Personal Notebook of Math Problems Recorded by Mathematician/Probabilist James G. Wendel, University of Michigan, 1951-1991

V

A Weakly Sequentially closed set which is not weakly closed.

Let $H = \ell_2$. In H let $S = \{f_N\}$, $f_N = (0,0,\dots,N^{1/4},0,0,\dots)$

$|f_N| \to \infty$, so no non-trivial weak limit points. But 0 is in weak closure of S.

Lemma: Let $g_i \in H$, $i=1,\dots,m$. $g_i = \{x_{in}\}$. For any n $\exists N \geq n \ni |x_{iN}| < \frac{1}{\sqrt{N}}$, $i=1,2,\dots,m$.

Proof: Suppose false. Then $\exists n \ni N \geq n \Rightarrow$ $\exists i_N \ni |x_{i_N N}| \geq \frac{1}{\sqrt{N}}$. Then $\sum_{i=1}^{m} |x_{iN}| \geq \frac{1}{\sqrt{N}}$.

Hence $h = \{\sum_{i=1}^{m} |x_{iN}|\}_{N=1}^{\infty} \notin H$, which is absurd. QED.

Now let $(g_1,\dots,g_m;\varepsilon)$ be a weak nbhd of (0). Choose $n \ni n^{1/4} > \frac{1}{\varepsilon}$. Choose $N \geq n$ by lemma. Compute (f_N, g_i).

We have $|(f_N, g_i)| = |N^{1/4}\bar{x}_{iN}| < N^{1/4}N^{-1/2} \leq n^{-1/4} < \varepsilon$.

QED.

Better — 4/1/69 (!) Let $S = \{x_n\}$, $x_n = \sqrt{n}\,u_n$, $\{u_n\}$ o.n. basis.

For any $y_1,\dots,y_k$ & $\varepsilon > 0$ must show $\exists n \ni$ $x_n \in N(y_1,\dots,y_k,\varepsilon)$, i.e. $|(\sqrt{n}\,u_n, y_i)| < \varepsilon$, $i=1,2,\dots,k$, i.e. $|(u_n, y_i)|^2 < \frac{\varepsilon^2}{n}$, $i=1,2,\dots,k$. If false then for all n $\sum_{1}^{k} |(u_n, y_i)|^2 \geq \frac{\varepsilon^2}{n}$. Sum on n, get $\sum_{1}^{k} \|y_i\|^2 \geq \sum_{1}^{\infty} \frac{1}{n} = \infty$, absurd

Simple proof of Weil's theorem: G a group, g a compact subgroup, invariant. H a closed invariant subgroup. Then $Hg/g \cong H/H\cap g$.

Proof: The algebra is clear. In fact, the 1-1 correspondence is given by

$$x(g) \leftrightarrow x(H\cap g). \quad \text{[Cosets.]}$$

Now let A be a set in Hg/g. A has the form $B(g)$, B a subset of H. Its image in $H/H\cap g$ has the form $B(H\cap g)$. We want to show: $B(g)$ closed in Hg $\equiv$ $B(H\cap g)$ closed in H. We use

Lemma: g compact, X closed $\Rightarrow$ gX closed.

Then Hg is closed, H is closed. Hence

Bg closed in Hg $\equiv$ Bg closed.

$B(H\cap g)$ closed in H $\equiv$ $B(H\cap g)$ closed.

So suppose Bg is closed. Then $B(H\cap g) = Bg \cap H$ is closed. (This follows, since if $x = bh$, $h \in H\cap g$, then $x \in Bg$, $x \in H$, since $b \in H$.) (If $x = bg_0 = h$ then, since $b \in H$ we have $g_0 \in g \cap H$, so that $x \in B(H\cap g)$.)

Conversely, if $B(H\cap g)$ is closed then $Bg = B(H\cap g)g$ is closed. <u>Q.E.D.</u>

On Hamel Bases.

1. $\exists$ basis of measure zero.

Proof: Let $E = \{x \mid x$ has no 7 in decimal exp.$\}$.

Every $y = x_1 + 7x_2$, $x_1 \in E$, $x_2 \in E$.

$\mu(E) = 0$. E contains a basis. For E contains a maximal set of lin. ind. (rationally) nos., which, if it does not span $-\infty, \infty$ cannot span E. Then can adjoin new $x \in E$ to it.

2. $\exists$ basis $B \ni \forall E,\ \mu(E) > 0 \Rightarrow B \cap E \neq \emptyset$.

(Hence in particular $\mu^*(B) = \infty$.)

Proof: Well order the closed sets F_α of pos. measure. Define $\{x_\alpha\}$, $x_\alpha \in F_\alpha$, inductively so as to make them rationally linearly independent. This is possible, for

$$S = \{\sum r_i x_{\alpha_i} \mid \alpha_i < \alpha\}$$

is a null-set, hence in F_α $\exists x \notin S$. (This assumes the continuum hypothesis; for the set of closed sets has power of continuum, and we need $\{\alpha \mid \alpha < \alpha_0\}$ countable.)

If the $\{x_\alpha\}$ are not yet a basis adjoin extra elements.

Now if $\mu(E) > 0$ we have $\exists F \subseteq E \ni \mu(F) > 0$. QED.

3. If B is measurable then $\mu(B) = 0$.

Proof: ~~[illegible]~~ Consider the sets $B + x/2$ for $x \in B$. These sets are all disjoint ~~except for the origin~~, + $\exists$ continuum no. of them. Hence since they all have equal measure they all have measure zero.

Theorem: Let G be l.c. with right invariant Haar measure. Let E and F be sets of positive measure. Then $\exists a \ni \mu(E \cap F_a) > 0$, where F_a is translation of F by a.

Proof: Consider $K_E * K_F$. We have

$$\int_{G \times G} K_E * K_F \, d(\mu \times \mu) = \int_G K_E \, d\mu \cdot \int_G K_F \, d\mu = \mu(E)\mu(F) > 0.$$

$\therefore$ $\int_G K_E(xy^{-1}) K_F(y) \, d\mu(y) > 0$ for some x.

~~So~~ $\cancel{K_E(xy^{-1}) K_F(y) > 0}$ ~~on a set of y of positive measure. But~~ $\cancel{K_E(xy^{-1}) = K_E}$

So $\int_G K_E(y^{-1}) K_F(yx) \, d\mu(y) > 0$ for some $x = x_0$

$K_F(yx_0) K_E(y^{-1}) > 0$ on set of pos. measure

$K_{Fx_0^{-1}}(y) K_{E^{-1}}(y) > 0$ " "

So $\mu(Fx_0^{-1} \cap E^{-1}) > 0$. Set $x_0^{-1} = a$.

Replacing E by E^{-1} gives the result.

cf. Halmos prob 4 pg 264.

Thm: A closed set E of real nos. of pos. Lebesgue measure has continuum no. of points. (with or without Continuum hypothesis!)

Proof: $\exists$ interval $[a,b] \ni m([a,b] \cap S) > 0$.

Consider $m([a, a+\varepsilon] \cap S) = f(\varepsilon)$. Cont fcn. 0 for $\varepsilon = 0$. Takes cont. no. of ~~[illegible]~~ values. For each $y = f(\varepsilon)$ let $x_y \in E$ be $\sup\{x \mid a \le x \le a+\varepsilon, x \in S\}$. $y_1 \ne y_2 \Rightarrow x_{y_1} \ne x_{y_2}$. QED.

Neyman Pearson Lemma.

$f, g \in \mathcal{L}(-\infty, \infty)$ and nonnegative.

A and B constants, $B \leq \int Ag$.

To find ~~the~~ a function $x \ni 0 \leq x \leq A$, measurable,

$\ni$ ~~$\int gx = B$~~ $\int gx = B$ and

$\int fx$ is maximum.

Solution: $\exists c \ni \begin{cases} x = 0 & \text{on } S_1 \\ x = \text{arb} & \text{on } S_2 \\ x = A & \text{on } S_3 \end{cases}$

where $S_2 = \{t \mid f(t) = c\,g(t)\}$

$S_1, S_3 = \{t \mid f(t) \gtrless c\,g(t)\}$.

Proof that max exists: $\int fx$ is bounded, so has sup. $\{x \mid \int gx = B,\ 0 \leq x \leq A\}$ is bounded set + weak* closed, hence weak* compact. Hence $\int fx$ has max.

Proof that max is attained by combination of at most 3 char functions:

Let $\Phi(x) = \left(\int fx, \int gx\right) \in E_2$.

R = Range of Φ is compact convex set in E_2.

Let e be an extreme point of R, consider $\Phi^{-1}(e)$. Compact (w^*) + convex. Has extreme points.

Let x_0 be one. I claim x_0 is an extreme point of $\{x \mid 0 \leq x \leq A\}$. If not write $x_0 = \frac{1}{2}(x_1 + x_2)$.

Then $\Phi(x_0) = \frac{1}{2}\Phi(x_1) + \frac{1}{2}\Phi(x_2) = e$.

So $\Phi(x_1) = \Phi(x_2) = e$. So $x_1, x_2 \in \Phi^{-1}(e)$. So $x_0 = x_1 = x_2$.

Let $f \in R$. f is comb. of at most 3 extreme points. ($n+1$ in n dimensions; n enough if set of extreme points is connected). In particular, solution point is such.

But extreme points in R come from extreme points in $\{x \mid \int gx = B,\ 0 \le x \le A\}$, which are characteristic functions.

Sharper analysis: Map the set S = set of x, $0 \le x \le A$, $\int gx = B$, onto the line by $x \xrightarrow{\Phi} \int fx$.
Let e be an extreme point of $R \equiv$ range. Then $\Phi^{-1}(e)$ has extreme points. Let x_0 be one. Then as before, x_0 is an extreme point of S. But in fact, any extreme point of S is an extreme point of $0 \le x \le A$. For suppose that there $\exists \beta \ni$ $m(t \mid \beta \le x(t) \le 1-\beta) > 0$. (Take $A \equiv 1$ for convenience)
Call set of such t, E.
~~Compute~~
Compute $\int_E x(t)\, g(t)\, dt$.

Find subset F of $E \ni \int_F g(t)\, dt = \frac{1}{2}\int_E x(t) g(t)\, dt$.

On F let
$$x_1(t) = x(t) + \beta$$
$$x_2(t) = x(t) - \beta$$

$E-F$ let
$$x_1(t) = x(t) - \beta$$
$$x_2(t) = x(t) + \beta.$$

Then $x = \frac{1}{2}(x_1 + x_2)$, $\int_F x_1 g = \int_F xg + \beta \int_F g\ )) = \frac{\beta}{2}\int x(t) g(t)$

$\int_{E-F} x_1 g = \int_{E-F} xg - \beta \int_{E-F} g$

QED

Generalizations.

Let C be a ~~closed~~ compact convex set in E^n.

Let $\mathfrak{X}$ consist of the set of measurable functions $x(t)$ $\ni$ $x(t) \in C$ a.e. ($x(t)$ vector)

$\mathfrak{X}$ is weak* compact in obvious topology.

The extreme points of $\mathfrak{X}$ are all those functions x $\ni$ ~~$x(t)$ is an extreme point of C, for almost every t~~ !

$x(t)$ lies in the closure of the set of extreme points of C for almost all t. (KARLIN)

[redacted]

Theorem: $e^{i\alpha f(x)}$ cont for each α; all x,
$\Rightarrow$ $f(x)$ is cont.

Proof: $\cancel{\int_0^\beta e^{i\alpha f(x_n)}\,d\alpha = e^{i\beta f(x_n)} - 1}$

$$\int_0^\infty e^{i\alpha f(x_n)}\, e^{-\alpha}\, d\alpha = \frac{1}{1 - i f(x_n)}$$

Let $x_n \to x$.

$$\int_0^\infty e^{i\alpha f(x)}\, e^{-\alpha}\, d\alpha = \lim_{x_n \to x} \frac{1}{1 - i f(x_n)}$$

also $$= \frac{1}{1 - i f(x)}$$ QED.!

(theorem & proof due to Karlin!)

Helson Problem: $R = \{\text{reals}\}$; $*$ a group operation on $R \ni R$ is top. ~~group~~ group, with euclidean topology. Then $\exists$ homeo. $\varphi \ni \varphi(x * y) = \varphi(x) + \varphi(y)$.

Another Helson Problem: Does boundedness of A on $L(G)$ follow from centralizing property?

$\rightarrow$ Consider $[a, b] * x$. This is compact + connected. So is an interval. Trivial to see it is $[a*x, b*x]$. Let μ be right invariant measure on R. Then

$$\mu([a, b]) = \mu([a*x, b*x]).$$

Let
$$\psi(x) = \mu([0, x]) \quad x > 0$$
$$= 0 \quad x = 0$$
$$= -\mu([x, 0]) \quad x < 0.$$

Then $\psi(x)$ is strictly $\uparrow$, cont., $\to \pm\infty$ with x.
So is homeo. and $\psi(b) - \psi(a) = \psi(b*x) - \psi(a*x)$.
Then $\psi(b) - \psi(e) = \psi(b*x) - \psi(x)$.
So $$\underbrace{\psi(b) - \psi(e)}_{\varphi(a)} + \underbrace{\psi(x) - \psi(e)}_{\varphi(x)} = \underbrace{\psi(b*x) - \psi(e)}_{\varphi(b*x)}.$$

QED.

Two Banach Space Theorems.

I. Let M be a strongly closed linear manifold in the space $\mathfrak{E}$ of bounded linear operators on $\mathfrak{X}$. Then $T \in M \equiv f(T) = 0$ whenever $f(M) = 0$, f being any strongly continuous linear functional on $\mathfrak{E}$.

Proof: Suppose $T \notin M$. Then $\exists \varepsilon;\ x_1, \cdots, x_n \in \mathfrak{X}$ $\ni \|T'x_i - Tx_i\| < \varepsilon,\ i = 1, \cdots, n \Rightarrow T' \notin M$. Hence $T' \in M \Rightarrow \sum_{i=1}^{n} \|T'x_i - Tx_i\| \geq \varepsilon$.

Form the Banach space $\mathfrak{X}^{(n)} = \{(y_1, \cdots, y_n)\}$, $y_i \in \mathfrak{X}$ with $\|(y_1, \cdots, y_n)\| = \sum_{i=1}^{n} \|y_i\|$.

Form $\mathfrak{M} \subset \mathfrak{X}^{(n)}$, $\mathfrak{M} = \{(T'x_1, T'x_2, \cdots, T'x_n) \mid T' \in M\}$.

$\overline{\mathfrak{M}}$ does not contain $(Tx_1, \cdots, Tx_n)$. Hence there is a bd. linear functional $y^* = (y_1^*, \cdots, y_n^*)$ such that $y^* \overline{\mathfrak{M}} = 0$, $y^*(Tx_1, \cdots, Tx_n) \neq 0$.

I.e.:
$$\sum_{i=1}^{n} y_i^* T'x_i = 0, \quad T' \in M,$$
$$\sum_{i=1}^{n} y_i^* T x_i \neq 0.$$

Since $\sum_{i=1}^{n} y_i^* T''x_i$, $T'' \in \mathfrak{E}$, is a strongly continuous linear ~~operator~~ functional we are done

II. Every f on E has form $f(T)=\sum_{i=1}^{n} x_i^* T x_i$
for some $x_i \in \mathfrak{X}$, $x_i^* \in \mathfrak{X}^*$, $i=1,2,\dots,n$. (T now general)

Proof: f is strongly continuous everywhere, hence at 0. Therefore $\forall \varepsilon>0$ $\exists$ $\delta, x_1, \dots, x_n \ni$
$\|Tx_i\|<\delta$, $i=1,\dots,n \Rightarrow |f(T)|<\varepsilon$.
Hence if $\|Tx_i\|=0$, $i=1,\dots,n$ we have $f(T)=0$. assume Tx_i maximal, so lin. ind.
So $f(T)$ depends only on $(Tx_1, \dots, Tx_n)$.
Its dependence is linear: $f(T)=\varphi(Tx_1,\dots,Tx_n)$.
If $\max_i \|T_\alpha x_i\| \xrightarrow[\alpha]{} 0$ then $f(T_\alpha)\to 0$. Hence

~~φ is bounded. So $\varphi(Tx_1,\dots,Tx_n)$~~ $=$
in $\mathfrak{X}^{(n)}$ form ~~$x^*Tx = \sum_i x_i^* T x_i$~~. $\mathfrak{M} = \overline{\{T(x_1,\dots,x_n)\}_{T\in E}}$
$\mathfrak{M}$ is a c.l.m. $\subseteq \mathfrak{X}^{(n)}$. φ is bd. on $\mathfrak{M}$, so has extension to all of $\mathfrak{X}^{(n)}$. Call extension x^*.
$x^* = (x_1^*, \dots, x_n^*)$; $x^*\xi = \sum_i x_i^* \xi_i$. So $\varphi(Tx)$
$= \sum_i x_i^* T x_i$ does it.

On max $\int_0^1 f(x(t))\,dt$, where f is fixed and cont, $x \in \mathfrak{X} = \{x \mid 0 \le x(t) \le 1, \text{ meas.}, \int_0^1 x(t)\,dt = c\}$.

Theorem: max exists.

Proof: $\varphi_x(f) = \int_0^1 f(x(t))\,dt$ is a bd. lin. fcnal on $C[0,1]$. So $\exists\, \alpha_x \ni \varphi_x(f) = \int_0^1 f(\xi)\,d\alpha_x(\xi)$.

$f \ge 0 \Rightarrow \int_0^1 f(x(t))\,dt \ge 0$. So $\alpha \uparrow$.

$\int_0^1 x(t)\,dt = c \Rightarrow \int_0^1 \xi\, d\alpha_x(\xi) = c$.

Conversely, given $\alpha \uparrow \ni \int_0^1 \xi\, d\alpha_x(\xi) = c$

$\exists\, x(t) \ni \int_0^1 f(x(t))\,dt = \int_0^1 f(\xi)\,d\alpha(\xi)$, $\int_0^1 x(t)\,dt = c$.

In fact the function inverse to α (properly defined!) does it.

Hence we have only to maximize $\int_0^1 f(\xi)\,d\alpha(\xi)$ over $A = \{\alpha \mid \|\alpha\| \le \|f\|, \alpha \uparrow, \int_0^1 \xi\, d\alpha(\xi) = c\}$. This is trivial. For Give A the w^* topology. A is closed. Since A is bd. we have A compact. $\int_0^1 f(\xi)\,d\alpha(\xi)$ is cont. Hence attains sup. QED.

Even if f is only u.s.c. max exists, for then $\int_0^1 f(\xi)\,d\alpha(\xi)$ is u.s.c. fcn. of α. Proof uses fact that A satisfies first countability axiom, which is the case for any bounded portion of the $\mathfrak{X}^*$ of a separable $\mathfrak{X}$, in w^* topology.

