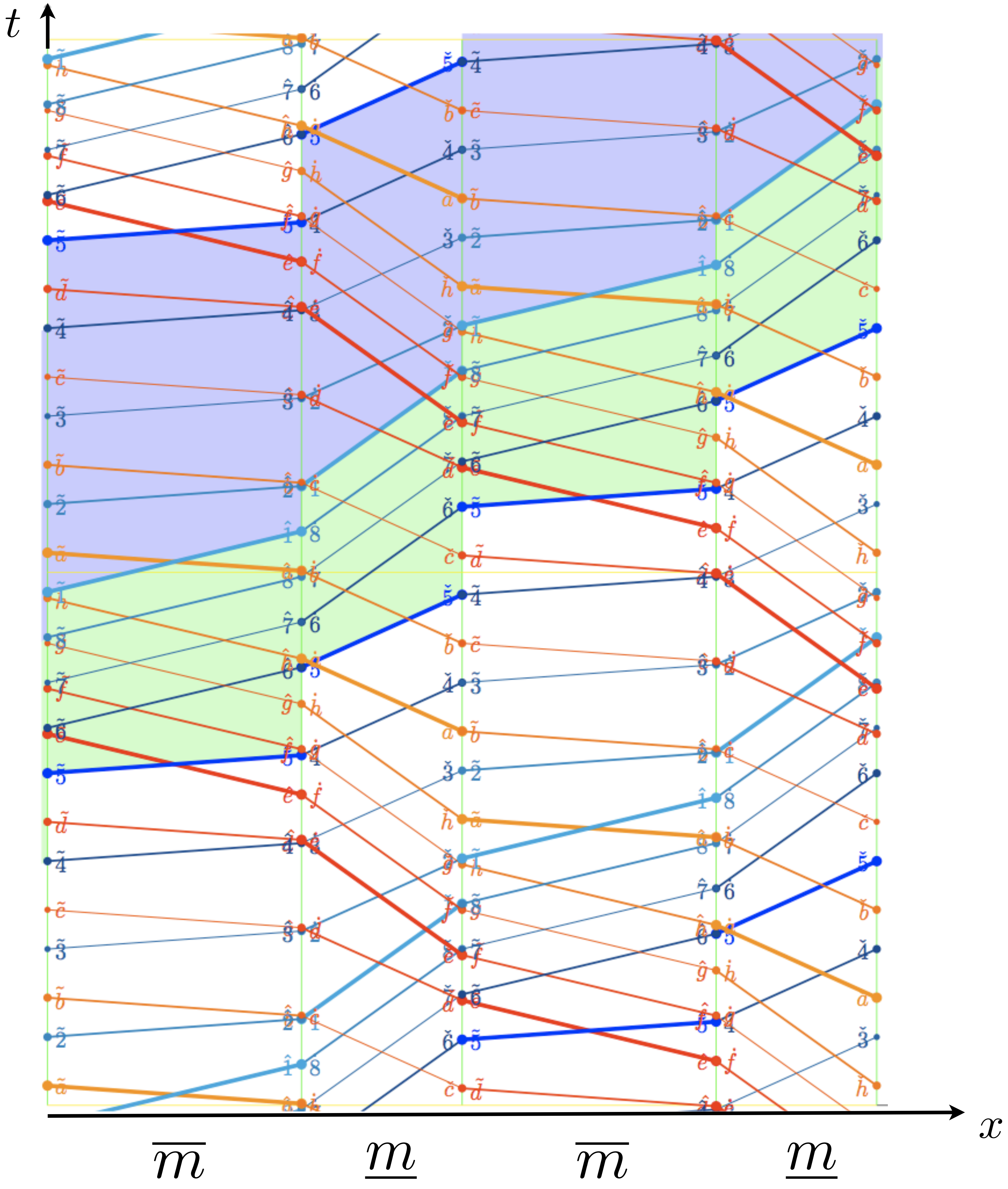
1,3-characteristics traverse 4-C's and 4-R's before returning





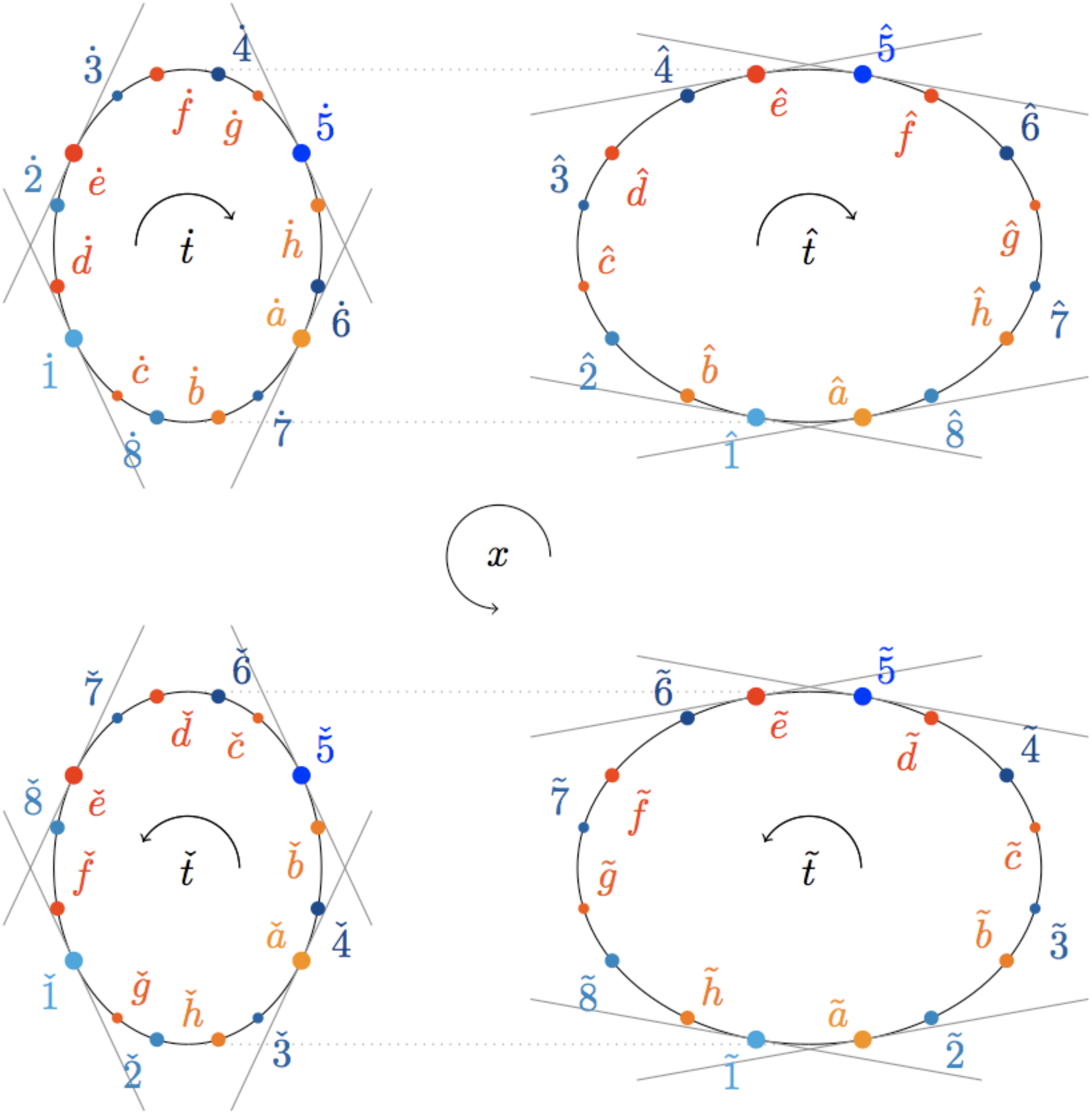
The simplest possible periodic structure