Let $f_n(\xi)$, cont, $\downarrow f(\xi)$. Then let $\alpha_m \to \alpha$, w^*.

We have: $\int_0^1 f_n(\xi)\,d\alpha_m(\xi) \xrightarrow[m]{} \int_0^1 f_n(\xi)\,d\alpha(\xi) \underset{n}{\searrow} \int_0^1 f(\xi)\,d\alpha(\xi)$.

$\int_0^1 f_n(\xi)\,d\alpha_m(\xi) \underset{n}{\searrow} \int_0^1 f(\xi)\,d\alpha_m(\xi)$.

To show $\int_0^1 f(\alpha)\,d\alpha(\xi) \geq \overline{\lim_{m\to\infty}} \int_0^1 f(\xi)\,d\alpha_m(\xi)$.

Let
$$a_{nm} = \int_0^1 f_n(\xi)\,d\alpha_m(\xi).$$
$$b_n = \int_0^1 f_n(\xi)\,d\alpha(\xi).$$
$$a_m = \int_0^1 f(\xi)\,d\alpha_m(\xi)$$
$$a = \int_0^1 f(\xi)\,d\alpha(\xi).$$

We have
$$\begin{cases} a_{nm} \to b_n \searrow a \\ a_{nm} \searrow a_m \end{cases}$$

To show $a \geq \overline{\lim_{m\to\infty}}\, a_m$.

Given $\varepsilon > 0$. Choose $n_0 \ni b_{n_0} < a + \varepsilon$.

Choose $m_0 \ni m \geq m_0 \Rightarrow a_{n_0 m} < b_{n_0} + \varepsilon$.

Then $m \geq m_0 \Rightarrow a_m \leq a_{n_0 m} < a + 2\varepsilon$.

$\therefore \overline{\lim_{m\to\infty}}\, a_m \leq a + 2\varepsilon$.

So $\overline{\lim_{m\to\infty}}\, a_m \leq a$. QED

14

Monthly problem Oct 1951 (appx).

$$\sum_m |b_m| < \infty$$

$$\lim_{\theta \to \infty} \sum_{m=1}^{\infty} b_m \cos\frac{\theta}{m} = 0$$

$$\Rightarrow \quad b_m \equiv 0.$$

Proof: $\sum_{m=1}^{\infty} b_m \cos f(m)\theta$ is a.p. fcn. of θ.

If it $\to 0$ at ∞ then it is $\equiv 0$.

So $\frac{1}{\theta}\int_0^\theta \left(\sum_{m=1}^{\infty} b_m \cos f(m)t\right)^2 dt = 0.$

But limit of this quantity as $\theta \to \infty$ is $\sum_{m=1}^{\infty} b_m^2$, providing all $f(m)$'s are distinct. Thus $\sum_{m=1}^{\infty} b_m^2 = 0$. Q.E.D.

An amusing example.

$f(x) \equiv 1$. δ finite subset of $[0,1)$.

$f_\delta(x) = 1$ for $x \in \delta$, $= 0$ for x midway (mod 1) between 2 successive points of δ, linear elsewhere.

$f_\delta(x) \xrightarrow[\delta]{} f(x)$, each x.

But $\int_0^1 f_\delta(x)\,dx = \frac{1}{2}$, $\int_0^1 f(x)\,dx = 1$.

Problem: $\lim_{n\to\infty} \sum_{j=0}^{n} x^{-j}\left(1 - \frac{j}{n}\right)^{-x}$

where $x > 1$, $x > 0$.

i.e. $\lim_{n\to\infty} \dfrac{x^n/n^x}{\sum_{j=1}^{n} \frac{x^j}{j^x}}$.

Integral equations. $(I-\lambda T)^{-1} = \dfrac{\sum_0^\infty \lambda^n S_n}{\sum_0^\infty \lambda^n a_n}$

S_n operators
a_n constants, depending on T.
$a_0 = 1$

$$\Rightarrow \left(\sum \lambda^n T^n\right)\left(\sum \lambda^n a_n\right) = \sum \lambda^n S_n$$

so $$S_n = \cancel{a_0} T^n + a_1 T^{n-1} + \cdots + a_{n-1} T + a_n I.$$

If numerator and denominator are to converge for all λ the a's must be pretty special! For in general best inequality for $\|S_n\|$ is e.g. $\|S_n\| \le C\|T\|^n$.

This is what is happening in Fredholm's solution of

$$u(x) = f(x) + \int_a^b K(x,y)\,u(y)\,dy$$

D. $f, u \in C([a,b])$ $K \in C([a,b]\times[a,b])$

The a_n are $$\frac{(-1)^n}{n!} \underbrace{\int\int\cdots\int}_{n} \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

and the S_n are essentially same with determinant bordered by: $K(x,y)$ $K(x,t_1) \cdots K(x,t_n)$
$K(t_1,y)$
$\vdots$
$K(t_n,y)$

By Hadamard's inequality the everywhere convergence follows, in the classical theory. Why!!

Sam says: because T is completely continuous singularities of $(\lambda I - T)^{-1}$ are poles. Such a function is the quotient of entire functions!

See
A.F. Ruston, Proc. Lon. Math. Soc. 53, 109
Smithies, Duke Math. J. 8. 107-130.

Feller: $u_n = f_n u_0 + f_{n-1} u_1 + \cdots + f_1 u_{n-1}$ $\quad n \ge 1$

where $u_0 = 1$

$f_n \ge 0$, $\sum_1^\infty f_n = 1$, $\sum_1^\infty n f_n < \infty$, ~~$\{f_n\}$ not~~

g.c.d. $\{n\} \ni f_n > 0$ is 1

$\Rightarrow \lim_n u_n = \dfrac{1}{\sum_1^\infty n f_n}$.

aperiodic case.

Sam's abstract formulation.

In (l) consider $I - \sum_1^\infty f_n T^n$, where $\|T\| \le 1$. Then this factors: equals

$$(I-T)\left(\sum_0^\infty g_n T^n\right) \qquad g_n = \sum_{\nu=n+1}^\infty f_\nu .$$

$$= (I-T)S$$

series converges from $\sum_1^\infty n f_n < \infty$

and S^{-1} exists. In fact, S^{-1} exists providing $\sum_0^\infty g_n \lambda^n \ne 0$ for $\lambda \in \sigma(T)$. Now $(1-\lambda)\sum_0^\infty g_n \lambda^n = -\sum_1^\infty f_n \lambda^n + 1$ does not vanish for $|\lambda| < 1$ since $\sum_1 f_n = 1$, and does not vanish for $|\lambda| = 1$, $\lambda \ne 1$ since $f_n \ge 0$. aperiodicity

Hence $\exists S^{-1}$.

Now let T be shift: $T\{x_n\} = \{x_{n-1}\}$ where $x_{-1} \equiv 0$. Then $S(I-T)u = u_0 = (1, 0, 0, \ldots)$ implies $(I-T)u = S^{-1}u_0 \in (l)$

so $\sum_0^\infty |u_n - u_{n-1}| < \infty$

and $\lim_n u_n$ exists. Value of $\lim_n u_n$ easily obtained. QED.

[redacted]

Survival games:

Finite game (a_{ij}). I has capital x, II has capital $1-x$. I plays strategy $\{p_i(x)\}$, II plays $\{q_j(1-x)\}$, these vectors depending only on x but changing with x as play proceeds.

$\pi(x)$ = prob. I survives.

Clearly, if $\pi(x)$ exists it satisfies

$$\pi(x) = \sum_{i,j} p_i(x) q_j(x) \pi(x+a_{ij}),$$

say

$$\pi(x) = \sum_k r_k(x) \pi(x+b_k)$$

where $\sum_k r_k(x) = 1$, $r_k(x) \geqq 0$.

$<\infty$ Properties? Existence of optimal p_i, q_j ??

"+ aperiodicity

Hausner-Wendel Games.

Let $p_1, p_2, \ldots, p_n$ be a ~~directed~~ set of points some pairs of which are connected by line segments with direction. The players each choose a point. If I chooses p_i & II p_j then I receives \$1 if $p_j \to p_i$, pays \$1 if $p_i \to p_j$, \$0 in all other cases.

This generalizes paper-scissors-stone.

Example $1 \to 2 \to 3 \to 4 \to 5 \to 6$.

Optimal strategies characterized by

$p_1 = p_4 = p_6$

where p_i now means probability of playing i.

Theorem proved by Hausner: Let $p_1, \ldots, p_n$ be points some pairs of which are connected by (nonoriented) line segments. Let an arbitrary subset be designated boundary elements. Then there exists one and only one real function on the p_i's taking prescribed boundary values & having the "average property" on the non-boundary points.

ity

18 A theorem, learned from Bellman.

Suppose A is <u>real</u> matrix and $A+A' \geqq 0$.
Then real parts of characteristic roots of A are $\geqq 0$.

Proof: Let $Ax = \lambda x$.
Then $A\bar{x} = \bar{\lambda}\bar{x}$

$$(Ax, \bar{x}) = \lambda(x, \bar{x})$$

Inner product is "real"; i.e. $(x, y) = \Sigma_i x_j y_j$.

$$(A'x, \bar{x}) = (x, A\bar{x}) = \bar{\lambda}(x, \bar{x})$$

$$\left((A+A')x, \bar{x}\right) = (\lambda+\bar{\lambda})(x, \bar{x})$$
$$= 2(Rl\,\lambda)(x, \bar{x})$$

QED.

Converse not true. E.g. $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$.

Another theorem from Bellman.

$\cancel{\Sigma a_{ij} x_i x_j \geqq 0}$ (a_{ij}) positive semi definite
(b_{ij}) " " "
$\Rightarrow$ $(a_{ij} b_{ij})$ " " "

Proof: $a_{ij} = \int u_i(t)\overline{u_j(t)}\, dG(t)$ $(*)$?

So $\Sigma a_{ij} b_{ij} x_i \overline{x_j} = \int \Sigma b_{ij} x_i u_i(t) \overline{x_j u_j(t)}\, dG(t)$

$\geqq 0$.

~~proof of (*)~~. First let E be a selfadjoint projection, $E^2 = E = E^*$. Then

$\geqq 0.$ $$e_{ij} = \sum_k e_{ik} e_{kj}$$

$$= \sum_k e_{ik} \bar{e}_{jk}$$

k $$= \int e_i(t)\, \bar{e}_j(t)\, d\varphi(t).$$

j. clearly $\varphi \uparrow$.

~~Now any self ad~~

~~and $\varphi \uparrow$ since $E \geqq 0$ (from $(Ex,x) = (E^2x,x) = (Ex, Ex)$)~~

D.. ~~Now if $A = A^*$ we have $A = \sum_i \lambda_i E_i$. If $A \geqq 0$ then $\lambda_i \geqq 0$. So $a_{ij} = \sum_i \lambda_i e_{i,ij}$~~

~~$$= \sum_i \lambda_i \int e_{i,i}(t)\, \bar{e}_{i,j}(t)\, d\varphi_i$$~~

(*) Proof: $\sqrt{A}$ exists, say $= B$; B is selfadjoint

Then ~~$a_{ij} = \sum b_{ij} b_{jk}$~~

~~$= \sum b_{ij} b_{kj}$~~

~~$= \int \ldots_i$~~

(t) $$a_{ij} = \sum_1 b_{ik} b_{kj}$$

$$= \sum b_{ik} \bar{b}_{jk}$$

$$= \int b_i(t)\, \bar{b}_j(t)\, d\varphi(t) \quad \text{in obvious way.}$$

20 Penny problem: 1 bad out of 12, to find in 3 weighings which it is and light or heavy.

J. Rosenbaum Jan 1947 Monthly:

I	1 2 3 4	5 6 7 8
II	1 2 3 5	4 9 10 11
III	1 6 9 12	2 5 7 10

F. J. Dyson, late '46 Math Gazette.

Let $M = \frac{3^n - 3}{2}$. Give integers from 1 through M their ternary expansions; assign the complements too, so now each integer has a pair of labels. Call ~~each~~ a label clockwise if first change is in cycle $0 \to 1 \to 2 \to 0$, if $0 \to 2 \to 1 \to 0$ call it anticlockwise.

E.g. 0000120 is clockwise.
{ 0210111 " anticlockwise.
{ 2012111 " clockwise complement of above.

Let $C(i, d) = \{p \mid$ clockwise label has d in i'th place, counting from left $\}$.

Permutation $0 \to 1 \to 2 \to 0$ shows $C(i,d) \to C(i, d+1)$ (mod 3)
so each $C(i,d)$ has $M/3$ pennies.

i'th weighing: ~~C(i,d)~~ $C(i,0)$ vs $C(i,2)$.
left right

record $a_i = 0$ or 2 for left or right + 1 for balance.

Bad penny has label $a_1 a_2 \ldots a_n$, and this is clockwise or anti according as penny is heavy or light!

$n=3$: 1345 2678
1678 291011
23811 56912

D: 1 2 3 4 5 6 7 8 9 10 11 12
R: 4 5 10 11 9 1 3 2 6 8 7 12

Hausner says: 19 equivalence classes!

Ku 1) 2;

Karlin problems:

1) A, B bounded operators on $\mathfrak{X}$.
$R(A) \supseteq R(B)$
A compact operator
$\Longrightarrow B$ compact !!

2) A closed operator on $\mathfrak{X}$.
$R(\lambda_0; A)$ has discrete spectrum
$\Longrightarrow A$ has discrete spectrum.

Proof of 1). Let $E_n = \{x \mid \exists y \ni \|y\| \leq n\|x\|,\ Ay = Bx\}$.
$$\mathfrak{X} = \bigcup_{n=1}^{\infty} E_n$$
$\therefore$ some $\overline{E}_n$ contains a sphere.

Clearly B is compact on E_n; for let $\|x_\nu\| \leq 1$, $x_\nu \in E_n$. We have y_ν corresponding, and $\|y_\nu\| \leq n$. So $\{Ay_\nu\}$ contains a convergent subsequence; $\therefore$ $\{Bx_\nu\}$ does also.

Consequently B is compact on $\overline{E}_n$. For if $x_\nu \in \overline{E}_n$ can select $x'_\nu \in E_n \ni \|x_\nu - x'_\nu\| < \frac{1}{\nu}$. $\{Bx'_\nu\}$ contains convergent subsequence. Therefore so does Bx_ν, since $\|Bx_\nu - Bx'_\nu\| < \frac{1}{\nu}\|B\|$.

So B is compact on a sphere. But then by linearity, B is compact on any sphere. Q.E.D.

Thoughts on quaternions – inspired by reading Klein

$\mathfrak{X}$ a 4-dimensional division algebra over R.

(1) $xy = 0 \Rightarrow x = 0$ or $y = 0$.

Consider $y \to xy$ as a linear transformation. Then x satisfies a quartic equation. Hence by (1) x satisfies a quadratic equation. If this is reducible then again by (1) $x = \lambda e$.

If $x \neq \lambda e$ we have

$$x^2 + 2ax + b = 0 \qquad a^2 < b$$
$$(x+ae)^2 = -(b-a^2)e$$
$$= -\lambda^2 e$$

Hence $\exists\ i \in \mathfrak{X} \ni i^2 = -1e$.
So $\mathfrak{X}$ has a basis e, i, j, k with $i^2 = j^2 = k^2 = -e$.

Consider ij. I claim this is lin. ind. of e, i, j.
In fact, if

then
$$\begin{cases} ae + bi + cj + dij = 0 \\ ai - be + cij - dj = 0 \\ aj + bij - ce - di = 0 \\ aij - bj - ci + de = 0 \end{cases}$$

$$\left.\begin{array}{l} ae + bi + cj + dij = 0 \\ -be + ai - dj + cij = 0 \\ -ce - di + aj + bij = 0 \\ +de - ci - bj + aij = 0 \end{array}\right\} \quad \text{mult by} \left.\begin{array}{c} a \\ -b \\ -c \\ d \end{array}\right\} \begin{array}{l} \text{add, get} \\ (a^2+b^2+c^2+d^2)e \\ \quad + 2(ad-bc)ij = 0. \end{array}$$

$\therefore$ If $a^2+b^2+c^2+d^2 \neq 0$, $ij = \lambda e$. $\therefore -j = \lambda i$ <u>absurd</u>.

Now I claim $(ij)^2 = -1$. In fact, let

$$(ij)^2 + aij + b = 0.$$
Then $$ijij + aij + b = 0 \qquad a^2 < 4b \text{ since } ij \neq \lambda e.$$

$$\begin{cases} ji + a + bij = 0 & (i \cdots j) \\ ij + a + bji = 0 & (ji \cdots) \end{cases}$$

$$ji - ij = b(ji - ij)$$

If $ij = ji$ we have $(ij)^2 = ijji = e$ so ~~$aij = -b - 1$~~. $ij = \pm 1e$ which is impossible.

[redacted]

Hence $b = 1$. Hence $ij + ji = -ae$

~~Then~~

$\cancel{j + iji = -ai}$

$\cancel{iji\,j = ai}$

Now let $J = \frac{-a}{\sqrt{4-a^2}} i + \frac{2}{\sqrt{4-a^2}} j$

Then

$$J^2 = -\frac{a^2+4}{4-a^2} e - \frac{2a}{4-a^2}(ij+ji)$$

$$= \frac{e}{4-a^2}(-a^2-4+2a^2)$$

$$= \frac{e}{4-a^2}(a^2-4) = -e.$$

and

$$iJ + Ji = \frac{a}{\sqrt{4-a^2}} e + \frac{2}{\sqrt{4-a^2}} ij + \frac{a\,e}{\sqrt{4-a^2}} + \frac{2}{\sqrt{4-a^2}} ji$$

$$= \frac{2a}{\sqrt{4-a^2}} e - \frac{2a}{\sqrt{4-a^2}} e = 0.$$

$$(iJ)^2 = \frac{1}{4-a^2}(ae + 2ij)^2$$

$$= \frac{1}{4-a^2}(a^2 + 4aij + 4(ij)^2)e$$

$$= \frac{1}{4-a^2}(a^2-4)e = -e$$

By above arg. applied to J in place of j we know that iJ is ind. of e, i, J.

Hence, changing notation, we has a basis $e, i, j, k \ni i^2 = j^2 = k^2 = -e$, $\left.\begin{matrix} ij = k \\ ji = -k \end{matrix}\right\}$ $\therefore$ $\begin{matrix} ik = -j, \\ ki = j \end{matrix}$

$\therefore jk = +i$
and $kj = -i$

24 A trivial theorem: ← This is equivalent to: $x = 0e + ai + \ldots + ck$

If $x, y \in \mathfrak{X}$, $x^2 = -\lambda^2 e$, $y^2 = -\mu^2 e$ then

$$(x+y)^2 = -\nu^2 e, \quad (ax)^2 = -a^2\lambda^2 e.$$

Clearly the set of $x \in \mathfrak{X}$ whose squares are of form $-\lambda^2 e$ is a linear 3-dim. space, say $\mathfrak{X}_0$.