$$\overline{m} > \underline{m}$$

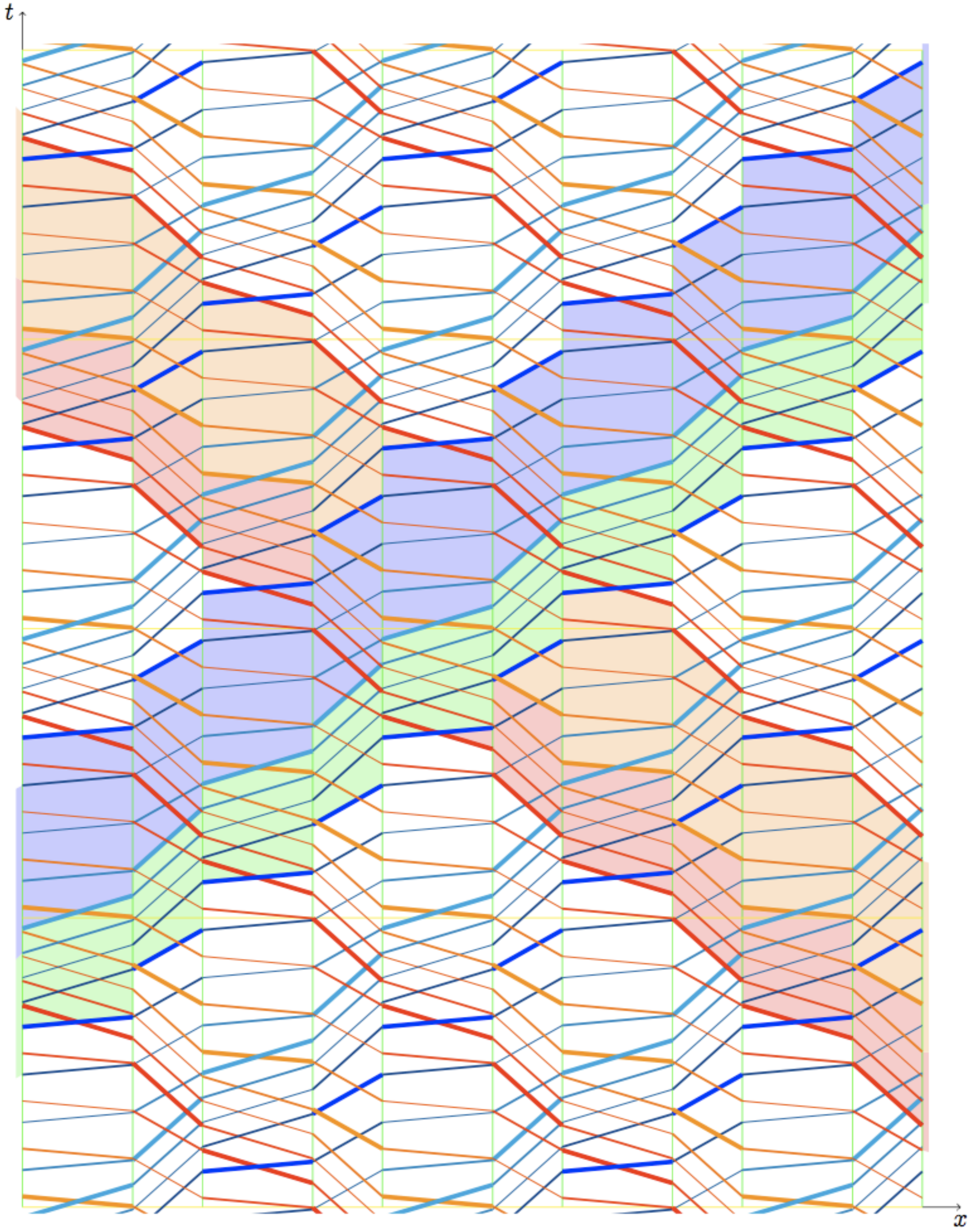


Labeling the states by numbers and letters

$\overline{m} > \underline{m}$



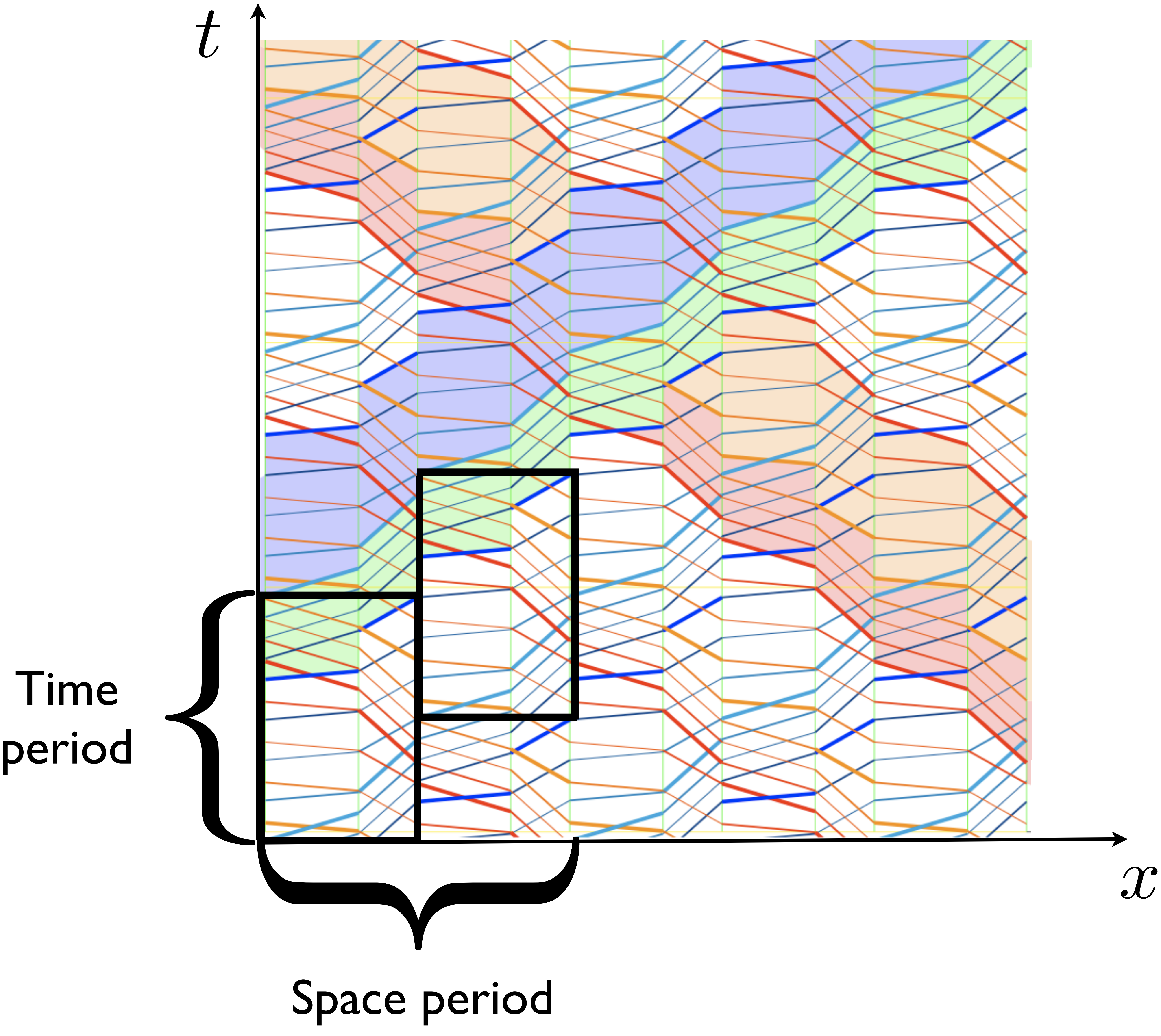
Ellipses showing periodicity in (z,u)-plane



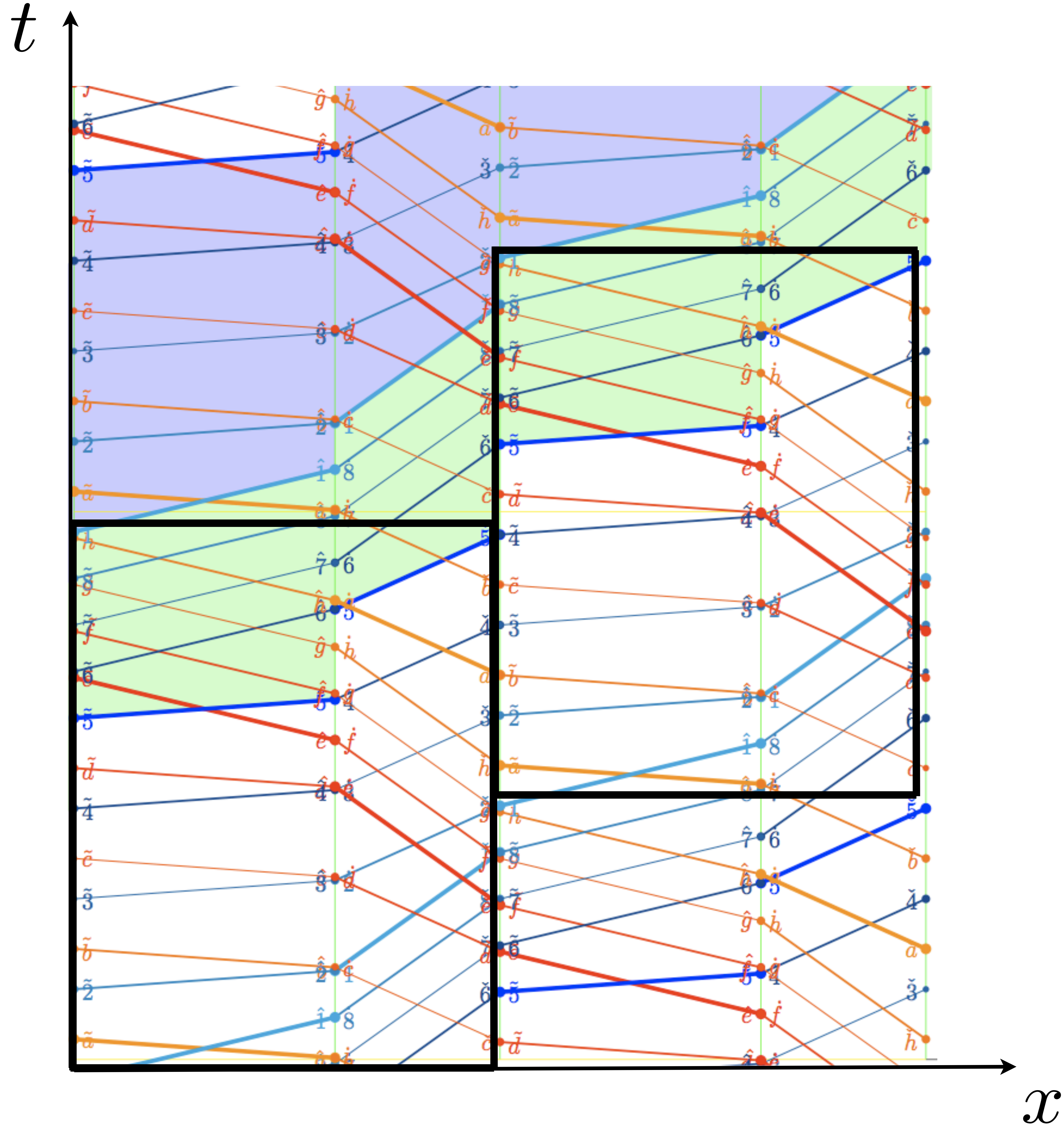
The global nonlinear periodic structure

The speed of the wave crests is like an effective
"Group-Velocity"