$x \in \mathfrak{X} \Rightarrow \exists!\ \lambda,\ x_0 \in \mathfrak{X}_0 \ni x = \lambda e + x_0$

Def: $\bar{x} = \lambda e - x_0$. Then $x \in \mathfrak{X}_0 \Rightarrow \bar{x} = -x$

Thm: $\overline{\lambda x} = \lambda \bar{x}$, $\overline{x+y} = \bar{x} + \bar{y}$, $\overline{xy} = \bar{y}\bar{x}$.

Def: $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$ $= (y, x)$ Note that this gives bona fide inner product wh. agrees with Euclidean inner product. $x \perp y$

Thm: $(x, yz) = (x\bar{z}, y)$. (But not from front!)

Rotations in 3-space.

Let T be a rotation through an angle ω about a vector $g \in \mathfrak{X}_0$. I claim:

$$Tx = p x \bar{p} \quad \text{where} \quad p = \cos\tfrac{\omega}{2} e + g,$$

the length of g being $\sin\frac{\omega}{2}$, so that $g\bar{g} = \sin^2\frac{\omega}{2}$, or $g^2 = -g\bar{g} = -\sin^2\frac{\omega}{2}$; so $p\bar{p} = 1$

In fact, ~~let~~ $x \perp g$, ~~then~~ $x \in \mathfrak{X}_0$

note that $p\mathfrak{X}_0\bar{p} = \mathfrak{X}_0$ for is linear, 1-1, onto

i) $(px\bar{p})^2 = px\bar{p}px\bar{p} = px^2\bar{p} = -\lambda^2 e\, p\bar{p} = -\lambda^2 e$

ii) $x = p(\bar{p}xp)\bar{p}$ and $\bar{p}xp \in \mathfrak{X}_0$ clearly.

~~Now let~~ $x \in \mathfrak{X}_0$, $x \perp g$. I.e. $x\bar{g} + g\bar{x} = 0$ I.e. $xg = -gx$

~~Since~~ $\bar{g} = -g$, $\bar{x} = -x$.

~~Instead~~

Instead try $pg\bar{p}$. This is $(\cos\frac{\omega}{2} e + g)\, g\, (\cos\frac{\omega}{2} e - g)$

$= \cos^2\frac{\omega}{2}\, g + g^2\cos\frac{\omega}{2} e - g^2\cos\frac{\omega}{2} e - g^2 g = g = Tg$

[redacted]

Now consider $(px\bar{p}, g)$. This is

~~$px\bar{p}\bar{g} + gpx\bar{p} = px\bar{p}\bar{g} + gpx\bar{p}$~~

~~$= (\cos\frac{w}{2}e + g)x(\cos\frac{w}{2}e - \dots$~~

~~$(px\bar{p}, g) = (px, gp) = (\cos\frac{w}{2}x + gx, g\cos\frac{w}{2} + g^2)$~~

~~$= \cos^2\frac{w}{2}(x, g) + (gx, g\cos\frac{w}{2}) + \cos\frac{w}{2}(x, g^2) + (gx, g^2)$~~

~~$= 0 - (x, \sin^2\frac{w}{2}\cos\frac{w}{2}e) + \cos\frac{w}{2}(x, -\sin^2\frac{w}{2}e)$~~

$(px\bar{p}, g) = (x, \bar{p}gp) = (x, g) = 0$ if $x \perp g$. Thus

$x \perp g \Rightarrow px\bar{p} \perp g$. Let $(x, x) = 1$. So $x^2 = -1$.

$x \perp g \Rightarrow xg = -gx$

Finally, $(px\bar{p}, x) = (px, xp) = ((\cos\frac{w}{2}e + g)x, x(\cos\frac{w}{2}e + g))$

$= \cos^2\frac{w}{2}(x, x) + \cos\frac{w}{2}(gx, x) + \cos\frac{w}{2}(x, xg) + (gx, \cancel{xg})$

$= \cos^2\frac{w}{2}\cancel{(x,x)} + \cos\frac{w}{2}(x, +gx) + \cos\frac{w}{2}(x, xg) - \sin^2\frac{w}{2}$

$= \cos 2\frac{w}{2} = \cos w$ $\qquad gx + xg = 0.$

QED

Hence $\bar{p}gp = g$, for [illegible] transform [illegible]

Cramer problem.

$$\Rightarrow \quad k^2 \int_k^\infty \overset{g(x)}{\downarrow}\, g(x)\,dx \leq \frac{4}{9}\int_0^\infty x^2 g(x)\,dx, \text{ all } k>0.$$

Sufficient to prove for any one $k>0$.

Take $k = 2/3$. Boils down to $\int_{2/3}^1 g(x)\,dx \leq \int_0^1 x^2 g(x)\,dx$.

For $g\uparrow$ eq. to $\int_0^1 x^2 g(x)\,dx \leq \int_{2/3}^1 g(x)\,dx$.

Special case of

$$\int_0^1 f(x)\,g(x)\,dx \leq \int_{1-\int_0^1 f(\xi)\,d\xi}^1 g(x)\,dx$$

where $0 \leq f \leq 1$, $g\uparrow$.

Sketch of proof: Since true for g = const (equality holds) can assume g vanishes at $1-\int_0^1 f(\xi)\,d\xi$. Then

$$\int_0^1 f(x)g(x)\,dx \leq \int_{1-\int_0^1 f(\xi)d\xi}^1 f(x)g(x)\,dx \leq \int_{1-\int_0^1 f(\xi)d\xi}^1 1\cdot g(x)\,dx \qquad \underline{\text{QED}}.$$

Ex. $\displaystyle\int_0^1 x^a x^b\,dx \leq \int_{1-\int_0^1 x^a dx}^1 x^b\,dx$

$$\frac{1}{a+b+1} \leq \left.\frac{x^{b+1}}{b+1}\right]_{1-\frac{1}{a+1}}^1$$

$$\frac{1}{a+b+1} \leq \frac{1}{b+1}\left\{1-\left(\frac{a}{a+1}\right)^{b+1}\right\}.$$

$$\frac{b+1}{a+b+1} \leq 1-\left(\frac{a}{a+1}\right)^{b+1} \quad\therefore\quad \left(\frac{a}{a+1}\right)^{b+1} \leq \frac{a}{a+b+1}$$

~~a useful example, showing limitations of uniform boundedness theorem if not careful.~~ 27

~~Adapted from von Neumann, Annalen v. 102 pg 370 + ff.~~

~~Let $\mathcal{H}$ = separable Hilbert space $= (l_2)$.~~
~~Let D be the following directed set~~

ddx. ~~$D = \{\alpha \mid \alpha = (m, n)$ where $m = 1, 2, \cdots$ and $n = 1, 2, \cdots, \infty\}$~~

~~(note $n = \infty$ is allowed, $m = \infty$ is forbidden).~~

~~Let T_α on $\mathcal{H}$ be defined by~~

~~$n \neq \infty$: $T_\alpha(x_1, \cdots, x_n, \cdots) = (0, \cdots, 0, x_m, \cdots, m x_n, \cdots)$~~

~~i.e. x_m in m'th place, $m x_n$ in n'th place, zeros elsewhere.~~

Let G be a nondiscrete l.c. group.
Form $L_1(G)$. Form $M(G)$ = set of measures on G = set of left centralizers on G. Topologies on $M(G)$:

st

$\mathcal{U}$: norm
$\mathcal{B}$: strong-bounded
$\mathcal{S}$: strong.

D. $\mathcal{S} \subset \mathcal{B} \subset \mathcal{U}$.

Example to show $\mathcal{S} \subset \mathcal{B}$ properly.
Let N be a strong neighborhood of 0, say

$$N = \{\mu \mid \|x_i \mu\| < \varepsilon,\ i = 1, 2, \ldots, n\}$$

[But D thinks of $\mathcal{B}$ as being defined by $\{\mu \mid \|x_i\mu\| < \varepsilon, \|\mu\| \leq c\}$ (which is wrong). So I don't know what this example is. On checking, this one can be used. Let $\alpha = (x_i, \varepsilon)$ directed set. μ_α chosen, take $K = n$.]

Then N contains μ of arbitrarily large norm.
Choose a nbhd V of $e \in G$ so small that

$$\int |x_i(g g_0^{-1}) - x_i(g)|\, dg < \frac{\varepsilon}{K}, \quad i = 1, \ldots, n,$$

$g_0 \in V$. Let $g_1, \ldots, g_{2K}$ be points of W, define μ = mass 1 at g odd, -1 at g even, where $W W^{-1} \subseteq V$. $\|\mu\| = 2K$.

But

$$\|x_i \mu\| \leq \sum_{j=1}^{K} \int |x_i(g g_{2j}^{-1}) - x_i(g g_{2j+1}^{-1})|\, dg$$

i

$i+1$

$$< K \frac{\varepsilon}{K} = \varepsilon \qquad \text{So } \mu \in N.$$

28 number of $n \times n$ matrices in $GF(2)$ which are nonsingular $= 2^{n(n-1)/2}(2^n-1)(2^{n-1}-1)\cdots(8-1)(4-1)(2-1)$.

Proof: We must find all ordered bases. There are 2^n vectors. $x_1 \neq 0$ can be chosen in 2^n-1 ways. Suppose $x_1, \dots, x_{k-1}$ $(k-1<n)$ chosen. x_k can be chosen $\neq 0$ and $\neq$ sum of any subset of $x_1, \dots, x_k$, so in $2^n - 2^{k-1}$ ways. QED.

This gives P_n (det of integers is odd). Monthly June-July 53 Prob.

Prob (May 53 monthly)

$$\lim_{n\to\infty}(\tanh 1 + \cdots + \tanh n - \log\cosh n).$$

Two partial answers:

i) $\log 2 + 2\sum_{j=1}^{\infty}(-1)^j \frac{1}{e^{j}-1}$

ii) $\frac{1}{2} - \int_0^\infty \frac{[x]-x+\frac{1}{2}}{\cosh^2 x}\,dx$

Don Darling prob.

$f \in L(-\infty,\infty)$, even, $\int f = 1$.

$$u_n = \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_1)f(x_2-x_1)\cdots f(x_n-x_{n-1})\,dx_n\,dx_{n-1}\cdots dx_1$$

$$= \frac{1}{2^{2n}}\binom{2n}{n}.$$

φ

Solution. $\int_{\text{whole space}} \varphi(\vec{x})\,d\vec{x} = 1$

$$= \sum_i \int_{M_i} \varphi(\vec{x})\,d\vec{x}$$

where $x_i >$ all other x_j for $\vec{x} \in M_i$.

By easy changes of variable we see that

$$\int_{T_i} \varphi = a_i a_{n-i} \qquad (a_0 = 1)$$

Thus
$$\sum_{i=0}^{n} a_i a_{n-i} = 1.$$

Let
$$\sum_0^\infty a_i \theta^i = f(\theta). \quad \text{Then } f(\theta)^2 = (1-\theta)^{-1}. \quad \underline{\text{QED.}}$$

Prob.

To solve $x^2+y^2 = u^2+v^2+1$ in integers.
First observe that $u+x$ and $v+y$ cannot both be odd. For mod 2 we would have $u \equiv x+1$, $v \equiv y+1$, $x^2+y^2 \equiv x^2+1+y^2+1+1$, absurd.
Next,
$$(x-u)(x+u) + (y-v)(y+v) = 1.$$
So $(x+u,\ y+v) = 1$ (are coprime).
Let m and n be any coprime integers, not both odd. Find a and b ∋
$$am + bn = 1.$$
We can assume that $a \equiv m\ (2)$, $b \equiv n\ (2)$.
For if a is odd, m even*, then necessarily b and n are odd, and we may replace a by $a+\lambda n$, b by $b-\lambda m$ with λ odd.
Let
$$\left.\begin{aligned} x+u &= m \\ x-u &= a \end{aligned}\right\} \rightarrow x = \frac{m+a}{2}, \quad u = \frac{m-a}{2}$$
and similarly for y, v. $\underline{\text{QED.}}$

[*note [illegible] cannot result, by above]

Ex.
$m = 4, \quad n = 23$
$a = 6, \quad b = -1$
$u = 1, \quad v = 12$
$x = 5, \quad y = 11$

$5^2 + 11^2 = 1^2 + 12^2 + 1.$

30 Calabi - quoted example of two disjoint convex sets in real (l_2) each dense in (l_2) and whose union is (l_2).

First let $A = \{(a_1, \cdots) \mid \underline{\lim} \sum_1^n a_k > 0\}$

$B = \{(b_1, \cdots) \mid \overline{\lim} \sum_1^n b_k \le 0\}$

$A \cap B = \phi$, A and B convex, A and B each dense in (l_2). Now if $A \cup B \neq (l_2)$ let $x \in (l_2) - (A \cup B)$. ~~I claim I can adjoin x to A or to B in such a way that the convex of the new~~

Consider $A' =$ convex of A, x

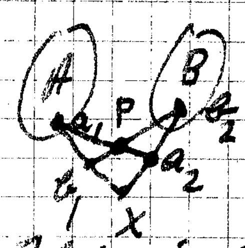
$B' =$ " " B, x.

Suppose $A' \cap B \neq \phi$, $B' \cap A \neq \phi$.

Then

$\exists \lambda_1 \exists a_1, b_1 \ni b_1 = \lambda_1 x + (1-\lambda_1) a_1$

$\exists \lambda_2 \exists a_2, b_2 \ni a_2 = \lambda_2 x + (1-\lambda_2) b_2$.



Form p as in diagram. $p \in A \cap B$! This is absurd. Hence one of $A' \cap B$ and $B' \cap A$ is empty. Adjoin x appropriately and proceed transfinitely. (QED)

Formula for p obtained by eliminating x:

$$\lambda_1 \lambda_2 x = \lambda_2 b_1 - \lambda_2 (1-\lambda_1) a_1 = \lambda_1 a_2 - \lambda_1 (1-\lambda_2) b_2$$

So $$p = \frac{\lambda_2 b_1 + \lambda_1 (1-\lambda_2) b_2}{\lambda_1 + \lambda_2 - \lambda_1 \lambda_2} = \frac{\lambda_1 a_2 + \lambda_2 (1-\lambda_1) a_1}{\lambda_1 + \lambda_2 - \lambda_1 \lambda_2}$$

Theorem (Special case of result of Kaplansky — quoted by Koch)

Let R be a compact connected topological ring. Then R is a zero ring: $xy = 0$ for all $x, y \in R$.

Proof: Let G be the additive group of R, is abelian. Let Γ be its character group. Since G is compact Γ is discrete. For each $x \in R$, $\gamma \in \Gamma$ define $\gamma_x \in \Gamma$ by $\gamma_x(y) = \gamma(xy)$ all $y \in R$. (γ_x is clearly a cont. char. on G). Holding γ fixed vary x in R. Then the function γ_x is continuous, hence constant. So $\gamma_x = \gamma_0$. So $\gamma(xy) = \gamma(0) = 1$ all $x, y \in R$.

[redacted]

But then $xy = 0$, since G possesses sufficiently many characters. QED.

A theorem of Shields — perhaps Rosen too.

A compact semigroup S has a nontrivial invariant integral if and only if kernel of S is a group; and in that case the integral comes from Haar measure on kernel.

(Non trivial means $I(1) = 1$; I is lin fcnal on $C(S)$, $I(\geq 0) \geq 0$, translation invariant.)

Proof consists in showing that any closed right (left) ideal has measure 1. Then kernel consists of single minimal two sided ideal in which circumstance K is known to be a group.

Theorem: No Banach space can have a countable (infinite) vector space basis.

Proof: Let $\mathfrak{X}$ be spanned by $e_1, e_2, \dots, e_n, \dots$.
Let $\mathfrak{X}_n$ = span of $e_1, e_2, \dots, e_n$.
This is closed and contains no sphere.
$\mathfrak{X} = \cup \mathfrak{X}_n$, contradicting 2nd cat. QED

Proof that $\mathfrak{X}_n$ is closed. Need only show: if $\sum_i a_i^\nu e_i \to 0$ then each $a_i^\nu \to 0$ as $\nu \to \infty$. Let $m^\nu = \max(1, |a_1^\nu|, |a_2^\nu|, \dots, |a_n^\nu|)$. $\frac{1}{m^\nu} \leq 1$, so $\frac{1}{m^\nu} \sum_i a_i^\nu e_i \to 0$. $\frac{a_i^\nu}{m^\nu}$ bounded, so can extract converging subsequence. Clearly then $\frac{a_i^\nu}{m^\nu}$ actually $\to 0$. Now I claim $m^\nu = 1$ for $\nu \geq \nu_0$. Otherwise $m^\nu = |a_i^\nu|$ some i infinitely often, contradicting $\frac{a_i^\nu}{m^\nu} \to 0$. ~~QED~~. Then $a_i^\nu \to 0$. QED.

Theorem (Littlewood, A Mathematician's Miscellany).

Let $a_1, a_2, \dots > 0$. Then

$$\overline{\lim_{n \to \infty}} \left(1 + \frac{a_{n+1}}{a_n}\right)^n \geq e, \text{ best possible.}$$

(next pg.)

3) Proof: Let $a_n = n^\alpha$, $\alpha \geq 1$.

Then $$\left(\frac{1+a_{n+1}}{a_n}\right)^n = \left(\frac{1+(n+1)^\alpha}{n^\alpha}\right)^n = \left(\left(1+\frac{1}{n}\right)^\alpha + \frac{1}{n^\alpha}\right)^n$$

$$= \left(1+\frac{\alpha}{n} + o\left(\frac{1}{n}\right) + \frac{1}{n^\alpha}\right)^n \begin{array}{l} \longrightarrow e^\alpha \text{ if } \alpha > 1 \\ \longrightarrow e^2 \text{ if } \alpha = 1 \quad ! \\ \longrightarrow \infty \text{ if } \alpha < 1 \end{array}$$

Sharper meaning of "best possible":

Let $a_n = n \log n$.

$$\frac{1+a_{n+1}}{a_n} = \frac{1+(n+1)\log(n+1)}{n \log n} = \frac{1+n\log(n+1)+\log(n+1)}{n\log n}$$

$$= \frac{1+n\log n + n\log\left(1+\frac{1}{n}\right) + \log(n) + \log\left(1+\frac{1}{n}\right)}{n \log n}$$

$$= 1 + \frac{1}{n} + \frac{1}{n\log n} + \frac{\log\left(1+\frac{1}{n}\right)}{\log n} + \frac{\log\left(1+\frac{1}{n}\right)}{n\log n}$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n\log n}\right)$$

$$= 1 + \frac{1}{n} + o\left(\frac{1}{n}\right) \quad \text{whose } n\text{'th power} \to e.$$

~~Now if $a_n \not\to \infty$, i.e. $a_{n_k} \leq M$ for $k=1,2,\ldots$~~

~~we have $\left(\frac{1+a_{n_k+1}}{a_{n_k}}\right)^{n_k} \geq \frac{(1+a_{n_k+1})^{n_k}}{M^{n_k}}$~~

Now consider $\frac{a_{n+1}}{a_n}$. If $\overline{\lim_{n\to\infty}} \left(\frac{a_{n+1}}{a_n}\right)^n \geq e$

we are done.