The characteristic=sound speed like a
"Phase-Velocity"



(1) Simplest structure is space-periodic



(1) Simplest structure is space-periodic

Inspection of the periodic structure indicates:

- Solution jumps between two entropy levels $\overline{m} > \underline{m}$
- Starting with time-periodic "initial data" U(t) at x=0, solution evolves through five operations before periodic return:

(1) $\overline{\mathcal{E}}$: Nonlinear evolution at $m = \overline{m}$

(2) $\mathcal{J}$: Jump from $m = \overline{m}$ to $m = \underline{m}$

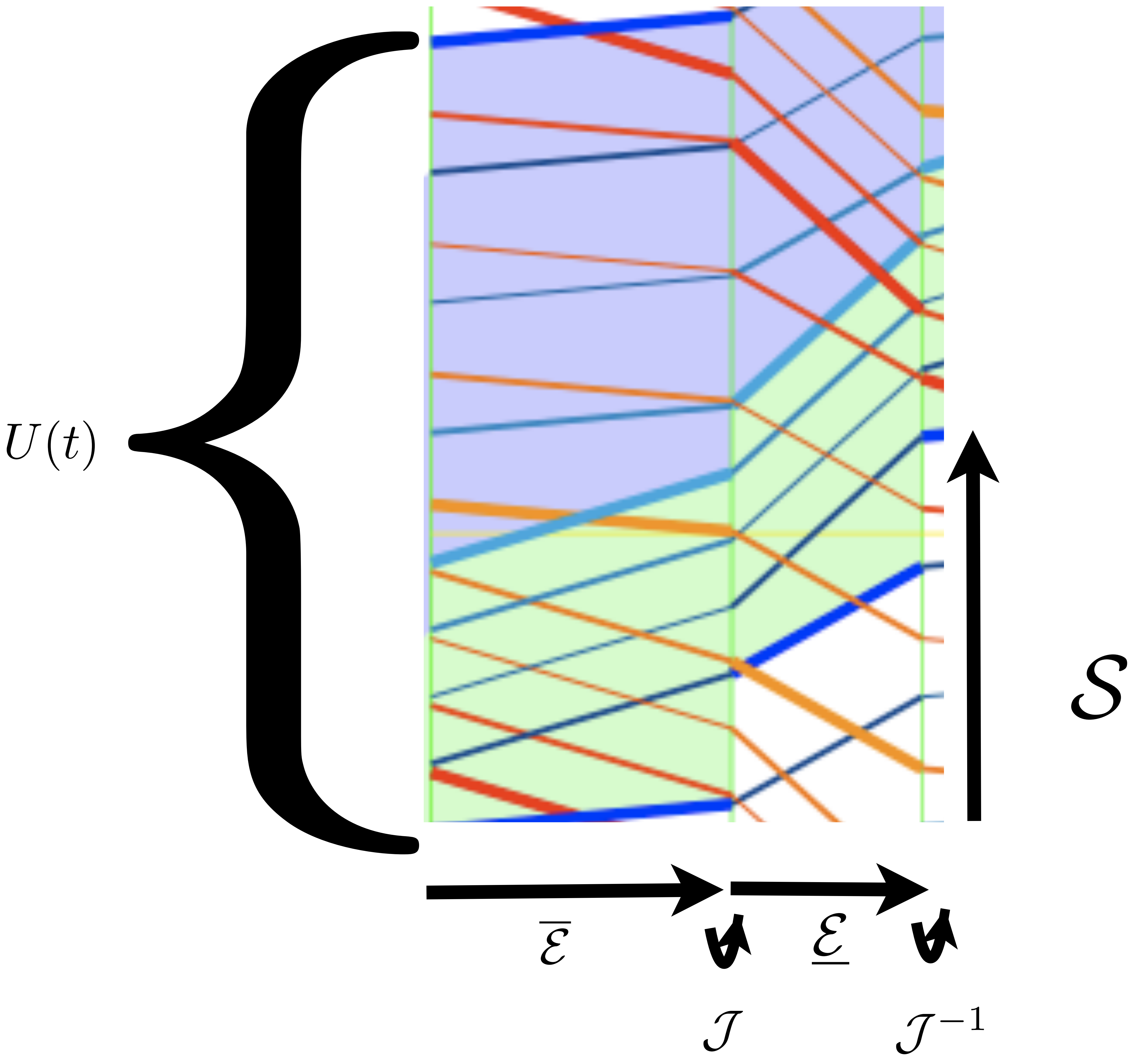
(3) $\underline{\mathcal{E}}$: Nonlinear evolution at $m = \underline{m}$

(4) $\mathcal{J}^{-1}$: Jump from $m = \underline{m}$ to $m = \overline{m}$

(5) $\mathcal{S}$: Half period shift

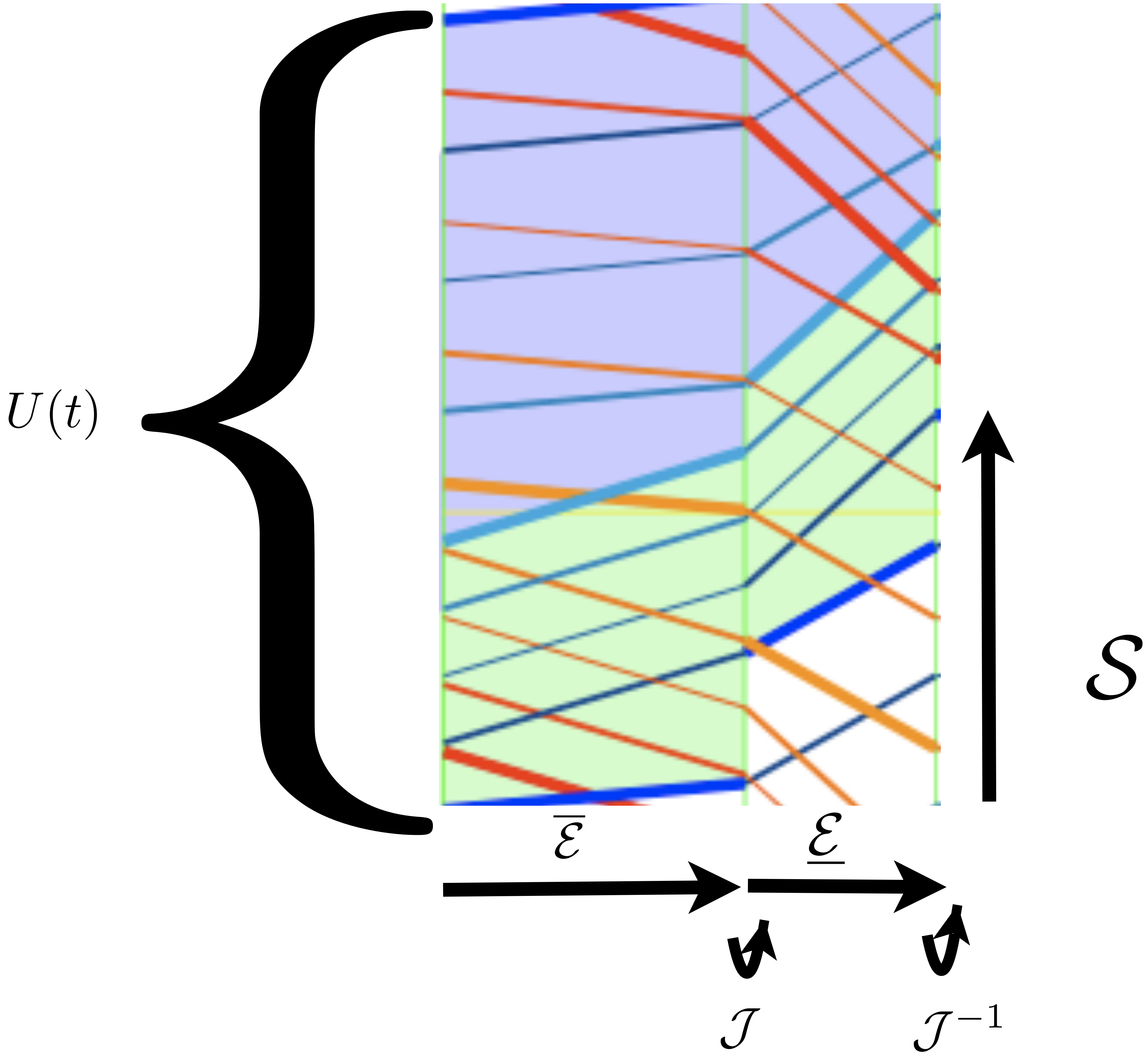
IMPOSING PERIODICITY BY PERIODIC RETURN

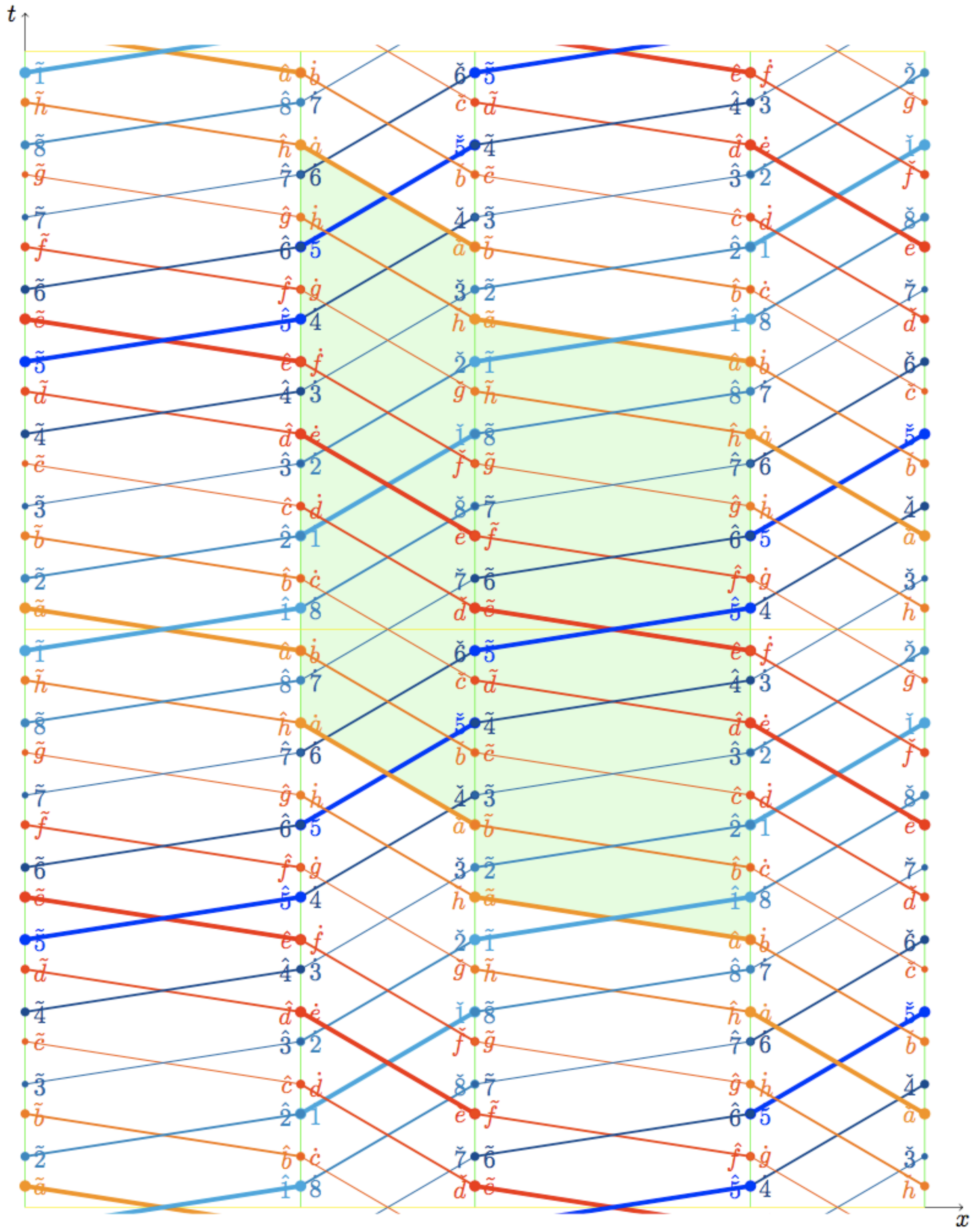
$$\mathcal{S}\cdot\mathcal{J}^{1}\cdot\underline{\mathcal{E}}\cdot\mathcal{J}\cdot\overline{\mathcal{E}}\;[U(\cdot)]=U(\cdot)$$

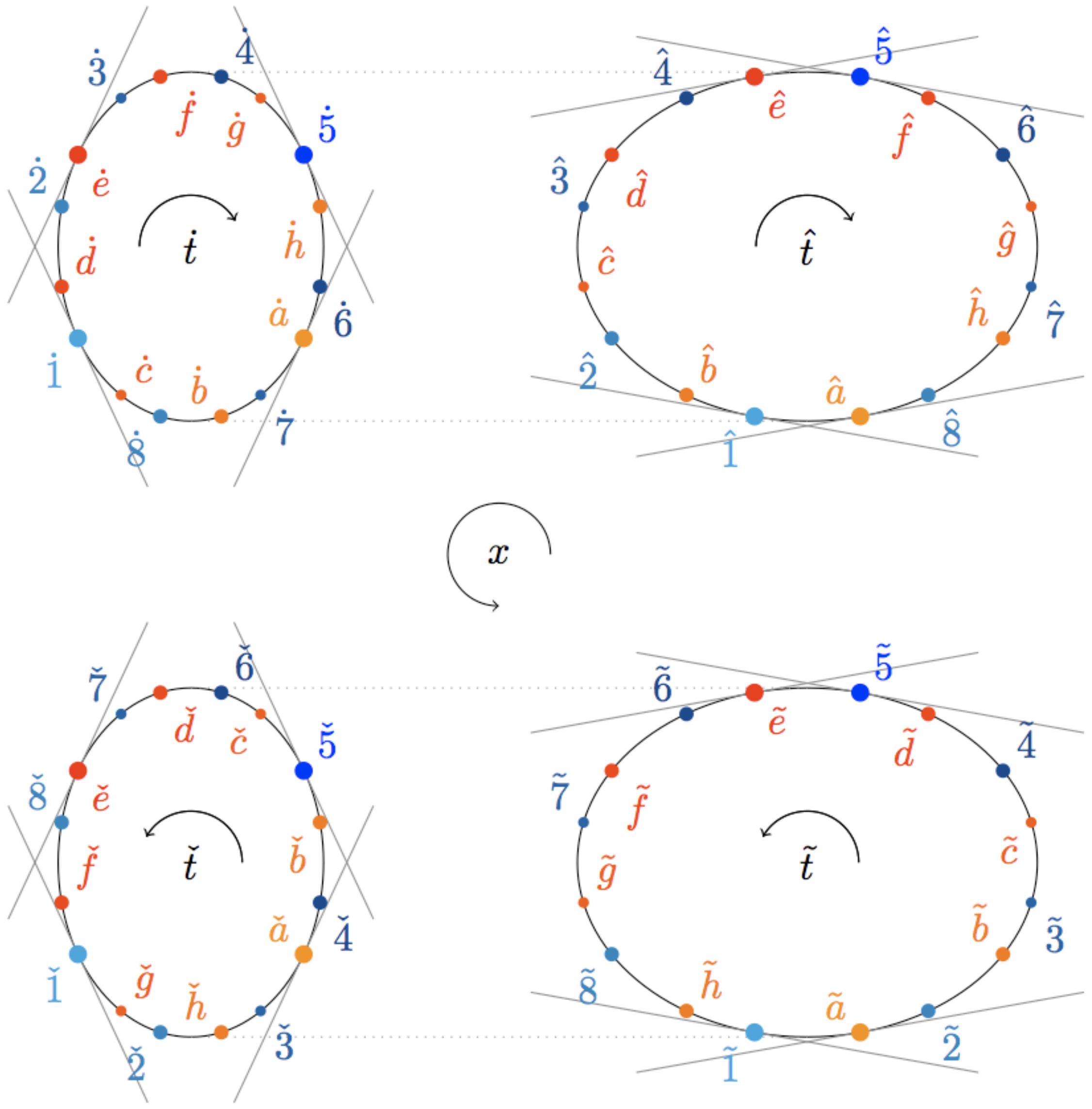


IMPOSING PERIODICITY BY
PERIODIC RETURN

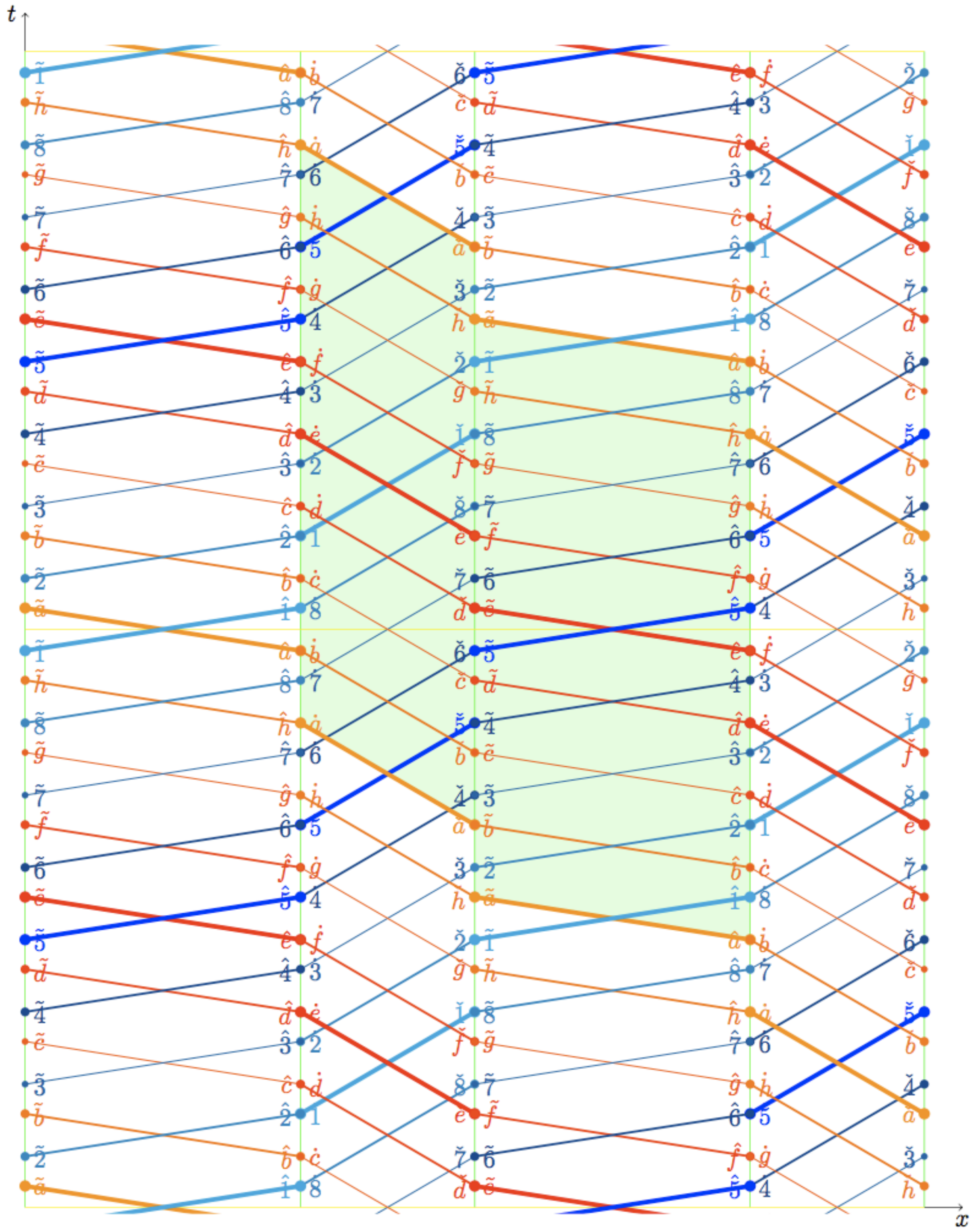
$$\mathcal{F}[U] = \left(\mathcal{S}\cdot\mathcal{J}^{-1}\cdot\bar{\mathcal{E}}\cdot\mathcal{J}\cdot\mathcal{E} - I\right)[U\cdot] = 0$$