Suppose then $\overline{\lim_{n\to\infty}} \left(\frac{a_{n+1}}{a_n}\right)^n < e$.

So
$$\overline{\lim_{n\to\infty}} \left(\frac{n}{n+1}\,\frac{a_{n+1}}{a_n}\right)^n < 1$$

Say
$$\left(\frac{a_{n+1}}{a_n}\right)^n \leq (1-\delta)\left(\frac{n+1}{n}\right)^n$$

$$\frac{a_{n+1}}{a_n} \leq (1-\delta)^{1/n}\left(1+\frac{1}{n}\right)$$

~~Now note: $x \log(1+x) \leq x^2$. Hence $(1+x)^x \leq e^{x^2}$.~~

So

$$\frac{a_{n+1}}{a_n} \leq (1-\delta)^{1/n} \cancel{e^{1/n^2}} \cdot \left(\frac{n+1}{n}\right)$$

$$\vdots$$

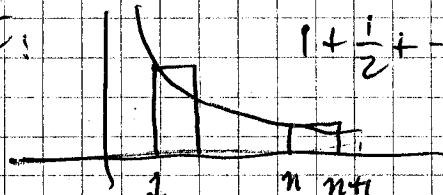
$$\frac{a_2}{a_1} \leq (1-\delta)\ e\ \left(\frac{2}{1}\right)$$

$$a_{n+1} \leq a_1 (1-\delta)^{1+\frac{1}{2}+\cdots+\frac{1}{n}} \cancel{e^{1+\frac{1}{4}+\cdots+\frac{1}{n^2}}}$$

$$= k\,(1-\delta)^{1+\frac{1}{2}+\cdots+\frac{1}{n}} \cdot (1+1)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)$$

$\cdot(n+1)$

Now note: $1+\frac{1}{2}+\cdots+\frac{1}{n} \geq \log(n+1)$.



So $a_{n+1} \leq k(1-\delta)^{\log(n+1)} \cdot (n+1)$

$$a_n \leq k\,(1-\delta)^{\log n} \cdot n$$

$$a_n \leq k\, n^{\log(1-\delta)} \cdot n \quad \text{since } \log(1-\delta) \leq -\delta$$

We have $a_n \leq k\, n^{-\delta} \cdot n$

Then $\left(\frac{1+a_{n+1}}{a_n}\right)^n \geq \left(\frac{1}{a_n}\right)^n \geq k^{-n}\, n^{n\delta - n}$ $\cancel{\xrightarrow[n\to\infty]{} \infty}$.

Patching:

If $\cancel{an}$ $\log(1-\delta) < -1$ still ok, i.e. if $1-\delta < \frac{1}{e}$,

$\delta > \frac{e-1}{e}$. Because then $a_n^{-n} \geq k^{-n}\, n^{n(-\log(1-\delta)-1)} \to \infty$

If $\log(1-\delta) > -1$ then $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^n > 0$.

$$\left(\frac{1+a_{n+1}}{a_n}\right)^n = \left(\frac{1+a_{n+1}}{a_{n+1}}\, \frac{a_{n+1}}{a_n}\right)^n = \left(1+\frac{1}{a_{n+1}}\right)^n \left(\frac{a_{n+1}}{a_n}\right)^n$$

$$\frac{1}{a_n} \geq k \frac{1}{n^{1-\delta}} \quad \text{So} \quad \left(1+\frac{1}{a_{n+1}}\right)^n \geq \left(1+\frac{k}{n^{1-\delta}}\right)^n \to \infty$$

and therefore

$$\lim_{n\to\infty} \left(\frac{1+a_{n+1}}{a_n}\right)^n = \infty.$$

pg 80

12/17/54 A curious integration problem — from $xy'' + (x^2-1)(y'-1) = 0$.

34 $$\int \frac{x^2-1}{x^2} e^{x^2/2}\,dx = -\frac{1}{x}(x^2-1)e^{x^2/2} + \int \frac{1}{x}\underbrace{\{2x + x(x^2-1)\}}_{x^2+1} e^{x^2/2}\,dx$$

$$dv = \frac{dx}{x^2}, \quad u = (x^2-1)e^{x^2/2}$$

$$\int \frac{x^2-1}{x^2} e^{x^2/2}\,dx = \left(x+\frac{1}{x}\right)e^{x^2/2} - \int \underbrace{\left(x+\frac{1}{x}\right)x}_{x^2+1} e^{x^2/2}\,dx$$

$$dv = \frac{x^2-1}{x^2}\,dx, \quad u = e^{x^2/2}$$

Adding, $2\int \frac{x^2-1}{x^2} e^{x^2/2}\,dx = \frac{2}{x} e^{x^2/2}$! $\therefore \int \frac{x^2-1}{x^2} e^{x^2/2}\,dx = \frac{e^{x^2/2}}{x} + c$

Can it be treated by series?

$$\int (1-x^{-2})\left(1+\frac{x^2}{2}+\frac{1}{2!}\frac{x^4}{4}+\cdots+\frac{1}{n!}\frac{x^{2n}}{2^n}+\cdots\right)dx$$

$$= \int\left\{-x^{-2}+\frac{1}{2}+\frac{3}{8}x^2+\cdots+\frac{1}{2^n n!}\left(1-\frac{1}{2(n+1)}\right)x^{2n}+\cdots\right\}dx$$

$$= +\frac{1}{x}+\frac{1}{2}x+\frac{1}{8}x^3+\cdots+\frac{1}{2^n n!}\frac{\cancel{2n+1}}{2n+2}\frac{x^{2n+1}}{\cancel{2n+1}}+\cdots \quad +c$$

$$= \frac{1}{x}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\cdots+\frac{1}{2^{n+1}(n+1)!}x^{2n+2}+\cdots\right)+c$$

$$= \frac{1}{x} e^{x^2/2} + c \quad !$$

3/24/55 (Collins Conversation).

1. An ordered space with order topology (nbhds are $\{x \mid a<x<b\}$) is complete iff every bounded set has compact closure. (Half rays are subbase.)

2. The relative topology on a subset of an ordered space need not agree with, in fact is generally stronger than (more open sets) the ~~topolo~~ order topology on the subspace itself.

Example $R \setminus [0,1]$ is homeomorphic to R in its order topology.

4/6/55 Koch quotes.

I. A connected space X can be linearly ordered continuously iff $X \times X$ is separated by diagonal. (Eilenberg '41 AmJ) (T_2)

II. A subset A of a T_2 locally compact X is locally compact in relative topology iff $A = O \cap C$ for some open O and closed C. (Hard 50% in Pollard's notes of Cartan's lectures on alg. top.)

Cor: A l.c. subgp of a T_2-gp is closed.

III. (Smith) ~~W~~ X is conn. iff any transitive relation containing a nbhd of diag is all of $X \times X$.

4/20/55 Recall of Koch - Collins - W. conversation

Tietze theorem: A closed in X normal, f cont on A to bd set B of reals $\Rightarrow$ f has cont. extension on all of X to closure of B.

Generalization: B need not be bounded.

Form $g = \arctan f$.

Extend g to G, Let $N = \{x \mid |G(x)| = \frac{\pi}{2}\}$.

Form continuous φ which is 0 on N, 1 on A.

Then $\varphi \cdot G$ maps X to $(-\frac{\pi}{2}, \frac{\pi}{2})$

Form $F = \tan \varphi G$. $F|A = f$, F cont. QED.

36

10/18/55 Conjectures of C. J. Titus.

1. Let $a_j = e^{i\theta} e^{i\theta_j}$ $\quad 0 = \theta_1 \leq \theta_2 < \dots < \theta_n < 2\pi$

$b_j = e^{i\varphi_j}$ $\quad 0 = \varphi_1 < \varphi_2 < \dots < \varphi_n < 2\pi$

and $\sum a_j = \sum b_j = 0$.

Then $\sum a_j \bar{b}_j \neq 0$!!

2. $\sum_0^n a_j x^j$ is bd. away from 0 for all n and a_j's as above for $|x| = 1$.

10/11/55 Result of Frank Spitzer

Let $x_1, x_2, \dots, x_n$ be reals.

Let $S(\sigma) = \max\{0, x_{\sigma(1)}, x_{\sigma(1)} + x_{\sigma(2)}, \dots\}$

Let $T(\tau) = \sum_c t(c)$

where τ is a permutation written as a product of disjoint cycles c and

$t(c) = \max\{0, \sum x_i\}$

where c is cycle on i's in some order.

Then: $\exists\ \varphi : 1\text{-}1 \ni S(\sigma) = T(\varphi(\sigma))$ for all σ.

10-11/55. Darling Problem. If $x_1, x_2, \dots$ are identically distributed independent random variables and $S_n = x_1 + \dots + x_n$ then if $P(S_n > 0) = P(S_n < 0)$ for every n is it true that the x_i's are symmetric?

Darling remarks on Kolmogoroff statistic.

Let $F_n(x)$ be the empiric distribution of n rectangular (on $[0,1]$) independent random variables. Want

$$G(a) = P(\,|F_n(x) - x| < a \text{ for all } x\,).$$

Let $X(t)$ be a Poisson process with parameter n. Then

$$G(a) = P\left(\,\left|\frac{X(t)}{n} - t\right| < a \;\middle|\; X(1) = n\right).$$

Problems (1) To characterize those ideals of measures (normalized nonnegative) on a group whose complements are semigroups. [Ideal in semigroup sense.]
Example (Cramer-Lévy): The set of non-normal distributions;
(Raikov): The set of non-poisson distributions.

(2) To understand Darmois' Theorem, that if X & Y are independent real r.v.'s such that for some nontrivial a, b, c, d $aX + bY$ & $cX + dY$ are also independent then X & Y are Gaussian.

(3) To prove the Lévy Cramer theorem by extremal property ≟ e.g. Weyl's inequality or, in fact, to give any essentially real variable proof.

(4) If x_n ind. 1 or 0 p or q then $S_n/n \to p$ almost surely.

Hence with probability 1 for any ε a last n exists $\ni$ $S_n/n - p \geq \varepsilon$.

What is distribution of n? Let N be number of $S_n/n - p \geq \varepsilon$. What is distribution of N?

Theorem: $x_1, x_2, \dots, x_n$ ind. 0-1-rectangular

$x_{(1)} > x_{(2)} > \dots > x_{(n)}$ their order-statistics.

Then $x_{(1)}, \quad \frac{x_{(2)}}{x_{(1)}}, \dots, \quad \frac{x_{(n)}}{x_{(n-1)}}$

are independent (though not identical).

11/17/55 Levy conversation.

(1) Let $0 < x < 1$, $\quad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$ with convergents

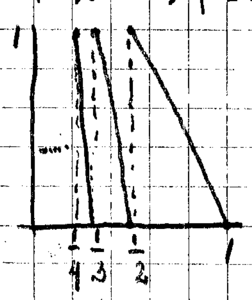
$\frac{p_n}{q_n}$. Then $y_n \equiv (q_n + q_{n-1}) \, | q_n x - p_n |$ is uniformly distributed on $(0,1)$, but these are not independent; probably "ultimately" so?

Does the empiric distribution of $y_1, y_2, \dots, y_n$ converge almost surely to the rectangular?

Example. $\frac{p_{-1}}{q_{-1}} = \frac{1}{0}, \quad \frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{[\frac{1}{x}]}$

$$y_0 = (1+0) \, | 1 \cdot x - 0 | = x.$$

$$y_1 = ([\tfrac{1}{x}] + 1) \, \left| [\tfrac{1}{x}] x - 1 \right|$$



(2) Let $\quad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n + z_n}}}$.

Celebrated problem of Gauss was to find limiting distribution of z_n if x is rectangular. Gauss' result was $\frac{\log(1+x)}{\log 2}$, but

not actually proved. Cf. Uspensky, Probability, Appendix
Argument begins with:

$$z_{n+1} = \frac{1}{z_n} - \left[\frac{1}{z_n}\right].$$

Let
$$f(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Let
$$E_\alpha = \{x \mid f(x) < \alpha\} = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+\alpha}, \frac{1}{n}\right)$$
$$0 < \alpha < 1.$$

Then
$$F_n(\alpha) = P(z_n < \alpha) = \int_{E_\alpha} dF_{n-1}(x) \ !$$

If $\lim F_n$ exists $= F$ then presumably
$$F(\alpha) = \int_{E_\alpha} dF(x)$$
$$= \sum_{1}^{\infty} \left(F\left(\frac{1}{n}\right) - F\left(\frac{1}{n+\alpha}\right)\right)$$
which is satisfied by $F(\alpha) = c \log(1+\alpha)$!
$$\cancel{c} \log(1+\alpha) = \sum_{1}^{\infty} \left(\cancel{c} \log\left(1+\frac{1}{n}\right) - \cancel{c}\log\left(1+\frac{1}{n+\alpha}\right)\right)$$
$$= \sum_{1}^{\infty} \left\{ \log \frac{(n+1)(n+\alpha)}{n(n+\alpha+1)} \right\}$$
$$= \lim_{N \to \infty} \left\{ \log \frac{(N+1)(1+\alpha)}{N+\alpha+1} \right\} \cancel{- \log \frac{1+\alpha}{1}}$$
$$= \log(1+\alpha).$$

See Karlin, Pac. J., Learning model stuff.

11/25/58. [illegible] problem.

How general can $P\{S_n > 0\}$ be?
i.e. characterizing sequences a_n for which there exist independent X_k with $S_n = X_1 + \cdots + X_n$ we have

$$a_n = P\{S_n > 0\}.$$

Is it true that $\lim \frac{a_1 + \cdots + a_n}{n}$ exists?

Example to show $P\{S_n > 0\}$ may be constant.

Let X_k be Cauchy and c a constant.

Form $S_n = \sum_1^n (X_k + c)$.

Then $\frac{S_n}{n}$ has same distribution as $X_1 + c$.

Hence $P\{S_n > 0\}$ is constant. By choice of c this constant may be chosen arbitrarily from $(0,1)$.

Let $S = \{1, 2, 3, \ldots\}$

$$p_1 \geq p_2 \geq p_3 \geq \cdots$$

Perform n independent trials. Let X_n = number of integers which appear. Find a useful expression for the distribution of X_n. Find $E(X_n)$, $V(X_n)$.

~~Clearly~~ $\cancel{P(X_n = k) = \sum_{j_1 < j_2 < \cdots < j_k} \{(p_{j_1} + p_{j_2} + \cdots + p_{j_k})^n - p_{j_1}^n - \cdots - p_{j_k}^n\}}$

~~but this is not useful.~~

Let $Y_j = 1$ if j appears in a trial, 0 otherwise.

$$X_n = \sum_{j=1}^{\infty} Y_j.$$

[redacted]

[redacted]

$$E(X_n) = \sum_1^\infty E(X_j)$$
$$= \sum_1^\infty P(j \text{ appears})$$
$$= \sum_1^\infty \{1 - P(j \text{ does not appear})\}$$
$$= \sum_1^\infty \{1 - (1-p_j)^n\}.$$

If this is $O(\log n)$ is p_j necessarily $O(a^j)$?

$$V(X_n) = \sum_1^\infty \{1-(1-p_j)^n\}(1-p_j)^n$$
$$- 2 \sum_{j=1}^\infty \sum_{k=j+1}^\infty \{(1-p_j-p_k+p_jp_k)^n - (1-p_j-p_k)^n\}$$

after some calculation.

2/18/56 (Catching up).

<u>Problem</u>: Find procedures for tossing one coin to determine outcomes A, B, C, D, E with equal probabilities 1/5. Find expected number of tosses, α.

1) Toss three times, assigning 5 of the 8 outcomes to subjects. Repeat until termination.

$$\alpha = 3 \cdot 8/5 = 4.8$$

2) Toss four times assigning 15 of the 16 in 3's. etc.

$$\alpha = 4 \cdot 16/15 = 4.27$$

3) $\underset{0}{|}\ A\ \underset{1/5}{|}\ B\ \underset{2/5}{|}\ C\ \underset{3/5}{|}\ D\ \underset{4/5}{|}\ E\ \underset{1}{|}$

Toss, coding $H=1$, $T=0$, getting (binary) $\omega = .110\ldots$. Let ω_n be ω chopped off at n. Let I_n be interval $(\omega_n, \omega_n + \frac{1}{2^n})$. Process continues past n if and only if one of the end-points lies in I_n, i.e. if and only if I_n meets two or more of the decision sets.

Let q_n = prob. not term. at n, $q_0 \equiv 1$.

Then $\alpha = \sum_1^\infty n(q_{n-1} - q_n)$

$= \sum_0^\infty q_n$

$q_0 = q_1 = q_2 = 1$; $q_n = \frac{4}{2^n}$ for $n \geq 3$. So $\boxed{\alpha = 4}$

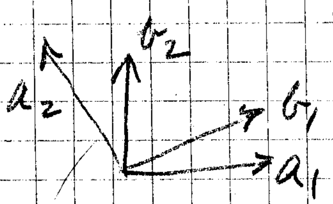
4) Toss until first H. If this occurs at trial 3, 4, 7, 8, 11, 12, ... : ~~an~~ event A; if not, toss twice more, assigning ~~each~~ one of the possible outcomes to B, C, D, E.

$\alpha = 2 + \frac{4}{5} 2 = 3.6$!

Uses $\frac{1}{5} = .001100110011 \cdots$

Is procedure 4 optimal? $3/$

Geometrical explanation of partial correlation.



$a_1 \perp b_2$

$a_2 \perp b_1$

$\angle a_1 b_1$ & $\angle a_2 b_2$ acute

$\Longrightarrow \angle a_1 a_2 + \angle b_1 b_2 = \pi$.

Application.

Evaluate $\frac{1}{2\pi\sqrt{\det V}} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(V^{-1}x,\, x)}\, d\vec{x}$

where $\vec{x} \in E_2$.

Let $y = \sqrt{V^{-1}}\, x$.