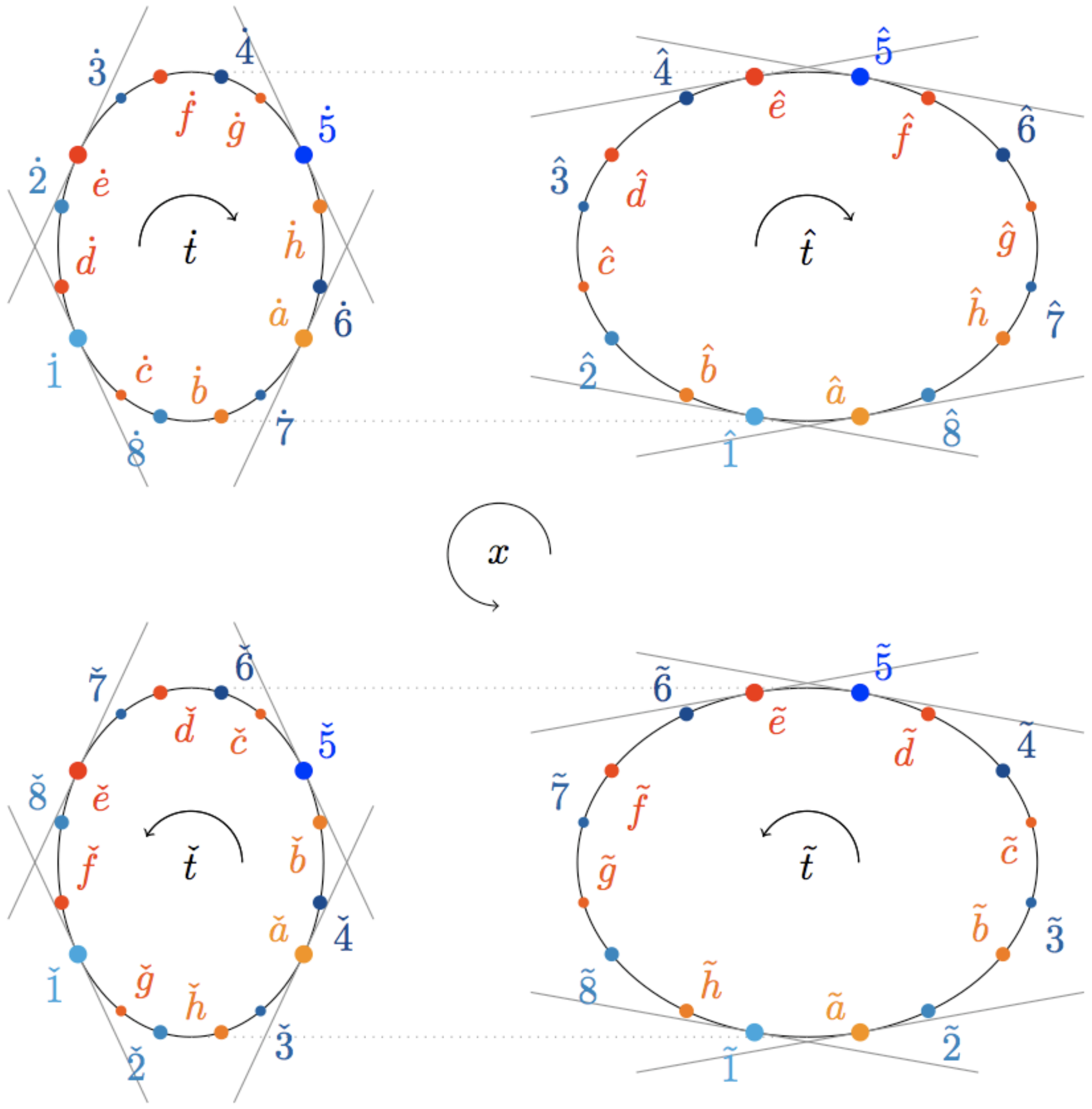






Linearized 1-mode exists with this pattern.

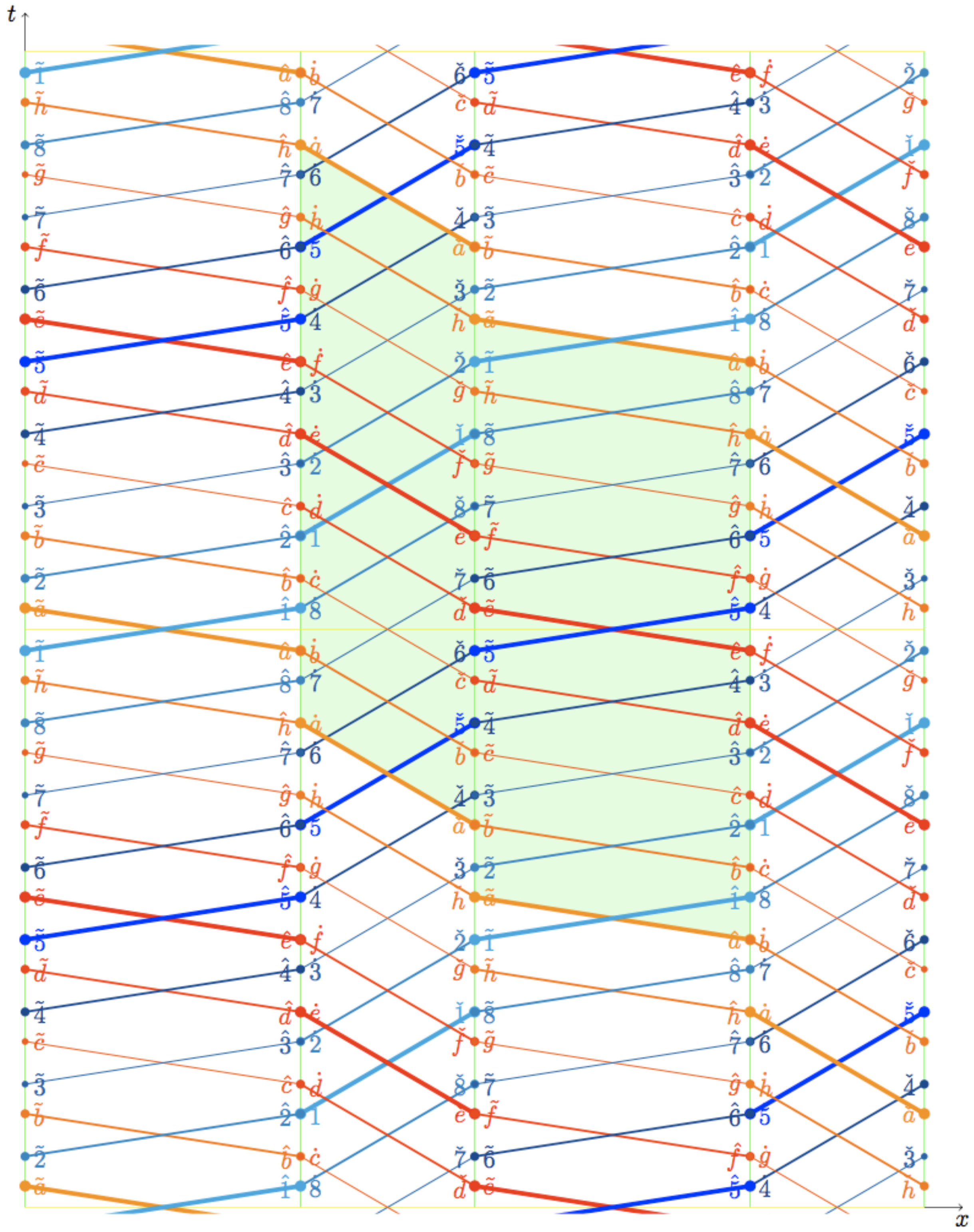


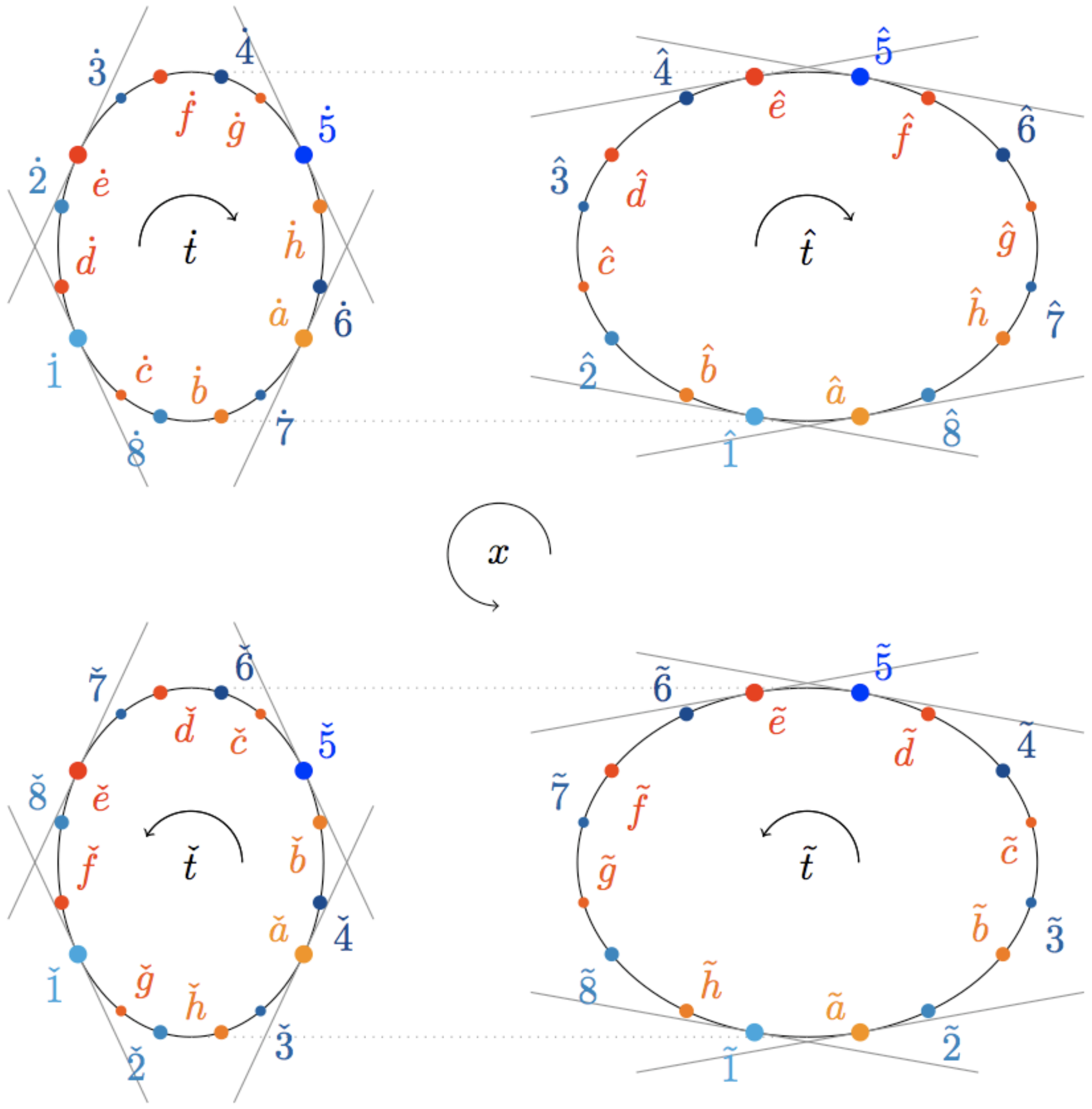


Linearized 1-mode exists with this pattern.

↓

Linear solutions should perturb to exact solutions of the nonlinear problem

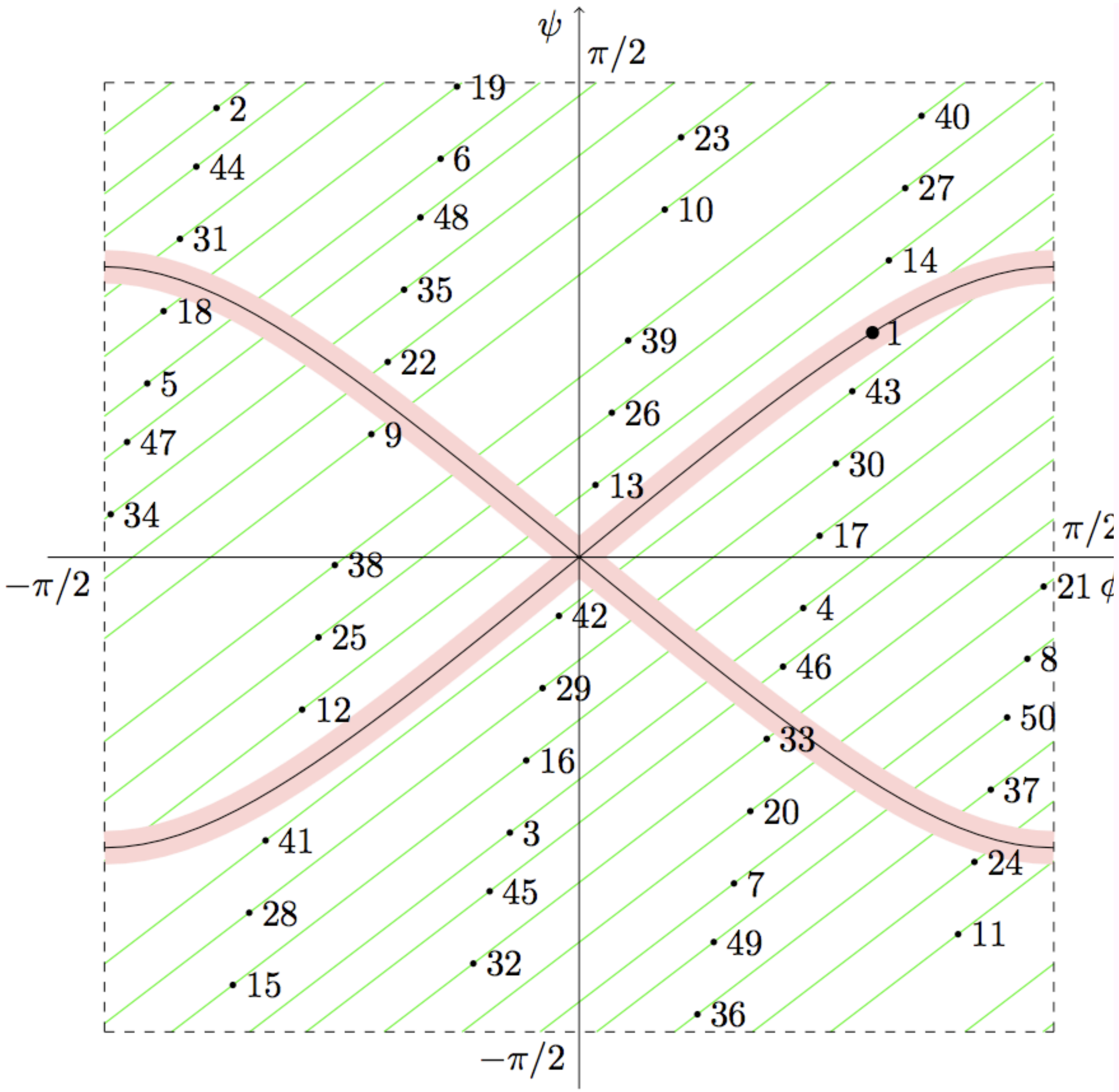




Linearized 1-mode exists with this pattern.

"We were never able to control the small divisors in a Nash-Moser Newton Method"

Numerical Plot of First 50 Eigenvalues–Case $\overline{\theta} \neq \underline{\theta}$



FROM A PRIOR TALK

- We have solved the Bifurcation Equation:
- It remains to solve the Auxiliary Equation:

$$\text{AUXILIARY EQUATION:} \quad \mathcal{P}\cdot\mathcal{F}_\epsilon[X\cdot Z + W_X(\epsilon)] = 0$$

$$\{W_X(\epsilon)\in\mathcal{K}^\perp\} \longmapsto \mathcal{P}\cdot\mathcal{F}_\epsilon[X\cdot Z + W_X(\epsilon)] \in \mathcal{R}$$

- The eigenvalues are not bounded away from zero, which leads to issues of small-divisors analogous to KAM theory.

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A COMPLETE PROOF OF EXISTENCE OF SPACE AND TIME PERIODIC SOLUTIONS OF COMPRESSIBLE EULER

with
R. Young, 2023

★ Step 3: Give a complete mathematical proof that linearized solutions perturb to nonlinear solutions.

The Details

THE PERIODIC TILE

Compressible Euler equations in Lagrangian Coordinates:

$$\rho_x + u_t = 0, \quad u_x - v(p, s)_t = 0$$

$$s = \text{specific entropy} \qquad v = \frac{1}{\rho} = \text{specific volume}$$

THE PERIODIC TILE

Compressible Euler equations in Lagrangian Coordinates:

$$\rho_x + u_t = 0, \quad u_x - v(p,s)_t = 0$$

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Evolve in x from $x = 0$ to $x = \ell$.

Theorem: $p(x,t)$ even in t and $u(x,t)$ odd in t is preserved under evolution in x.

THE PERIODIC TILE

We restrict to nonlinear evolution from $x = 0$ to $x = \ell$ for T-periodic solutions with t-symmetries

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Condition (2) is analogous to (1) observing that even/odd periodic functions are also even/odd about the half period T/2.

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Condition (1) extends solutions by reflection across $x = 0$

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(Turns out: (BC) gives periodicity by "Projection"…

…instead of having to impose "Periodic Return"!)