Integral becomes

$$\frac{1}{2\pi} \iint_Q e^{-\frac{1}{2}\|y\|^2}\, dy$$

This is equivalent to

$$\left[\frac{n}{2^r}\right] = \left[\frac{k}{2^r}\right] + \left[\frac{n-k}{2^r}\right] \qquad r = 1, 2, \dots$$

And thus to

$$(n)_r = (k)_r + (n-k)_r \,, \quad r = 1, 2, \dots$$

Where $(x)_r$ = r'th binary digit of n.

$$= \left[\frac{n}{2^{r-1}}\right] - 2\left[\frac{n}{2^r}\right].$$

Hence # of k for which this is possible is the number of subsets of s things, where s = # of ones in binary representation of n!!

<u>Much</u> simpler: Consider $(1+x)^n$ mod 2.

Clearly

$$(1+x)^{2^k} \equiv 1 + x^{2^k} \mod 2$$

(induction).

Then

$$(1+x)^n \equiv (1+x)^{\Sigma 2^{k_i}} \quad \text{with } \Sigma 2^{k_i} = n \text{ and } k_i \text{ all distinct}$$

$$\equiv \prod (1 + x^{2^{k_i}})$$

$$= \sum_a x^a \quad a \text{ ranging over } \Sigma 2^{k_i} \text{ formed for subsets of } \{i\}.$$

QED

3/15/56 Darling remark. If $\Sigma a_n = \infty$, $a_n > 0$

Then $\sum_1^N \frac{a_n}{S_n} \sim \log S_N$

$$S_n = \sum_1^n a_\nu.$$

more generally $\sum_1^N \varphi(S_n)\, a_n \sim \int^{S_N} \varphi(t)\, dt$

under suitable conditions on φ.

3/28/56 $\int_0^1 (ax)(bx)\,dx = \frac{1}{12ab} + \frac{1}{4}$
if a & b are relatively prime integers > 0.
Thus covariance of (ax) & (bx) is $\frac{1}{12ab}$
Here ( ) means fractional part of.

4/5/56 Let V be the covariance matrix of the multinomial distribution. Then every eigenvalue of V is $\leq 1/2$ $\to n=1$.

Proof: We have $(Vx, x) = \sum p_i (x_i - \bar{x})^2$
where $\bar{x} = \sum p_i x_i$. Then

$$\sum p_i (x_i - \bar{x})^2 \leq \sum p_i \left(x_i - \frac{x_{max} + x_{min}}{2}\right)^2$$
$$\leq \sum p_i \left(\frac{x_{max} - x_{min}}{2}\right)^2$$
$$= \frac{1}{4}(x_{max} - x_{min})^2$$
$$\leq \frac{1}{4}(x_{max} - x_{min})^2 + \frac{1}{4}(x_{max} + x_{min})^2$$
$$= \frac{1}{2}x_{max}^2 + \frac{1}{2}x_{min}^2$$
$$\leq \frac{1}{2}(x, x) \qquad \text{QED.}$$

This result is best possible in the sense that if $p_1 = p_2 = 1/2$, all others zero, then $\lambda = 1/2$ is eigenvalue; take $x = \begin{pmatrix} 1 \\ -1 \\ \text{arb} \\ \vdots \end{pmatrix}$.

5/23/56 (more on above). Solving $Va = \lambda a$, $\sum a_i = 0$ we find
$$a_i = \lambda \frac{a_i}{p_i} + \sum_1^k p_j a_j$$
$$a_i = \frac{c}{1 - \frac{\lambda}{p_i}} \qquad c = \sum_1^k p_j a_j$$

So if p_i's distinct eigenvalues are precisely the zeros of $\sum \left(1 - \frac{\lambda}{p_i}\right)^{-1} = 0$.

[redacted]

3/28/52 $\int_0^1 (ax)(bx)\,dx = \frac{1}{12ab} + \frac{1}{4}$

if a & b are relatively prime integers > 0.

Thus covariance of (ax) & (bx) is $\frac{1}{12ab}$.

Here $(\ )$ means fractional part of.

4/5/56 Let V be the covariance matrix of the multinomial distribution. Then every eigenvalue of V is $\leq 1/2$. (n=1.)

Proof: We have $(Vx, x) = \sum p_i (x_i - \bar{x})^2$ where $\bar{x} = \sum p_i x_i$. Then

$$\sum p_i (x_i - \bar{x})^2 \leq \sum p_i \left(x_i - \frac{x_{max} + x_{min}}{2}\right)^2$$

$$\leq \sum p_i \left(\frac{x_{max} - x_{min}}{2}\right)^2$$

$$. \quad = \frac{1}{4}(x_{max} - x_{min})^2$$

$$\leq \frac{1}{4}(x_{max} - x_{min})^2 + \frac{1}{4}(x_{max} + x_{min})^2$$

$$= \frac{1}{2} x_{max}^2 + \frac{1}{2} x_{min}^2 .$$

$$\leq \frac{1}{2}(x, x) \qquad \text{QED.}$$

The result is best possible in the sense that if $p_1 = p_2 = 1/2$, all others zero, then $\lambda = 1/2$ is eigenvalue; take $x = \begin{pmatrix} 1 \\ -1 \\ \text{arb} \\ \vdots \end{pmatrix}$.

5/23/56 (more on above). Solving $Va = \lambda a$, $\sum a_i = 0$ we find

$$a_i = \frac{\lambda a_i}{p_i} + \sum_1^k p_j a_j$$

$$\therefore \quad a_i = \frac{c}{1 - \frac{\lambda}{p_i}} \qquad c = \sum_1^k p_j a_j$$

So if p_i's distinct eigenvalues are precisely the zeros of $\sum_1^k \left(1 - \frac{\lambda}{p_i}\right)^{-1} = 0$.

If several p_i coincide easy to take care of this too.

numerical example: $p_1 = .5$, $p_2 = .3$, $p_3 = .2$

$$\frac{1}{1-2\lambda} + \frac{1}{1-\frac{10}{3}\lambda} + \frac{1}{1-5\lambda}$$

$$100\lambda^2 - 62\lambda + \cancel{3}9 = 0$$

$$\lambda = .01\left(31 \pm \sqrt{\cancel{8}61}\right)$$

$$= ~~[illegible]~~$$

$$= .01\,(31 \pm 7.8102)$$

$$= .3881,\ .2319$$

10/5/56 Some Darling problems.

Let $x_1, \dots, x_n$ be r.v.'s. $\ni$

$\sum \alpha_i x_i$ is unif. dist. on $(-1, 1)$

whenever $\sum \alpha_i^2 = 1$.

What distribution have the x's?

~~now~~ The hypotheses imply that

$$E\left(e^{i\sum t_j x_j}\right) = \varphi(t_1, t_2, \dots, t_n) = \frac{\sin\sqrt{t_1^2+\cdots+t_n^2}}{\sqrt{t_1^2+\cdots+t_n^2}}$$

Expanding this we find that for $n \geq 4$ some variances must be < 0. Hence there is no such distribution for $n \geq 4$.

For $n = 3$ we guess & verify that we have just the uniform distribution over the surface of the unit sphere.

This from

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{it_3\cos\varphi}\sin\varphi\,d\varphi\,d\theta = \frac{\sin t_3}{t_3}$$

~~For n = 0~~

In the case $n=3$ can prove from the expansion also that $\sigma^2(X^2+Y^2+Z^2)=0$, so $X^2+Y^2+Z^2 = \text{const}$, clearly unity.

What about case $n=2$? Of what distribution is

$$\frac{\sin\sqrt{t^2+u^2}}{\sqrt{t^2+u^2}},$$

the characteristic function? Ans: (Darling)

$$f(r,\theta) = \frac{\text{const}}{\sqrt{1-r^2}}.$$

Earlier Darling prob.

$X_1, X_2, \ldots$ ident., ind. When do $\{\alpha_n\}$, a, exist $\ni$ $\overline{\lim_{n\to\infty}}\, \alpha_n X_n = a$?

Soln (For $\underline{\lim}$ instead). We have $\underline{\lim}\, b_n = b$

$\Longleftrightarrow$ $\left.\begin{array}{l} \infty\text{'many } b_n < b+\varepsilon \\ \text{only finitely '' '' } < b-\varepsilon \end{array}\right\}$ for every $\varepsilon>0$.

Hence (Borel Cantelli) $\left.\begin{array}{l} \sum_1^\infty F\left(\frac{a-\varepsilon}{\alpha_n}\right) < \infty \\ \sum_1^\infty F\left(\frac{a+\varepsilon}{\alpha_n}\right) = \infty \end{array}\right\}$ all $\varepsilon>0$.

E.g. $N(0,1)$: $\alpha_n = \frac{1}{\sqrt{\log n}}$, $a=\sqrt{2}$, using

$$F(x) \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}\,|x|} \text{ as } x\to-\infty.$$

If $A_1, A_2, \ldots$ pairwise ind. and $\sum P(A_n) = \infty$ then $P(\overline{\lim} A_n) = 1$.

Proof by applying Tchebycheff inequality to

$$N_n = \# \text{ of } A_k \text{ occurring}, \; 1 \le k \le n$$

$$\sum \frac{\sigma_n^2}{n^2} < \infty \Rightarrow \frac{1}{n^2} \sum \sigma_k^2 \to 0$$

Spec. case of

$$\sum_{j=1}^{\infty} \frac{x_j}{a_j} < \infty \Rightarrow \frac{1}{a_n} \sum_{j=1}^{n} x_j \to 0$$

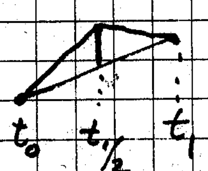
in case $a_j \to \infty$ monotonically.

~~and~~ ~~For proof~~ Set $x_j = a_j y_j$, $\sum^{n} y_j = S_n$.
apply Toeplitz type summability theorem.

$x(t)$ Wiener process on $[0,1]$.

Show: $S_n \equiv \sum_{k=0}^{2^n-1} \left(x\left(\frac{k+1}{2^n}\right) - x\left(\frac{k}{2^n}\right) \right)^2 \to 1$ with probability 1.

We have the interpolation process



$$x(t_{1/2}) = \frac{1}{2}\left(x(t_0) + x(t_1)\right) + \sqrt{\frac{(t_{1/2} - t_0)(t_1 - t_{1/2})}{t_1 - t_0}} \; \xi$$

where ξ is $N(0,1)$ & ind. of all existing $x(t)$.

Hence $\left(x(t_1) - x(t_{1/2})\right)^2 + \left(x(t_{1/2}) - x(t_0)\right)^2 =$

$$\tfrac{1}{2}\left(X(t_1) - X(t_0) \right)^2 + 2\,\frac{(t_{1/2}-t_0)(t_1 - t_{1/2})}{t_1 - t_0}\,\xi^2$$

If $t_1 - t_0 = \frac{1}{2^n}$ then coeff of ξ^2 is

$$2 \cdot 2^{-2(n+1)} \cdot 2^n = 2^{-(n+1)}.$$

Hence $S_{n+1} = \frac{1}{2} S_n$ + Sum of 2^n independent terms of form $\frac{\xi^2}{2^{n+1}}$.

Hence by induction we can represent the situation as follows: ~~Σ_n has the~~

Let $\xi_1, \xi_2, \ldots$ be ind $N(0,1)$.

Then $S_n = \dfrac{\xi_1^2 + \cdots + \xi_{2^n}^2}{2^n}$

Hence by SLLN $S_n \longrightarrow E(\xi^2) = 1$ w.pr. 1.

QED

$x(t)$ Wiener on $[0, \infty)$. $t_n = 2^{n^2}$.

~~Show~~: $\alpha_n = Pr(\text{no zeros in } (t_n, t_{n+1}))$.

Show : $\sum \alpha_n < \infty$.

Proof : $$\alpha_n = \frac{2}{\sqrt{2\pi t_n}} \int_0^\infty e^{-x^2/2t_n}\,dx\ \frac{2}{\sqrt{2\pi}} \int_0^{\frac{x}{\sqrt{t_{n+1}-t_n}}} e^{-u^2/2}\,du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-x^2/2}\,dx \int_0^{x\sqrt{\frac{t_n}{t_{n+1}-t_n}}} e^{-u^2/2}\,du.$$

But $$\sqrt{\frac{t_n}{t_{n+1}-t_n}} = \sqrt{\frac{2^{n^2}}{2^{(n+1)^2} - 2^{n^2}}} \sim 2^{-n}$$

Then $$\sum_n \alpha_n \cong \frac{2}{\pi} \int_0^\infty e^{-x^2/2}\,dx \sum_n \int_0^{x2^{-n}} e^{-u^2/2}\,du$$

$$< \frac{2}{\pi}\int_0^\infty e^{-x^2/2}\,dx \cdot \sum_1^\infty x 2^{-n} < \infty \quad \underline{QED}.$$

---

c.f. of $\int_0^1 x(t)^2\,dt$

We have $\quad x(t) = \sqrt{2}\sum_0^\infty \xi_j \dfrac{\sin (j+\frac{1}{2})\pi t}{[(j+\frac{1}{2})\pi]}$

Then
$$\int_0^1 x(t)^2\,dt = \sum_0^\infty \frac{\xi_j^2}{((j+\frac{1}{2})\pi)^2}$$

and
$$\text{c.f.} = \left\{\prod_{j=0}^\infty \left(1 - \frac{2it}{(j+\frac{1}{2})^2\pi^2}\right)\right\}^{-1/2}$$
$$= (\cos\sqrt{2it})^{-1/2}.$$

3/18/51. Darling on Stirling's formula.

$$n! = \int_0^\infty e^{-x} x^n\,dx.$$

$e^{-x}x^n$ max at $x = n$
inflections at $x = n \pm \sqrt{n}$.

So put $x = n + t\sqrt{n}$
$$n! = e^{-n} n^n \sqrt{n}\int_{-\sqrt{n}}^\infty e^{-\sqrt{n}t}\left(1+\frac{t}{\sqrt{n}}\right)^n dt$$

integrand $\le \begin{cases} e^{-\frac{1}{2}t^2} & \text{on } t<0 \\ e^{-t}(1+t) & \text{on } t>0. \end{cases}$

now $e^{-\sqrt{n}t}\left(1+\frac{t}{\sqrt{n}}\right)^n \to e^{-t^2/2}$ as $n\to\infty$.
(take logs)

$$\therefore \frac{n!}{(\frac{n}{e})^n\sqrt{n}} \to \int_{-\infty}^\infty e^{-t^2/2}\,dt = \sqrt{2\pi} \quad !!$$

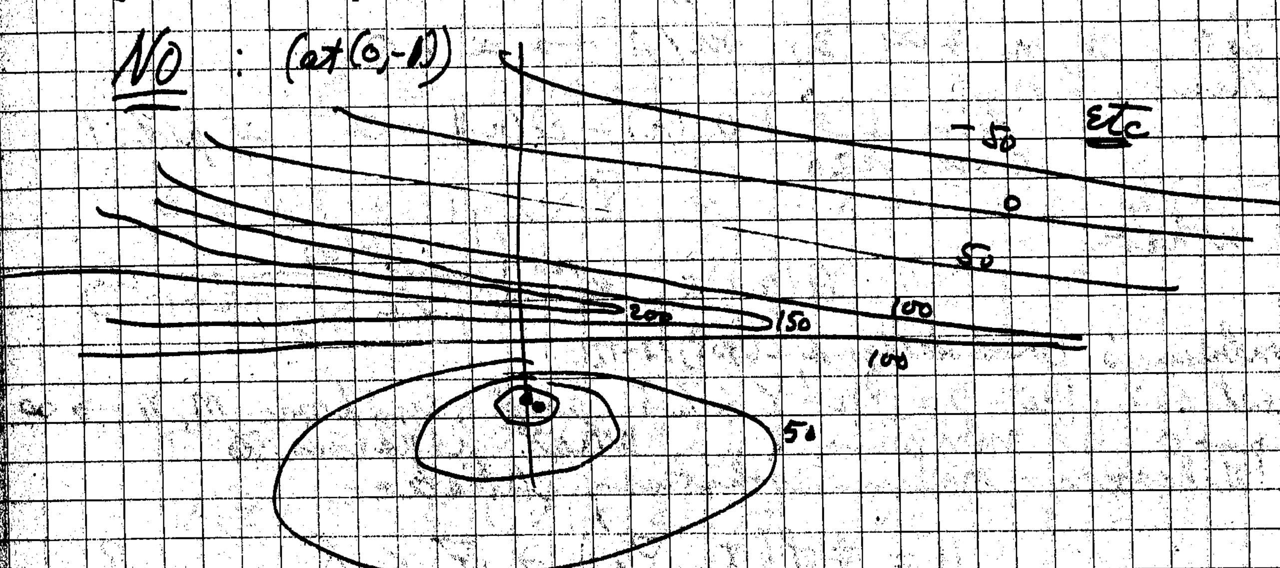
~~Working Prob. $x(t)$ = tied down Wiener; easy to show Markovian.~~

~~Find $P_r\{\max x(t) \le z \mid x(1)=0\}$.~~

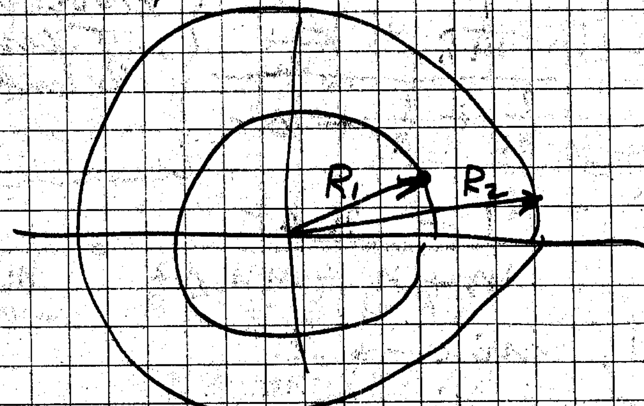
Prob. If $f(x,y) \in C^\infty$ & $f_x = f_y = 0 \Rightarrow x=y=0$ with $(0,0)$ a local min.

Is $(0,0)$ an abs. min?

<u>NO</u> : (at $(0,-1)$)



Generation of Poisson Process in plane.



Get dist. of R_1,

then Cond. dist. of R_2 given R_1

Etc.

6/13/57 Blum & Weiss AMS 28, 242-246, 1957. 6

$X_1, \dots, X_m$ rect. $Y_1, Y_2, \dots, Y_n \sim g$, all on $[0,1]$.

S_i = # of X's betw. each pair of ordered Y's.
(beginning at 0 and ending at 1, in addition to the actual Y's.)
$i = 1, 2, \dots, n+1$.

$$Q_n(r) = \frac{\#\text{ of } S\text{'s} = r}{n+1}$$

Let $m, n \to \infty \ni m/n \to \alpha > 0$.
Then
$$\lim_{n\to\infty} \sup_{r \geq 0} |Q_n(r) - Q(r)| = 0 \quad \text{w. prob. } 1.$$

where
$$Q(r) = \alpha^r \int_0^1 \frac{g^2(y)}{[\alpha + g(y)]^{r+1}}\, dy.$$

Birnbaum's [Berk Stat Symp I (1955) 13-17].
Question: Put $p = \Pr\{Y < X\}$ and suppose $m, n \to \infty$, $p \to 0 \ni mnp \to \lambda$. What happens to Wilcoxon-Mann-Whitney statistic
$$U = \#\text{ of } (X_j, Y_k) \ni Y_k < X_j \quad ?$$

$$A = \frac{1}{m} \qquad B = \frac{1}{n} + \Delta^2$$

? Generalize to linear hyp. problem?
σ unknown?
Δ unknown?

$$\frac{AB}{A+B} = \frac{\frac{1}{mn} + \frac{\Delta^2}{m}}{\frac{1}{m} + \frac{1}{n} + \Delta^2}$$

$$= \frac{1 + n\Delta^2}{m + n + mn\Delta^2} = \frac{1}{m}\left\{1 - \frac{n}{m+n+mn\Delta^2}\right\}$$

$\Delta \to \infty$, $\lambda \to 1$, min $\to \frac{1}{m}$

$\Delta \to 0$, $\lambda \to \frac{m}{m+n}$, min $\to \frac{1}{m+n}$

$n \to \infty$, $\lambda \to \frac{m\Delta^2}{1+m\Delta^2}$, min $\to \frac{\Delta^2}{1+m\Delta^2}$

6/14/57 $X_1, \ldots, X_m$ $N(a, \sigma^2)$

$Y_1, \ldots, Y_n$ $N(a+\delta, \sigma^2)$ δ random. $E(\delta) = 0$.

$$E(\lambda \bar{X} + (1-\lambda)\bar{Y}) = a.$$

What choice of λ will minimize $\sigma^2(\lambda\bar{X}+\cdots)$?

$$\bar{X} \sim N(a, \frac{\sigma^2}{m})$$

$$\bar{Y} \sim N(a+\delta, \frac{\sigma^2}{n}) \text{ given } \delta.$$

$$Z = \lambda\bar{X} + (1-\lambda)\bar{Y} \sim N(a+(1-\lambda)\delta, \tau(\lambda)^2)$$

$$\tau(\lambda)^2 = \left\{\frac{\lambda^2}{m} + \frac{(1-\lambda)^2}{n}\right\}\sigma^2.$$

$$\sigma^2(Z) = E(\sigma^2(Z|\delta)) + \sigma^2(E(Z|\delta))$$

$$V(Z) = E(V(Z|\delta)) + V(E(Z|\delta))$$

$$= \left[\left\{\frac{\lambda^2}{m} + \frac{(1-\lambda)^2}{n}\right\} + (1-\lambda)^2\Delta^2\right]\sigma^2$$

where $V(\delta) = \sigma^2\Delta^2$

$$\frac{\sigma^{-2}}{2}\frac{d}{d\lambda}V(Z) = \frac{\lambda}{m} - (1-\lambda)\left[\frac{1}{n} + \Delta^2\right] = 0$$

$$\boxed{\lambda = \frac{m + mn\Delta^2}{m+n+mn\Delta^2}}$$

$$\min \; A\lambda^2 + B(1-\lambda)^2$$

$$A\lambda = B(1-\lambda)$$

$$\lambda = \frac{B}{A+B} \qquad \min = \frac{AB^2 + A^2B}{(A+B)^2}$$

$$1-\lambda = \frac{A}{A+B} \qquad = \frac{AB}{A+B}$$

6/17. Let $x_1 + x_2 + \cdots + x_n = 1$ with $x_i \geq 0$, $i = 1, \ldots, n$, uniformly distributed.

Let $y_1, \ldots, y_n$ be their order-statistics, $y_1 > y_2 > \cdots > y_n$.

Find $E(y_i)$, $i = 1, \ldots, n$. simply!

Let $x_i \sim \theta e^{-\theta x}$, $x > 0$

y_i as above. $T = \Sigma x_i$. Problem amounts to: find $E(y_i \mid T=1)$.

Now let $\alpha : x_i \to \alpha x_i$, $i = 1, 2, \ldots, n$.

Then $\alpha : y_i \to \alpha y_i$

& $T \to \alpha T$.

From conditional Monte Carlo paper,

$$E(y_i \mid T=1) = E(y_i^* w^*)$$

where $$y_i^* = \frac{y_i}{\Sigma x_i}$$

$$w^*(x) = \frac{1}{\sigma(1)} \mu\left(\frac{1}{\Sigma x_i}\right) \frac{\Pi f\left(\frac{x_i}{\Sigma x_i}\right)}{\Pi f(x_i)} \frac{1}{(\Sigma x_i)^n}$$

$$f(x) = \theta e^{-\theta x}$$

$$\Pi f(x_i) = \theta^n e^{-\theta \Sigma x_i}$$

$$\Pi f\left(\frac{x_i}{\Sigma x_i}\right) = \theta^n e^{-\theta}$$

$$w^*(x) = \frac{1}{\sigma(1)}\, \mu\left(\frac{1}{\sum x_i}\right) \frac{\theta^n e^{-\theta}}{\theta^n e^{-\theta \sum x_i}} \frac{1}{(\sum x_i)^n}$$

$$\sigma = \text{density of } T = \frac{\theta^n t^{n-1} e^{-\theta t}}{(n-1)!}\,(dt)$$

$$= \underbrace{\frac{\theta^n t^n e^{-\theta t}}{(n-1)!}}_{\sigma}\, \frac{dt}{t}$$

So

$$\sigma(1) = \frac{\theta^n e^{-\theta}}{(n-1)!}$$

Thus

$$w^* = \frac{(n-1)!}{\theta^n \cancel{e^{-\theta}}}\, \mu\left(\frac{1}{\sum x_i}\right) \frac{\cancel{e^{-\theta}}}{e^{-\theta \sum x_i}} \frac{1}{(\sum x_i)^n}$$

$$\int_0^\infty \mu(\alpha)\, \frac{d\alpha}{\alpha} = 1$$

$$\int_0^\infty \mu\left(\frac{1}{\alpha}\right) \frac{d\alpha}{\alpha} = 1 \text{ too.}$$

$$\mu\left(\frac{1}{\alpha}\right) = \frac{\theta^n \alpha^n e^{-\theta\alpha}}{(n-1)!} \quad \text{will do it.}$$

But then

$$\mu\left(\frac{1}{\sum x_i}\right) = \frac{\theta^n (\sum x_i)^n e^{-\theta \sum x_i}}{(n-1)!}$$

and finally $w^*(x) = 1$!!!

Thus $\boxed{E(y_i \mid T=1) = E\left(\frac{y_i}{\sum x_i}\right)}$

actually equally good for functions of the y's.

6/17. Let $x_1 + x_2 + \cdots + x_n = 1$ with $x_i \geq 0$, $i=1,\ldots,n$, uniformly distributed.

Let $y_1, \ldots, y_n$ be their order-statistics, $y_1 > y_2 > \cdots > y_n$.

Find $E(y_i)$, $i=1,\ldots,n$. Simply!

Let $x_i \sim \theta e^{-\theta x}$, $x > 0$

y_i as above. $T = \Sigma x_i$. Problem amounts to: find $E(y_i \mid T=1)$.

Now let $\alpha : x_i \to \alpha x_i$, $i=1,2,\ldots,n$.

Then $\alpha : y_i \to \alpha y_i$

& $T \to \alpha T$.

From conditional Monte Carlo paper,

$$E(y_i \mid T=1) = E(y_i^* w^*)$$

where

$$y_i^* = \frac{y_i}{\Sigma x_i}$$

$$w^*(x) = \frac{1}{\sigma(1)} \mu\left(\frac{1}{\Sigma x_i}\right) \frac{\Pi f\left(\frac{x_i}{\Sigma x_i}\right)}{\Pi f(x_i)} \frac{1}{(\Sigma x_i)^n}$$

$$f(x) = \theta e^{-\theta x}$$

$$\Pi f(x_i) = \theta^n e^{-\theta \Sigma x_i}$$

$$\Pi f\left(\frac{x_i}{\Sigma x_i}\right) = \theta^n e^{-\theta}$$

$$w^*(x) = \frac{1}{\sigma(1)}\ \mu\left(\frac{1}{\sum x_i}\right)\ \frac{\cancel{\theta^n}\, e^{-\theta}}{\cancel{\theta^n}\, e^{-\theta \sum x_i}}\ \frac{1}{(\sum x_i)^n}$$

$$\sigma = \text{density of } T = \frac{\theta^n t^{n-1} e^{-\theta t}}{(n-1)!}\ (dt)$$

$$= \underbrace{\frac{\theta^n t^n e^{-\theta t}}{(n-1)!}}_{\sigma}\ \frac{dt}{t}$$

So

$$\sigma(1) = \frac{\theta^n e^{-\theta}}{(n-1)!}$$

Thus

$$w^* = \frac{(n-1)!}{\theta^n \cancel{e^{-\theta}}}\ \mu\left(\frac{1}{\sum x_i}\right) \frac{\cancel{e^{-\theta}}}{e^{-\theta \sum x_i}}\ \frac{1}{(\sum x_i)^n}$$

$$\int_0^\infty \mu(\alpha)\ \frac{d\alpha}{\alpha} = 1$$

$$\int_0^\infty \mu\left(\frac{1}{\alpha}\right)\ \frac{d\alpha}{\alpha} = 1 \text{ too.}$$

$$\mu\left(\frac{1}{\alpha}\right) = \frac{\theta^n \alpha^{n} e^{-\theta\alpha}}{(n-1)!} \quad \text{will do it.}$$

Get then

$$\mu\left(\frac{1}{\sum x_i}\right) = \frac{\theta^n (\sum x_i)^n e^{-\theta \sum x_i}}{(n-1)!}$$

and finally $w^*(x) = 1$!!!

Thus: $\boxed{E(y_i \mid T=1) = E\left(\frac{y_i}{\sum x_i}\right)}$

actually good equally for functions of the y's.

$$E\left(\frac{\max x_i}{\Sigma x_i}\right) = \varphi_n \qquad x_i \sim e^{-x},\ x \geq 0.$$

$$E\left(\frac{y_1}{\Sigma x_i}\right) = \varphi_n$$

$$E\left(\frac{y_2}{\Sigma x_i}\right) =$$

Dist. of $\dfrac{y_i}{\Sigma x_i}$ does not involve θ.

So $\dfrac{y_i}{\Sigma x_i}$ is ind. of Σx_i, the sufficient statistic.

So $E\left(\dfrac{y_i}{\Sigma x_i}\right) = \dfrac{E(y_i)}{E(\Sigma x_i)} = \boxed{\dfrac{1}{n} E(y_i)}$.

How to find $E(y_i)$ simply? This (& more) I did in seminar last year. (See notes). Result is

$$\boxed{E(y_i) = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{i}}$$

6/18-21/57 Musings on Grenander-Rosenblatt & especially the theorem:

Let $f(\lambda) \geq 0$ on $|\lambda| \leq \pi$, with $\int_{-\pi}^{\pi} f < \infty$.

Let $\varphi_n : \varphi_n(\lambda) = e^{in\lambda}$. $-\infty < n < \infty$

Let $\mathfrak{M}_n = [\varphi_\nu \mid \nu \leq n]$

~~Then~~ Inner product is $(\varphi, \psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\lambda)\overline{\psi(\lambda)} f(\lambda) d\lambda$

Then $\mathfrak{M}_n \neq \mathfrak{M}_{n+1}$, all n

or $\mathfrak{M}_n = \mathfrak{M}_{n+1}$, all n

according as $\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty$

or $= -\infty$.

Example: $f(\lambda) = I_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$

Then $(\varphi_n, \varphi_m) = (\varphi_{n-m}, \varphi_0) = \begin{cases} \frac{1}{2} & m = n \\ 0 & n-m \text{ even} \neq 0 \\ \frac{(-1)^{[(n-m)/2]}}{(n-m)\pi} & n-m \text{ odd} \end{cases}$

Why is $\varphi_0 = \sum_{1}^{\infty} b_\nu \varphi_{-\nu}$??

(convergence in norm)

(Rambling:) Let $\mathcal{O} = \{\varphi_{-(2\nu+1)} \mid \nu \geq 0\}$

$\mathcal{E} = \{\varphi_{-2\nu} \mid \nu \geq 1\}$

Then ~~vectors~~ φ's of $\mathcal{O}$ & $\mathcal{E}$ are mutually $\perp$

but it seems that $\angle(\mathcal{O}, \mathcal{E}) = 0°$,

by reference to Hardy, Littlewood, Polya #299 pg 22

and some calculation.

~~Also then~~ [Does $\mathcal{O} \cap \mathcal{E} = (0)$?]

$\Theta \cap \mathcal{E} = [0]$ is absurd if $\varphi_0 = \sum b_\nu \varphi_{-\nu}$,
for then
$$\underbrace{\varphi_0 - \sum_1^\infty b_{2\nu} \varphi_{-2\nu}}_{\in\,\Theta} = \underbrace{\sum_0^\infty b_{2\nu+1} \varphi_{-(2\nu+1)}}_{\in\,\mathcal{E}}$$
(or rather after translating down a notch.)

Krein refs from Doob (Doklady $\underline{45}$, 139; $\underline{46}$, 91; $\underline{46}$, 306 — AS 262 .P53) do not seem helpful.

Kellogg pg 59
$$J = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{p_1^2\cos^2\varphi + q_1^2 \sin^2\varphi}} = \int_0^{\pi/2} \frac{d\psi}{\sqrt{p_2^2 \cos^2\psi + q_2^2 \sin^2\psi}}\,!$$
where
$$p_2 = \frac{p_1+q_1}{2}, \quad q_2 = \sqrt{p_1 q_1} \quad (p_1 > q_1).$$
Iterating they approach a common limit α, so that
$$J = \frac{\pi}{2\alpha}, \quad \alpha = \frac{\pi}{2J}$$
$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\frac{p_1^2+2p_1q_1+q_1^2}{4}\cos^2\psi + p_1 q_1 \sin^2\psi}}$$

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{p_1^2\cos^2\varphi + q_1^2 \sin^2\varphi}} = \frac{1}{p_1}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{\cos^2\varphi + \frac{q_1^2}{p_1^2}\sin^2\varphi}}$$

[redacted]

$$= \frac{1}{p_1} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \left(1 - \frac{q_1^2}{p_1^2}\right) \sin^2\varphi}}$$

$$= \frac{1}{p_1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2\varphi}}$$

$$k_1^2 = 1 - \frac{q_1^2}{p_1^2} \qquad q_1^2 = p_1^2(1 - k_1^2)$$

$$\frac{q_1^2}{p_1^2} = 1 - k_1^2$$

$$k_2^2 = \cancel{1 - q_2^2} \quad 1 - \frac{q_2^2}{p_2^2}$$

$$= 1 - \frac{4 p_1 q_1}{(p_1 + q_1)^2}$$

$$= 1 - \frac{4 \cancel{p_1^2} \sqrt{1 - k_1^2}}{\left(1 + \sqrt{1 - k_1^2}\right)^2}$$

$$= \frac{\left(1 - \sqrt{1 - k_1^2}\right)^2}{\left(1 + \sqrt{1 - k_1^2}\right)^2}$$

$$k_2 = \frac{1 - \sqrt{1 - k_1^2}}{1 + \sqrt{1 - k_1^2}}$$

$$p_2 = p_1 \frac{1 + \sqrt{1 - k_1^2}}{2}$$

6/27 Tits problem.

$$\sum x_i = 1 \qquad x_i \geq 0.$$

Define $\quad x_1' = \frac{1}{n}$

$$x_i' = \frac{x_i}{n\bar{x}} \quad \text{for } i \geq 2, \qquad \bar{x} = \frac{\sum_2^n x_i}{n-1}$$

Then repeat on index 2, etc., n, 1, 2, ..., ...
Does thing converge to $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$?

<u>Solution</u>: Define a sequence of vectors ξ by

$$\xi_0 = x = (x_1, \ldots, x_n)$$

$$\xi_1 = (\bar{x}, x_2, \ldots, x_n) \qquad \bar{x} = \frac{\sum_2^n x_i}{n-1}$$

$$\xi_2 = (\bar{x}, \bar{\bar{x}}, x_3, \ldots, x_n)$$

$\vdots$

cyclically.

Then $\lim \xi_n$ exists $\neq 0$, <u>equal components</u> <u>since variance</u> <u>tends to zero.</u>
So $\lim \frac{\xi_n}{\varphi(\xi_n)}$ exists

where $\cancel{\varphi : \xi \to \Sigma}$ $\varphi(\xi) = (\xi, \mathbf{1})$,
$\mathbf{1} = (1, \ldots, 1)$.

But the mapping $\frac{\xi}{\varphi(\xi)} \to \frac{\xi'}{\varphi(\xi')}$

corresponds to $\qquad x \to x'$. <u>QED</u>

Projections (not orthogonal) $P_1, P_2, \dots, P_n, P_1, \dots, P_n, \dots$

variance : $\|\xi - \frac{1}{n}(\xi, 1)\,\underline{1}\,\|^2$

$$= \|\xi\|^2 - \frac{1}{n}(\xi, 1)^2.$$

When is

$$\|P\xi\|^2 - \frac{1}{n}(P\xi, 1)^2 \le \|\xi\|^2 - \frac{1}{n}(\xi, \underline{1})^2 \ ?$$

$$\frac{1}{n}\left\{(\xi,1)^2 - (P\xi, 1)^2\right\} < \|\xi\|^2 - \|P\xi\|^2 \quad ?$$

of course $\|P\xi\|$ need not be $\le \|\xi\|$ since $P \neq P^*$

Note that $P1 = 1$.

$$(P\xi, P\xi) - (P\xi, u)^2 < (\xi,\xi) - (\xi, u)^2.$$

$$(P^*P\xi, \xi) - (P\xi, u)^2 < (\xi,\xi) - (\xi, u)^2.$$

$\cancel{(\xi, u)^2}$ $\qquad u = \frac{1}{\sqrt{n}}\,1$

Difference $= \left((I - P^*P)\xi, \xi\right) - \left\{(\xi, u)^2 - (P\xi, u)^2\right\}$.

$$= \mathcal{E}\left((I - P^*P)\xi, \xi\right) - \mathcal{E}\,(\xi + P\xi, u)(\xi - P\xi, u)$$

$$P: \begin{pmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \qquad P^*1 = \begin{pmatrix} 0 \\ \frac{n}{n-1} \\ \vdots \\ \frac{n}{n-1} \end{pmatrix} = n\,Q_2\,1$$

$$P^*: \begin{pmatrix} 0 & 0 & & \\ \frac{1}{n-1} & 1 & & \\ \vdots & & \ddots & \\ \frac{1}{n-1} & & & 1 \end{pmatrix}$$

$$P^*P = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \frac{1}{n-1} & 1 & & \\ \vdots & & \ddots & \\ \frac{1}{n-1} & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \left[\frac{1}{(n-1)^2} + 1\right] & \frac{1}{(n-1)^2} & \cdots & \frac{1}{(n-1)^2} \\ * & & & \\ & & & 1 \end{pmatrix}$$

$$I - P^*P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{1}{(n-1)^2} & -\frac{1}{(n-1)^2} & \cdots & -\frac{1}{(n-1)^2} \\ & & \ddots & \\ & * & & \end{pmatrix}$$

$P^*P = Q_1 + \frac{1}{n-1} Q_2$ where the Q's are commuting orthogonal projections.