THE PERIODIC TILE

Theorem: Assume a smooth solution starts from T-periodic initial data satisfying

$$p(0,t) = 0 \text{ even},$$
$$u(0,t) = 0, \text{ (acoustic)}$$

and evolves to satisfy (BC) at $x = \ell$.

Then the solution extends by reflection to a 4ℓ-periodic solution of compressible Euler.

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The resulting solution is:

$$T\text{-periodic is time}$$
$$4\ell\text{-periodic in space}$$

THE EXISTENCE THEOREM

The Reflection Principle reduces the problem of existence of periodic solutions of compressible Euler to the following boundary value problem (BVP):

Compressible Euler: $s = s(x), \quad 0 \leq x \leq \ell.$

$$\rho_x + u_t = 0,$$
$$u_x - v_p(p, s(x))p_t = 0. \qquad \text{(Euler)}$$

Boundary Conditions:

$$x = 0 : \quad p(0,t) = \text{ even } T\text{-periodic function of } t$$
$$u(0,t) = 0 \qquad \text{(IC)}$$

$$x = \ell : \quad p(\ell, t + T/2) = p(\ell, t)$$
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Sturm-Liouville theory implies ω_k are isolated, and grow linearly with wave number k.

Theorem [TY 2023]: Each non-resonant linear pure-tone solution perturbes to a 1-parameter family of non-linear pure-tone solutions of (BVP) with the same time period T.

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Theorem [TY 2023]: Linearizing CE around a quiet state $p = \bar{p}$, $u = 0$, $s_0(x)$, Sturm-Liouville theory determines a sequence of pure-tone solutions of the linear (BVP):

$$p(x,t) = \bar{p} + \phi_k(x) \cos(\omega_k t)$$
$$u(x,t) = \psi_k(x) \sin(\omega_k t)$$

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where ϕ_k and ψ_k are the eigenfunctions of the Sturm-Liouville problem, $k = 1, 2, 3, ...$.

Our Main Theorem establishes that each non-resonant linearized pure-tone solution perturbs to a one parameter family of pure-tone solutions of the nonlinear compressible Euler equations, with the same frequency and time period

$$T_k = \frac{2\pi}{\omega_k}.$$

THE EXISTENCE THEOREM

DEFN: A k-mode is non-resonant if ω_k is not a rational multiple of any other eigen-frequency,

$$\frac{\omega_j}{\omega_k} \notin Q \quad \text{for all } j \neq k$$

Restrict to entropy profiles within the set

$$\mathcal{B} \equiv \left\{ s = s(x) \in L^1[0,\ell] \;\middle|\; \sigma \in L^1,\ \log\sigma \in BV \right\}$$

Theorem [TY2023]: For $s(\cdot) \in \mathcal{B}$, every non-resonant linearized k-mode perturbes to a 1-parameter family of pure-tone solutions of (BVP) for the nonlinear compressible Euler of the form

$$p(x,t) = \overline{p} + \alpha \cos(\omega_k t)\,\varphi_k(x) + O(\alpha^2)$$

$$u(x,t) = \alpha \sin(\omega_k t)\,\psi_k(x) + O(\alpha^2).$$

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$$x = \ell : \qquad U = \begin{pmatrix} p \\ u \end{pmatrix}, \qquad \frac{\mathcal{I} - \mathcal{R}}{2} \mathcal{S}^{-T/4} U(\ell, \cdot) = 0 \qquad \text{(BC)}$$

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Define

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$x = 0 :$ $p(0, t) =$ even T-periodic function of t

$u(0, t) = 0$ (IC)

Define

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Nonlinear evolution from $x = 0$ to $x = \ell$

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$$\mathcal{F}[U] \equiv \underbrace{\tfrac{\mathcal{I} - \mathcal{R}}{2}\, \mathcal{S}^{-T/4}}\, \underbrace{\mathcal{E}\, U(\cdot)}$$

Imposes (BC) at $x = \ell$ by $\mathcal{F}[U] = 0.$

Nonlinear evolution from $x = 0$ to $x = \ell$

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Simpler Condition:

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projection onto even

“Kills odd u and leaves even p unchanged”

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Thus (A) implies u=0 and (BC) at $x = \ell$, so is special case.

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Thus define: $\mathcal{F}_A[U] \equiv \frac{\mathcal{I}-\mathcal{R}}{2} \, \mathcal{E}\left[U(\cdot)\right].$

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Thus define: $\mathcal{F}_A[U] \equiv \frac{\mathcal{I}-\mathcal{R}}{2} \mathcal{E}\,[U(\cdot)].$

$\mathcal{F}_a[U] = 0$ implies $u = 0$ and (BC) at $x = \ell$.

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Thus define: $\mathcal{F}_A[U] \equiv \frac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{E}\,[U(\cdot)]$.

$\mathcal{F}_a[U] = 0$ implies $u = 0$ and (BC) at $x = \ell$.

Theorem [TY 2023]: Assume T-periodic data $U(t)$ satisfies (IC) and $\mathcal{F}_a[U] = 0$. Then $U(t)$ evolves to a T-periodic solution of the compressible Euler equations satisfying $u = 0$ at both $x = 0$ and $x = \ell$.

THE NONLINEAR FUNCTIONAL

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These turn out to be the even mode solutions of (★) !

PROOF OF EXISTENCE

The problem of existence of spacetime periodic solutions of compressible Euler is now reduced to the problem of solving

$$\mathcal{F}[U] \equiv \frac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{S}^{-T/2}\,\mathcal{E}\,[U(\cdot)] = 0$$

starting from $\mathrm{T\text{-}periodic}$ initial data $U(\cdot)$ at $x = 0$ satisfying:

$$U(t) = \left(\begin{array}{c} p(t) \\ u(t) \end{array}\right) = \left(\begin{array}{c} even \\ 0 \end{array}\right). \qquad \text{(IC)}$$

We perturb about a "quiet" state (reversible solution of Euler)

$$p = \bar{p},\ u = 0,\ s = s(x)$$

Linearizing $\mathcal{E}$ about $p = \bar{p},\ u = 0,\ s = s(x)$ yields

$$p_{tt} - \sigma^2(x)p_{xx} = 0 \qquad \text{(L)}$$

which can be solved by separation of variables.

PROOF OF EXISTENCE

The linearize operator which goes with

$$\mathcal{F}[U] = \tfrac{\mathcal{I}-\mathcal{R}}{2}\, \mathcal{S}^{-T/2}\, \mathcal{E}\,[U(\cdot)]$$

is thus

$$D\mathcal{F}[V] = \tfrac{\mathcal{I}-\mathcal{R}}{2}\, \mathcal{S}^{-T/2}\, \mathcal{L}\,[V(\cdot)]$$

where $\mathcal{L}\,[V(\cdot)]$ denotes linearized evolution by (L) from

$x = 0$ to $x = \ell$.

Solutions of (L) with boundary conditions (IC),(BC) solve

$$D\mathcal{F}[V] = \tfrac{\mathcal{I}-\mathcal{R}}{2}\, \mathcal{S}^{-T/2}\, \mathcal{L}\,[V(\cdot)] = 0$$

which can be solved by separation of variables and Sturm-Liouville Theory:

THE LINEARIZED PROBLEM

Separating variables in the linear wave equation yields a Sturm-Liouville system in x.

Square roots of the Sturm-Liouville eigenvalues give the eigenfrequencies ω_k of pure tone linearized periodic solutions.

Sturm-Liouville theory implies ω_k are isolated and grow linearly with k.

Theorem [TY 2023] Assume the $s(x) \in \mathcal{B}$ where

$$\mathcal{B} \equiv \left\{ s \in L^1[0,\ell] \;\middle|\; \sigma(x) = \sqrt{-v_p(\bar{p}, s(x))} \in L^1,\ \log \sigma(\cdot) \in BV \right\}.$$

Then the linear boundary value problem (IC),(BC) admits the following family of pure tone periodic solutions:

$$p(x,t) = \bar{p} + \phi_k(x)\cos(\omega_k t)$$

$$u(x,t) = \psi_k(x)\sin(\omega_k t) \qquad k = 1, 2, 3, \ldots$$

where ϕ_k and ψ_k are the Sturm-Liouville eigenfunctions.

THE NONLINEAR PROBLEM

DEFN: A k-mode is non-resonant if ω_k is not a rational multiple of any other eigen-frequency,

$$\frac{\omega_j}{\omega_k} \notin Q \quad \text{for all } j \neq k$$

Theorem:[TY 2023] All non-resonant linearized k-modes perturb to periodic solutions of the nonlinear compressible Euler equations with the same space/time periods.

In Lagrangian coordinates the solutions take the form

$$p(x,t) = \bar{p} + \alpha\,\phi_k(x)\cos(\omega_k t) + O(\alpha^2),$$

$$u(x,t) = \alpha\,\psi_k(x)\sin(\omega_k t) + O(\alpha^2).$$

α = amplitude = perturbation parameter

Proof: It suffices to prove that linearized mode solutions of

$D\mathcal{F}[V] = 0$ perturb to solutions of $\mathcal{F}[U] = 0$ by IFT

The Bifurcation Problem

THE NONLINEAR PROBLEM

For ease of expression we introduce notation:

$$y(x,t) = p(x,t) + u(x,t)$$

so the even part is p and the odd part is u:

$$p = \frac{\mathcal{I}+\mathcal{R}}{2} y, \qquad u = \frac{\mathcal{I}-\mathcal{R}}{2} y$$

In terms of y the periodic tile problem is

$$\mathcal{F}_P(y_0) \equiv \frac{\mathcal{I}-\mathcal{R}}{2} \mathcal{S}^{-T/4} \, \mathcal{E} \, y_0 = 0.$$

and the acoustic boundary value problem is:

$$\mathcal{F}_A(y_0) \equiv \frac{\mathcal{I}-\mathcal{R}}{2} \, \mathcal{E} \, y_0 = 0.$$

THE NONLINEAR PROBLEM

The Main Point:

The linearized operator factors out of the nonlinear operators!