$I - P^*P = \underbrace{I - Q_1}_{= Q_3} \cancel{+} - \frac{1}{n-1} Q_2$ and $(I - Q_1) Q_2 = 0$.

$$(Q_3 \xi, \xi) \cancel{+} - \frac{1}{n-1} (Q_2 \xi, \xi) - (\xi, u)^2 + (\xi, P^* u)^2$$

$$(Q_3 \xi, \xi) - \frac{1}{n-1} (Q_2 \xi, \xi) - (\xi, u)^2 + n^2 (\xi, Q_2 u)^2.$$

Procedure for sampling Poisson process in plane.

Points at (R_n, θ_n). θ_n ind., unif on $(0, 2\pi)$.

$F(r_n \mid R_{n-1})$ $\quad f(R_1) = \lambda A e^{-\lambda V}$ Start.

$= Pr\{R_n < r_n \mid R_{n-1}\}$

$= 1 - Pr\{\text{ring is empty}\}$

$= 1 - e^{-\lambda\pi(r_n^2 - R_{n-1}^2)}$, $\quad r_n > R_{n-1}$

So $f(r_n \mid R_{n-1}) = 2\lambda\pi r_n e^{-\lambda\pi(r_n^2 - R_{n-1}^2)}$.

Same works in k-dim space, with

$$F(r_n \mid R_{n-1}) = 1 - e^{-\lambda(V(r_n) - V(R_{n-1}))}$$

$$f(r_n \mid R_{n-1}) = \lambda A(r_n) e^{-\lambda(V(r_n) - V(R_{n-1}))}$$

where V and A denote volume & surface area of k-dim sphere of indicated radius.

What is probability that sphere of radius a contains exactly k points?

$$Pr\{R_k < a < R_{k+1}\}$$

$$= \int_0^a dr_k \int_a^\infty dr_{k+1}\, f(r_{k+1} \mid r_k)\, f(r_k)$$

~~Pr~~ $e^{\lambda V(r_k)} f(r_k)$

$$= \int_0^a dr_k \int_a^\infty \lambda A(r_{k+1}) e^{-\lambda V(r_{k+1})}\, dr_{k+1}$$

$$= \int_0^a e^{\lambda V(r_k)} f(r_k)\, dr_k \cdot \int_a^\infty \lambda A(r_k) e^{-\lambda V(r_k)}\, dr_k$$

$$= e^{-\lambda V(a)} \int_0^a e^{\lambda V(r)} f_k(r)\, dr$$

(changing notation slightly).

Probability that sphere contains at least k points is

$$\int_0^a f_k(r)\, dr$$

i.e. $Pr\{R_k < a\}$.

Hence

$$\boxed{\int_0^a f_k(r)\, dr - \int_0^a f_{k+1}(r)\, dr = e^{-\lambda V(a)} \int_0^a e^{\lambda V(r)} f_k(r)\, dr}$$

$$f_k(a) - f_{k+1}(a) = f_k(a) - \lambda A(a) \cancel{} e^{-\lambda V(a)} \int_0^a e^{\lambda V} \ldots$$

Put $e^{-\lambda V(a)} \int_0^a e^{\lambda V(r)} f_k(r)\, dr \equiv g_k(a)$.

So

$$e^{\lambda V(a)} f_k(a) = \frac{d}{da}\, g_k(a)\, e^{\lambda V(a)}$$

$$f_k(a) = g_k'(a) + \lambda A(a)\, g_k(a).$$

Equ thus becomes

$$\boxed{g'_{k+1} + \lambda A g_{k+1} = \lambda A g_k}$$

check - does $\dfrac{e^{-\lambda V(a)} (\lambda V(a))^k}{k!}$

Satisfy ?

$$\frac{d}{da} \frac{e^{-\lambda V} \lambda^{k+1} V^{k+1}}{(k+1)!}$$

$$= \frac{e^{-\lambda V} \lambda^{k+1} \cancel{(k+1)} V^k A}{\cancel{(k+1)}!} - \frac{\lambda A e^{-\lambda V} \lambda^{k+1} V^{k+1}}{(k+1)!}$$

✓ ✓

$V(r)$ $\boxed{f_k(r)\, dr}$

λv $(r)\, f_k(r)\, dr$.

Connection with linear Poisson process.

Let $\{T_n\}$ be a Poisson process on the line, with parameter λ.

Let $R_n = V_k^{-1}(T_n)$

and let "θ_n" be unif over sphere.

Then $(R_n, "\theta_n")$ is Poisson process in k-space. I know it's true - how to prove it formally, and is it worth doing?

Linear poisson process: a random measure $\mu(E,\omega) = \mu(E)$ ($m(E)$ finite & positive) $\ni$

$$Pr\{\omega \mid \mu(E,\omega) = k\} = e^{-\lambda m(E)} \frac{(\lambda m(E))^k}{k!}$$

and for disjoint E_i the r.v.'s $\mu(E_i)$ are independent.

$$\mu(E) = \int_0^\infty I_E(t)\, dx(t)$$

where $x(t)$ is linear poisson process normalized so as to be left-continuous.

$$E\left(e^{i\theta \mu(A)}\right) = E\left(e^{i\theta \int_0^\infty I_A(t)\,dx(t)}\right)$$

$$= e^{\lambda m(A)(e^{i\theta} - 1)}$$

whenever A is disjoint union of intervals, and so at least for every open set.

Now in k-space. Let B be an open set of radii. Form $n = \mu(V^{\#}B)$. Let S be an open set on the unit sphere. Define

$$\nu(B \times S) = j \quad \text{w. prob.} \quad \binom{n}{j} p^j (1-p)^{n-j}$$

where $p = \dfrac{\text{area of } S}{\text{area of sphere}}$

Then $\nu(B \times S)$ has poisson dist. with parameter $\lambda V(B \times S)$.

In fact, n has poisson dist. with parameter

$$\lambda\, m(V^{-1}(B)) = \lambda \int_{V^{-1}(B)} dt = \lambda \int I_{V^{-1}B}(t)\, dt$$

$$= \lambda \int I_B(V(t))\, dt$$

$$= \lambda \int I_B(t)\, dV(t) = \lambda\, Vol(B)$$

etc.

---

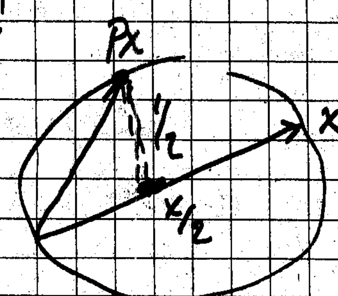
x, $\|x\| = 1$.

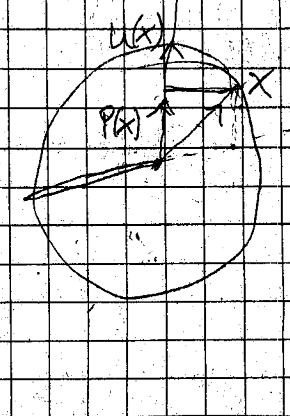
Px = projection of x on Ux (U unitary)

$= (x, Ux)\, Ux$

Then $\|Px - \frac{1}{2}x\| = \frac{1}{2}$!!

So $\{Px \mid U \text{ unitary}\}$ is sphere having x as diameter !!





Maruyama, Gisiro. Fourier Analytic Treatment of some problems on the sums of random variables.
Nat. Sci. Rep. Ochanomizu Univ. 6 (1955) 7-24
MR 18 (1957 #4) 341. Cf also Chung & Pollard PAMS / 1952 303-7.

$\{X_n\}$ ind., non lattice, ident. $m = E(X_1) > 0$.

Let $N_x = \#$ of X_n in $(x, x+h]$.

Then $\lim_{x\to\infty} E(N_x) = \frac{h}{m}$.

~~(2) $\lim_{n\to\infty} \Pr\{M_n \le \sqrt{n}\,\frac{h}{\sigma}\, x\}$~~

NOW: I ask, does N_x have a limiting distribution as $x \to \infty$?

Darling's identity: $e^{i\xi \sum_0^n a_k} = 1 + \sum_0^n (1 - e^{-i\xi a_j})\, e^{i\xi \sum_j^n a_k}$

An amusing Markoff chain.

$\{P_i\}$ a finite network, connected.

Transitions to neighbors equally likely.

Let N = total number of edges.

$n(i)$ = " " " " ending at P_i.

Then $\frac{n(i)}{N}$ = stationary distributions.

24

1/31/57 Theorem: $x > 0$, $e^x = P_n(x) + R_n(x)$.

$$0 < \frac{1}{P_n(x)} - e^{-x} < |P_n(-x) - e^{-x}|.$$

Proof: n even. To show

$$\frac{1}{P_n(x)} - e^{-x} < P_n(-x) - e^{-x}$$

$$\Leftrightarrow \quad 1 < P_n(x)\,P_n(-x).$$

Write $P_n = E + O$, note $E' = O$, $O' = E - \frac{x^n}{n!}$

$$\Leftrightarrow \quad 1 < E^2 - O^2.$$

Since $E(0) = 1$ & $O(0) = 0$ need only prove

$$\frac{d}{dx}(E^2 - O^2) \geq 0 \quad \text{for } x > 0$$

This is $2(EE' - OO')$.

$$= 2 \cdot O \cdot (E - O')$$

$$= 2 \cdot O \cdot \frac{x^n}{n!} > 0$$

n odd. Now $O' = E$, $E' = O - \frac{x^n}{n!}$.

Want

$$\frac{1}{P_n(x)} - e^{-x} < e^{-x} - P_n(-x)$$

i.e. $$1 < 2e^{-x} P_n(x) - P_n(x)\,P_n(-x)$$

i.e. $$1 < 2e^{-x}(E + O) - (E^2 - O^2).$$

$\frac{1}{2}\frac{d}{dx}$ is $$e^{-x}(E' \cancel{+O'}) - e^{-x}(\cancel{E} + O)$$

$$- (EE' - OO')$$

$$= (O - E')(E - e^{-x})$$

$$= \frac{x^n}{n!}\big((>1) - (<1)\big) > 0$$

QED

8/7/51

Note on sq roots by Newton's Method.

$$x_1 = \frac{x_0^2 + a}{2x_0}$$

So
$$\frac{x_1}{\sqrt{a}} = \frac{(x_0/\sqrt{a})^2 + 1}{2(x_0/\sqrt{a})}$$

or
$$y_1 = \frac{y_0^2 + 1}{2y_0}$$
, Newton for 1.

Taking

$y_0 = 1.5$

get
$y_1 = 1.0833\cdots$
$y_2 = 1.0032051$
$y_3 = 1.0000051^+$

Hence for $.01 \leq a < 1$

find $n = 1, 2, \cdots, 5 \ni$

$$a_{n-1} \leq a < a_n$$

where $a_n = .01\,(1.5)^{2n}$

and use $x_0 = .1\,(1.5)^n$

Then three iterations will give

$$1 \leq \frac{x_3}{\sqrt{a}} < 1.0000051^+.$$

In fact, if $y_0 = \frac{a+b}{a-b}$ then $y_1 = \frac{a^2+b^2}{a^2-b^2}$

$$\therefore\ y_n = \frac{a^{2^n} + b^{2^n}}{a^{2^n} - b^{2^n}}$$
$a = y_0 + 1$, $b = y_0 - 1$.

So
$$y_n = \frac{(y_0+1)^{2^n} + (y_0-1)^{2^n}}{(y_0+1)^{2^n} - (y_0-1)^{2^n}}$$

For $y_n = 1 + \varepsilon$ must have $y_0 = \frac{1+\delta}{1-\delta}$,

with
$$\delta = \sqrt[2^n]{\frac{\varepsilon}{2+\varepsilon}}$$

A Trivial Strong Law.

$\{X_n\}$ ident, ind, $0 < E(X_n) = m < \infty$.

$c > 0$. N_c = least $n \ni S_n \geq c$.

Then $\dfrac{N_c}{c} \to \dfrac{1}{m}$ w. prob. 1.

Pf. $\dfrac{S_n}{n} \to m$ w. prob. 1.

$\therefore \dfrac{n}{S_n} \to \dfrac{1}{m}$ " "

$\therefore \dfrac{N_c}{S_{N_c}} \to \dfrac{1}{m}$ " " since N_c is a subseq. that $\to \infty$ w. pr. 1

and $\dfrac{N_c - 1}{S_{N_c - 1}} \to \dfrac{1}{m}$ " "

Now $\dfrac{S_{N_c}}{c} \geq 1 \geq \dfrac{S_{N_c - 1}}{c}$

So $\underline{\lim}_{c \to \infty} \dfrac{S_{N_c}}{c} \geq 1 \geq \overline{\lim}_{c \to \infty} \dfrac{S_{N_c} - 1}{c}$

and therefore

$$\underline{\lim}_{c \to \infty} \frac{N_c}{c} \geq \frac{1}{m} \geq \overline{\lim}_{c \to \infty} \frac{N_c - 1}{c}$$

$$\therefore \lim_{c \to \infty} \frac{N_c}{c} = \frac{1}{m} \quad \text{w. prob. 1.}$$

A Reference.

Kozakiewicz Fundamenta 31, 160–178, (1938)

$F_{n,m}(z,z) \to F(z)$ at all cont pts

$\Longleftrightarrow$

$X_n \to X$ in prob.

8/14/57 Howard Reinhardt remark:

u unif on sphere of unit radius in E^n

$\chi^2_{(n)}$ ind. of u

$$\Rightarrow \quad u\sqrt{\chi^2_{(n)}} \sim N(0, I).$$

In particular, if $x \sim N(0, I)$

then $u = \frac{x}{\|x\|}$ is unif

So $\frac{x}{\|x\|}\sqrt{\chi^2_{(n)}} \sim N(0, I)$

Darling Problem: $X = \begin{cases} 1 & p \\ -N & q \end{cases}$; $E(X) > 0$.

$Z = 0$ if all $S_n \geq 0$

$= j$ if first negative S_n is $-j$.

Then

$p_j = \Pr\{Z = j\} = q/p$, $j = 1, 2, \ldots, N$!!

Proved by simple argument showing

$p_1 \leq p_2 \leq \cdots \leq p_N \leq p_1$,

just transforming sequences; cf letter to Darling Summer 1957.

also proved by Wald identity.

E^7

Problem : $\sigma \subset [1, 2, \dots, n]$
(11/7/57)

$$x_\sigma \geq 0, \quad \sum_\sigma x_\sigma = 1$$

$$y_\sigma = \sum_{\tau \supseteq \sigma} x_\tau$$

Note inversion

$$x_\sigma = \sum_{\tau \supseteq \sigma} (-1)^{\tau - \sigma} y_\tau$$

$$z_\sigma = \prod_{i \in \sigma} y_{\{i\}}$$

$$E_\sigma : \; y_\sigma = z_\sigma$$

For $\sigma = \phi, \{1\}\{2\}, \dots, \{n\}$ E_σ automatic.
Call these σ trivial.

Let S be a subset of nontrivial σ.

$\forall S$: Show : $\exists \{x_\sigma\} \ni E_\sigma$ holds for $\sigma \in S$ or trivial, does not hold for any other σ.

The concrete model is : $x_\sigma = \Pr \left\{ \begin{array}{l} i \in \sigma \Rightarrow A_i \text{ occurs} \\ i \notin \sigma \Rightarrow A_i \text{ does not occur} \end{array} \right\}$

and are trying to prove the independence of the eqns, $2^n - n - 1$ in number, defining mutual independence.

Example. $n = 3$. $S = \{\sigma\} = \{\{1, 2, 3\}\}$ (just one σ in S).

$$x_\phi = 11/40$$

$$x_1 = x_2 = x_3 = 1/40$$

$$x_{12} = x_{13} = x_{23} = 7/40$$

$$x_{123} = 5/40$$

11/8 Solution of problem. Let initially all $x_\sigma = \frac{1}{2n}$. Perturb each unwanted y_σ by adding to it $\frac{1}{2^{2n}}$. Then the x_σ remain nonnegative. QED.

4/27/58 Theorem [Apr 15, 1954].

If $\varphi(t)$ is c.f. of i.d. then

$$\log \varphi(t) - \frac{1}{2}\int_{t-1}^{t+1} \log \varphi(\tau)\, d\tau$$

is positive definite.

Pf. $\varphi(t)^s$ $(s>0)$ makes sense and equals

$$\int_{-\infty}^{\infty} e^{itx}\, dF_s(x).$$

Then

$$\varphi(t)^s - \frac{1}{2}\left\{ \varphi(t+u)^s + \varphi(t-u)^s \right\}$$

$$= \int_{-\infty}^{\infty} e^{itx} \{1 - \cos xu\}\, dF_s(x)$$

is pos. def. fcn. of t.

Then $2\left[\varphi(t)^s - \frac{1}{2}\int_{-1}^{1} \varphi(t+u)^s\, du\right]$ is too.

So

$$\frac{\varphi(t)^s - 1}{s} - \frac{1}{2}\int_{-1}^{1} \frac{\varphi(t+u)^s - 1}{s}\, du \text{ is too.}$$

So $\quad \log\varphi(t) - \frac{1}{2}\int_{-1}^{1} \log\varphi(t+u)\,du \quad$ is too.

QED.

The point of this was to get a new proof of the Lévy formula. We have to solve an eqn of the form

$$\log\varphi - \frac{1}{2}\int_{-1}^{1}\log\varphi = \int e^{itx}\,dG(x)$$

The known thing satisfies this and the question of uniqueness remains.