Theorem: [TY 2023] The nonlinear operators $\mathcal{F}_P$ and $\mathcal{F}_A$ factor as:

$$\mathcal{F}_P \equiv \underbrace{\tfrac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{S}^{-T/4}\,\mathcal{L}}_{D\mathcal{F}_P(\bar{p})}\mathcal{N}, \qquad \mathcal{F}_A \equiv \underbrace{\tfrac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{L}}_{D\mathcal{F}_A(\bar{p})}\mathcal{N}.$$

where $\mathcal{N} = \mathcal{L}^{-1}\mathcal{E}$ is bounded invertible.

Consider general case: $\mathcal{F} \equiv \mathcal{F}_P = \frac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{S}^{-T/4}\mathcal{L}\mathcal{N}$

THE NONLINEAR PROBLEM

Consider general case: $\mathcal{F} \equiv \mathcal{F}_P = \underbrace{\frac{\mathcal{I}-\mathcal{R}}{2} \mathcal{S}^{-T/4} \mathcal{L}}_{D\mathcal{F}(\bar{p})} \mathcal{N}$

Fix a non-resonant k-mode solution of linearized operator:

$$D\mathcal{F}(\bar{p})[Y_k] = 0$$

$$Y_k(t) = \underbrace{\bar{p} + \phi_k(0) \cos(\omega_k t)}_{p(0,t)} + \underbrace{\psi_k(0) \sin(\omega_k t)}_{u(0,t)=0}$$

$$= \bar{p} + \phi_k(x) \cos(\omega_k t)$$

Now express arbitrary i-data $y(t)$ as F-series using fixed period

$$T \equiv T_k = \frac{2\pi}{\omega_k}$$

THE NONLINEAR PROBLEM

Theorem: [TY 2023] The linearized operators respect j-modes, and

$$D\mathcal{F}(\bar{p})[\cos(j\tfrac{2\pi}{T}t)] = \delta_j(T)\sin(j\tfrac{2\pi}{T}t)$$

$$D\mathcal{F}(\bar{p})[\cos(k\tfrac{2\pi}{T}t)] = 0$$

where $\delta_j(T) = a_j \neq 0$ if Y_k is non-resonant.

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Conclude:

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Conclude:

$$\mathcal{F}[y(\cdot)] = \underbrace{\frac{\mathcal{I}-\mathcal{R}}{2}\,\mathcal{S}^{T/4}\,\mathcal{L}}_{\boxed{D\mathcal{F}(\bar{p})}}\,\mathcal{N}\,\underset{\substack{\uparrow\\ \boxed{y(t)=\sum_{j=0}^{\infty} c_j\cos(j\frac{2\pi}{T}t)}}}{[y(\cdot)]}$$

$D\mathcal{F}(\bar{p})$ is invertible on complement of k-mode kernel with small divisors $\delta_j \equiv \delta_j(T)$.

Precisely what's needed to apply Implicit Function Theorem and Liapunov-Schmidt.

THE NONLINEAR PROBLEM

Define:

$$\mathcal{H}_1 := \left\{ z + \alpha\, \mathrm{c}(k \tfrac{2\pi}{T_k} t) \;\middle|\; z, \alpha \in \mathbb{R} \right\} \quad \text{and}$$

$$\mathcal{H}_2 := \left\{ \sum_{j \neq k} a_j\, \mathrm{c}(j \tfrac{2\pi}{T_k} t) \;\middle|\; \sum_{j \neq k} a_j^2\, j^{2s} < \infty \right\},$$

So the Domain is $H^s = \mathcal{H}_1 \oplus \mathcal{H}_2$

The Range is

$$\mathcal{H} := \left\{ \beta\, \mathrm{s}(k \tfrac{2\pi}{T_k} t) \right\} \oplus \mathcal{H}_+,$$

$$\mathcal{H}_+ := \left\{ y = \sum_{j \neq k} a_j\, \mathrm{s}(j \tfrac{2\pi}{T_k} t) \;\middle|\; \|y\| < \infty \right\},$$

with norm

$$\|y\|^2 := \beta^2 + \sum_{j>1} a_j^2\, \delta_j^{-2}\, j^{2s}$$

$$j>1$$

(Here a_j are the Fourier coefficients)

THE NONLINEAR PROBLEM

Define the Projection

$$\Pi : \mathcal{H} \to \mathcal{H}_+ \quad \text{by} \quad \Pi\Big[\beta\, \mathrm{s}(k\tfrac{2\pi}{T_k}t) + \sum_{j\neq k} a_j\, \mathrm{s}(jt)\Big] := \sum_{j\neq k} a_j\, \mathrm{s}(jt),$$

which projects onto all but the k-mode.

The Liapunov-Schmidt Method:

Auxiliary Equation:

$$\Pi\,\mathcal{F}(y^0) = 0, \quad \text{with} \quad y^0 = \bar{p} + z + \alpha\, \mathrm{c}(k\tfrac{2\pi}{T_k}t) + W,$$

Solve for

$W(\alpha, z) \in \mathcal{H}_2,$

Bifurcation Equation:

$$\begin{aligned} f(\alpha, z) &= \Big\langle \mathrm{s}(k\tfrac{2\pi}{T_k}t), \mathcal{F}(y^0) \Big\rangle \\ &= \langle \sin(k\tfrac{2\pi}{T}t), \mathcal{F}(\bar{p} + z + \alpha \cos(k\tfrac{2\pi}{T}t) + W(\alpha, z)) \rangle = 0 \end{aligned}$$

Both follow from the IFT.

Solve for

$z = z(\alpha)$

THE NONLINEAR PROBLEM

Solution of the Auxiliary Equation follows directly from Implicit Function Theorem in Banach Spaces:

Lemma 18. *If the k-mode is nonresonant, there is a neighborhood* $\mathcal{U} \subset \mathcal{H}_1$ *of the origin and a unique* C^1 *map*

$$W : \mathcal{U} \to \mathcal{H}_2, \quad \textit{written} \quad W\big(\overline{p} + z + \alpha\, \mathrm{c}(k\tfrac{2\pi}{T_k}t)\big) =: W(\alpha, z) \in \mathcal{H}_2,$$

such that, for all $z + \alpha\, \mathrm{c}(k\frac{2\pi}{T_k}t) \in \mathcal{U}$, *we have a solution of the auxiliary equation* (7.35), *given by*

$$\Pi\, \mathcal{F}\Big(\overline{p} + z + \alpha\, \mathrm{c}(k\tfrac{2\pi}{T_k}t) + W(\alpha, z)\Big) = 0.$$

Moreover, the map $W(\alpha, z)$ *satisfies the estimate*

$$W(\alpha, z) = o(|\alpha|),$$

uniformly for z *in a neighborhood of 0.*

THE NONLINEAR PROBLEM

Solution of the Bifurcation Equation follows classical IFT:

$$f(\alpha, z) = \left\langle \mathrm{s}(k\tfrac{2\pi}{T_k}t), \mathcal{F}(y^0) \right\rangle$$

$$= \langle \sin(k\tfrac{2\pi}{T}t), \mathcal{F}(\bar{p} + z + \alpha\cos(k\tfrac{2\pi}{T}t) + W(\alpha, z)) \rangle = 0$$

We must go to the second derivative:

$$g(\alpha, z) := \frac{1}{\alpha} f(\alpha, z), \quad \alpha \neq 0,$$

$$g(0, z) := \frac{\partial f}{\partial \alpha}(0, z).$$

It suffices to prove

$$\frac{\partial g}{\partial z}\Big|_{(0,0)} \neq 0, \quad \text{which is} \quad \frac{\partial^2 f}{\partial z\, \partial \alpha}\Big|_{(0,0)} \neq 0.$$

For this we must evaluate $\frac{\partial^2}{\partial z\, \partial \alpha} \mathcal{E}^\ell y^0 \Big|_{(0,0)}$

which can be explicitly calculate using Sturm Liouville apparatus.

THE NONLINEAR PROBLEM

The following theorem gives existence of a one parameter family of solutions of compressible Euler satisfying (IC) and (BC), which perturb an arbitrary non-resonant k-mode of the Linearized Equations:

Proof: [TY 2023] There exists a function $W(\alpha, z)$ of the Auxiliary Equations and a function $z = z(\alpha)$ of the Bifurcation Equation such that

$$\mathcal{F}[\bar{p} + z(\alpha) + \cos\left(k\frac{2\pi}{T_k}t\right) + W(\alpha, z(\alpha))] = 0,$$

where $z(\alpha)$ and $W(\alpha, z(\alpha))$ are order $O(\alpha^2)$.

COMMENTS

Important Insight: We construct solutions at the constant state $\bar{p} + z$, but only require uniform estimates for the small divisors of a fixed Linearized operator at the fixed constant state $\bar{p}$.

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This factoring overcomes a fundamental problem associated with Nash-Moser which inverts a different linearized operator, for a different constant state, at each step of a Newton Method.

Such a factoring does NOT happen when periodicity is imposed by the periodic return condition,

$$(\mathcal{F}-I)[U]=0.$$

CONCLUDING REMARKS

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Q: Is this shock-free regime the actual regime of ordinary sounds of speech and musical tones heard in nature?

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Q: Math Question: Do the quasi-periodic mixed modes of the linearized theory perturb like pure modes do?

END

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Thank you!