See Titchmarsh* on the integral equation $\quad f - \frac{1}{2}\int f = 0.$

* Fourier Integral. pg 300 – 298

$$\varphi(t) = e^{iat}$$
$$\log = iat$$

$$iat - \frac{1}{2}\int_{-1}^{1} ia(t+u)\,du = 0$$

$$\varphi(t) = e^{-\frac{1}{2}t^2}$$
$$-t^2/2$$

$$-t^2/2 - \frac{1}{4}\int_{-1}^{1}(t+u)^2\,du$$

$$-\frac{t^2}{2} - \frac{1}{12}\left((t+1)^3 - (t-1)^3\right)$$

$$\cancel{-\frac{t^2}{2}} + \frac{1}{4}\int_{-1}^{1}(\cancel{t^2} + \cancel{2tu} + u^2)\,du = \text{const}$$

$$\varphi(t) = \cancel{e^{(1-e^{it})}} \quad e^{(e^{it}-1)}$$

$$e^{it}\cancel{-1} - \frac{1}{2}\int_{-1}^{1}(e^{i(t+u)}\cancel{-1})\,du$$

$$e^{it} - \frac{1}{2}e^{it}\,\frac{e^{i}-e^{-i}}{i} = (1-\sin 1)e^{it}$$

6/25/58 The problem of "peaks".

Let $i_1, i_2, \ldots, i_n$ be a random permutation of $1, 2, \ldots, n$. Say that j is a *peak* in case all i_r to left of j are less than j. Let $X_j = \begin{cases} 0 & j \text{ not a peak} \\ 1 & j \text{ a peak} \end{cases}$

Then the X_j are mutually ind. &

$$\Pr(X_j = 1) = \frac{1}{n-j+1}, \quad j = 1, 2, \ldots, n$$

Pf: $X_n \equiv 1$.

$$X_{n-1} = \begin{cases} 1 & \text{if } n-1 \text{ to left of } n \\ 0 & \text{" } \text{ " } \text{ rt. " " } \end{cases}$$

two equally likely positions.

Similarly, $n-2$ can go in 3 " " only one of which makes $n-2$ a peak, etc.

Cor: $S_n = \sum_1^n X_i$ = # of peaks has dist. given by

$$E(t^{S_n}) = \prod_{j=0}^{n-1} (t+j) / n!$$

This bears on the problem of a strategy for the problem of finding the largest number in a set of n distinct. Look at them one by one and declare "largest" (stop looking) or "not largest" (continue looking). Clearly it is only necessary to look at peak values.

7/26/58 Spitzer's problem – but see Desiré André, Jahresberichte 1879 & thereabouts.

$p_n = \Pr\{\text{random perm. of } 1,2,\dots,n \text{ is zigzag}\}$

Clearly

$$p_n = \frac{1}{n} \sum_{k=0} p_{2k}\, p_{n-2k-1} \qquad (p_0 = 1).$$

So

$$P(x) - 1 = \int_0^x E(\xi)\, P(\xi)\, d\xi \qquad (E(\xi) = \text{even powers})$$

So

$$P(x) = \sec x + \tan x \qquad !!!$$

12/24/58 Shield's problem. $\mu(C - \bigcup_1^\infty C_n) = 0$, where C = unit circle and C_n's are disjoint nonzero circles. Prove Σ radius C_n diverges.

μ = Lebesgue measure in E^2
m = " " " E^1

Wesler's tentative solution.

Let $A_n = \{x \mid (x,y) \in C_n \text{ for some } y\}$

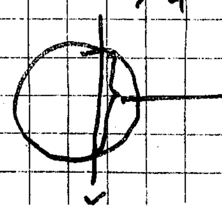
Then $m(A_n) = 2r_n$ (r_n = radius of C_n).

If $\Sigma\, m(A_n) < \infty$ then $m(\overline{\lim}\, A_n) = 0$,

i.e. $m\{x \mid \cancel{(x,y)}\ x \text{ in } \infty\text{'ly many } A_n\} = 0$

i.e. $m\{x \mid \text{chord of } x \text{ meets } \infty\text{'ly many } C_n\} = 0$.

This is surely absurd! (In fact, probably for almost every x ~~chord~~ the chord of x meets ∞'ly many C_n.)


chord of x.

Griffin says: if a ~~line~~ chord meets only finite number of circles & total length is cut off then it [illegible]

∃ only countable # of pts of tangency of pair of circles

te.

9/59 (Darling)

$\varphi(u,v) = e^{-\sqrt{u^2+v^2}} = e^{-\rho}$ is a cf analytic in θ (!) but the corresponding distribution with density $f(x,y) = (2\pi)^{-1}(1+x^2+y^2)^{-3/2}$ does not have many moments and so does not have a moment generating function.

(Darwin) (Darling)

log normal distribution is not determined by its moments. Indeed, its density

$$x^{-1} \exp -\frac{1}{2\sigma^2}(\log x - \mu)^2$$

is large enough at ∞ that one can add e.g. $c e^{-x^{1/4}} \sin x^{1/4}$ all of whose moments vanish.

Problem: Find a distribution determined by its moments whose cf does not have analytic extension near zero.

(Darling) (Wyman Richardson)

Probable solution: $f(x) = ce^{-x^{3/4}}$ (cf. Titchmarsh Fourier Integral pg 320); question seems to reduce to: can

$$\int_{-\infty}^{\infty} x^n dF(x) = \int_{0}^{\infty} x^n dG(x) \quad (F, G, \text{dfs.})$$

when $F(x) > 0$ for all x.

(Darling)

A trivial lemma: Let $\{E_n\}$ be events, $1 > c > 0$, n_0 positive integer; suppose that

$$(1) \quad \Pr\left\{ \bigcup_{\nu = n+n_0}^{\infty} E_\nu \,\middle|\, E_n \right\} < c$$

Then $\Pr\{ \overline{\lim_{n\to\infty}} E_n \} = 0$.

<u>Pf.</u> We show $\sum_1^\infty \Pr\{E_n\} < \infty$.

Let $F_n = \bigcap_{\nu \geq n} E'_\nu$ (prime for complement).

Clearly (1) is equivalent to

$$\Pr\{ F_{n+n_0} \cap E_n \} > (1-c)\Pr\{E_n\}.$$

$$F_{n+n_0} \cap E_n \subseteq F_{n+n_0} - F_n$$

$$\therefore \quad (1-c)\Pr\{E_n\} \leq \Pr\{F_{n+n_0} - F_n\}$$

Sum on $n = n_0, 2n_0, \ldots$

then on $n = n_0+1, 2n_0+1, \ldots$

and so on. <u>QED</u>.

(Wyman Richardson)

A queer trick: $i = 1, \ldots, k$ $\quad j = 1, \ldots, n_i$ $\quad x_{ij}$ have same dist. for $j = 1, \ldots, n_i$ and are ind. To test if they are normal with the same variance (but means μ_i) work the following trick on each batch: form $\frac{x_{i1} - x_{i2}}{\sqrt{2}}, \frac{x_{i1} + x_{i2} - 2x_{i3}}{\sqrt{6}}, \ldots$ getting $n_i - 1$ <u>ind.</u> normal (under hypothesis) equal-variance. Then perform χ^2 goodness-of-fit to all of these simultaneously, $i = 1, 2, \ldots, k$.

10/19/59 2/25/60

$$G(p,q) = \frac{1}{(1-p-q)(1+p)(1+q)} = \sum c_{xy} p^x q^y$$

As $x, y \to \infty$, $y/x \to \lambda$, $c_{xy} \sim c\binom{x+y}{y}$ and

$c_1\binom{x+y}{x} < c_{xy} < c_2\binom{x+y}{y}$, c_1, c_2 fns of λ.

Generalize? (Darling, from Schutzenberger).

12/22/59 Possible problems for Eisenman (and others).

1) Iterated log for Cauchy etc (stable, inf div.) processes or seq's of partial sums.

2) Cesaro summability of $\Pr\{S_n > 0\}$? What kinds of numbers can these be? If all $= 1/2$ is variable symmetric? 3/11

3) Occupation times for partial sums in general sets?

4) Asymp. dist. of last $n \ni \left|\frac{S_n}{n} - \mu\right| > \varepsilon$, $\varepsilon \to 0$?

5) For Wiener process, asymp. dist. of last $t \ni x(t) \geq (1+\delta)\sqrt{t \log\log t}$, $\delta \to 0$. And discrete analogue.

6) dist. of $\sqrt{c}\left(\frac{N_c}{c} - \frac{1}{\mu}\right)$, $c \to \infty$? N_c = first $n \ni S_n > c$.

7) place of max of $\frac{S_n}{\sqrt{n}}$.

8) Central limit thm Error term for e.g. X = two incommensurate values. (Gap here between lattice and non-lattice cases.)

9) "Properties of almost all numbers", Don's math club talk.

?/60 Problem (McLaughlin) $g(x) = \dfrac{\sin nx}{n \sin x} \Rightarrow$

$$\sum_{j=1}^{n} g\left(x - \frac{j\pi}{n}\right) g\left(\frac{j\pi}{n} - y\right) = g(x-y)$$

Problem (old thing - new details). $f(0) = f(1) = 0$, $f(x)$ cont. α is a <u>chord</u> of f if $\exists x \ni f(x) = f(x+\alpha)$. M_f is the measure of the set of chords of f. Show $M_f \geq 1/2$. (In fact, if α is not a chord then $1-\alpha$ is a chord! Pf by extending f to be periodic and showing that every number is a chord.)

8/11/60 Darling's Math Club Talk — Oct 1957

$f(n)$ a given fcn.

A_n : H's on trials $n, n+1, \cdots, n+f(n)$.

B_n : H's for $\geq f(n)$ consecutive trials $j \leq n$.

$$\Pr\{\overline{\lim}\, A_n\} = \begin{cases} 1 & \sum p^{f(n)} = \infty \\ 0 & < \infty \end{cases}$$

$$\Pr\{\overline{\lim}\, B_n\} = \begin{cases} 1 & \sum n p^{f(n)} = \infty \\ 0 & < \infty \end{cases}$$

$$\Pr\{\underline{\lim}\, B_n\} = \begin{cases} 1 & \sum e^{-nq p^{f(n)}} < \infty \\ 0 & = \infty \end{cases}$$

10/19/59 2/25/60

$$G(p,q) = \frac{1}{(1-p-q)(1+p)(1+q)} = \sum c_{xy} p^x q^y$$

As $x, y \to \infty$, $y/x \to \lambda$, $c_{xy} \sim c\binom{x+y}{y}$ and

$c_1\binom{x+y}{x} < c_{xy} < c_2\binom{x+y}{y}$, c_1, c_2 fns of λ.

Generalize? (Darling, from Schützenberger).

12/22/59 Possible problems for Eisenman (and others).

1) Iterated log for Cauchy etc (stable, inf div.) processes or seq's of partial sums.

2) Cesaro summability of $Pr\{S_n > 0\}$? What kinds of numbers can there be. If all $= 1/2$ is variable symmetric? 3/11

3) Occupation times for partial sums in general sets?

4) Asymp. dist. of last $n \ni \left|\frac{S_n}{n} - \mu\right| > \varepsilon$, $\varepsilon \to 0$?

5) For Wiener process, asymp. dist. of last $t \ni x(t) \geq (1+\delta)\sqrt{t \log\log t}$, $\delta \to 0$. And discrete analogue.

6) dist. of $\sqrt{c}\left(\frac{N_c}{c} - \frac{1}{\mu}\right)$, $c \to \infty$? N_c = first $n \ni S_n > c$.

7) place of max of $\frac{S_n}{\sqrt{n}}$.

8) Central limit thm error term for e.g. X = two ~~va~~ incommensurate values. (Gap here between lattice and non-lattice cases.)

9) "Properties of almost all numbers", Don's math club talk.

4/60 From Ted Harris' book.

$$f(x) = \sum_0^\infty a_n x^n, \quad a_n \geqq 0, \quad f(0) > 0, \quad f(1) = 1$$

$$\Longrightarrow: \quad g(x) = (f(x) - x)^{-1} = \sum_0^\infty b_n x^n, \quad b_n \geqq 0.$$

Pf: $\dfrac{1-f(x)}{1-x} = a_1 + a_2(1+x) + \cdots$ has non neg coeffs.

$$\left|\frac{1-f(x)}{1-x}\right| < 1 \text{ for } |x| < x_0 \text{ since } \frac{1-f(x)}{1-x} \underset{x\to 0}{\to} 1 - f(0) < 1$$

So $g(x) = \dfrac{1}{1-x} \dfrac{1}{1 - \frac{1-f(x)}{1-x}} = \dfrac{1}{1-x} \sum_0^\infty \left\{\dfrac{1-f(x)}{1-x}\right\}^n$ has nonneg coeffs. QED.

A probabilistic interpretation.

Suppose $p(x) = \sum_{-1}^\infty p_j x^j, \quad p_j = Pr\{X = j\}$

$$= E(x^X)$$

If $E(X) < 0$ then $M = \max_{0 \leq n < \infty} S_n$ is finite w.pr. 1.

Let $\qquad g(x) = E(x^M).$

We have $\quad E(x^M) = \sum_{-1}^\infty E(x^M \mid X_1 = j) p_j$

$$= \sum_{-1}^\infty E(x^{(M+j)^+}) p_j \qquad u^+ = \max(u, 0).$$

$$= \left(\sum_0^\infty p_j x^j\right) g(x) + p_{-1} P \frac{g(x)}{x} \qquad P x^n = \begin{cases} 1 & n \leq 0 \\ x^n & n > 0 \end{cases}$$

$$\therefore \quad g(x) = (p(x) - p_{-1} x^{-1}) g(x) + p_{-1} P \frac{g(x)}{x}$$

Hence $\quad [p(x) - 1] g(x) = p_{-1} [I - P] \dfrac{g(x)}{x} \qquad I = \text{identity}$

$$P([p(x) - 1] g(x)) = 0$$

$(p(x) - 1) g(x) = \dfrac{a}{x} - a$ since lowest term is of degree -1.

Hence $\qquad g(x) = \dfrac{\frac{a}{x} - a}{p(x) - 1}$

$$= \frac{a(1-x)}{x(p(x) - 1)}$$

Since $g(1) = 1$ we have $a = -E(\mathfrak{X})$,

$$g(x) = \frac{-E(\mathfrak{X})(1-x)}{x\,p(x) - x}$$

and $xp(x)$ may be identified with $f(x)$.

4/60 Inversion integrals are improper at zero.

$\int_0^\infty \frac{\operatorname{Im}\varphi(t)}{t}\,dt$ need not exist at 0 (or at ∞)

Pf: Consider $\varphi(t) = \sum_1^\infty e^{itn} p_n$

So $\operatorname{Im}\varphi(t) = \sum_1^\infty p_n \sin nt$.

Clearly $\int_0^1 \frac{\left|\sum_1^N p_n \sin nt\right|}{t}\,dt$ exists.

~~So if Suppose~~ $\cancel{\int_0^1 \frac{\left|\sum_1^\infty p_n \sin nt\right|}{t}\,dt}$

1. Let $p = \{p_n\}$ range over (ℓ_1), all $p_n \geq 0$, (ℓ_1)-norm.
Form $f_p(t) = \sum p_n \sin nt$.

Theorem: $\{p \mid \int_0^1 t^{-1}|f_p(t)|\,dt < \infty\}$ is of first category.

Pf: Let $\mathcal{F}_N = \{p \mid \int_0^1 t^{-1}|f_p(t)|\,dt \leq N\}$.

I claim $\mathcal{F}_N$ is closed. In fact, if $p_n \to p$ in norm then $f_{p_n}(t) \to f_p(t)$ for every t; supposing $p_n \in \mathcal{F}_N$ and applying Fatou we have $\int_0^1 t^{-1}|f_p(t)|\,dt$ $\leq \varliminf \int_0^1 t^{-1}|f_{p_n}(t)|\,dt \leq N$, so $p \in \mathcal{F}_N$. Next I claim that $\mathcal{F}_N$ contains no sphere. For suppose $\exists\, p, \delta \ni p \in \mathcal{F}_N$ & $\|g - p\| \leq \delta \Rightarrow g \in \mathcal{F}_N$. ~~Let then~~

Pick $g = g_n = p + \delta\varepsilon_n$ $(\varepsilon_n = (0, 0, \ldots, \underset{n^{\text{th}}}{1}, 0, \ldots))$
Form $f_g(t) = f_p(t) + \delta \sin nt$.

We have $|\delta \sin nt| \leq |f_p(t)| + |f_{g_n}(t)|$
and so $\int_0^1 t^{-1}|\delta \sin nt|\,dt \leq N + N = 2N$.

But $n \to \infty$ gives a contradiction

5/18/60 Theorem: X r.v., φ its cf, $Rl\,\varphi(t) \to 0$ as $t\to\infty$. Then $\cos nX$ has arcsine law as limiting distribution!

Proof: $$e^{\lambda\cos x} = e^{\frac{1}{2}\lambda e^{ix}}\ e^{\frac{1}{2}\lambda e^{-ix}}$$

$$= \sum_0^\infty \left(\frac{\lambda}{2}\right)^n \frac{e^{nix}}{n!}\ \sum_0^\infty \left(\frac{\lambda}{2}\right)^n \frac{e^{-nix}}{n!}$$

$$= \sum_0^\infty \left(\frac{\lambda}{2}\right)^{2n} \frac{1}{(n!)^2} + \sum_1^\infty G_n(\lambda)\cos nx$$

Hence $$e^{i\xi\cos x} = \sum_0^\infty \frac{(i\xi)^{2n}}{2^{2n}(n!)^2} + \sum_1^\infty H_n(\xi)\cos nx$$

$$e^{i\xi\cos Nx} = \sum_0^\infty \frac{(i\xi)^{2n}}{2^{2n}(n!)^2} + \sum_1^\infty H_n(\xi)\cos nNx$$

$$E\{e^{i\xi\cos Nx}\} = \sum_0^\infty \frac{(-1)^n\xi^{2n}}{2^{2n}(n!)^2} + \sum_1^\infty H_n(\xi)\,Rl\,\varphi(nN)$$

$$\longrightarrow \sum_0^\infty \frac{(-1)^n\xi^{2n}}{2^{2n}(n!)^2} \quad \text{as } N\to\infty.$$

But the latter is $J_0(\xi) = E(e^{i\xi y})$, y arcsine

[cf Whittaker & Watson §17.1 Ex. 3

$$e^{iz\cos\phi} = J_0(z) + 2i\cos\phi\, J_1(z) + 2i^2\cos 2\phi\, J_2(z) + \cdots \quad]$$

5/26/60 (p_{ij}) a markov chain, (p_i) its stationary distribution. Let the initial distribution be (p_i). Let τ be the time when a fixed state i is entered for the first ~~time~~ time, $\tau = 0, 1, 2, \ldots$. Find distribution of τ.

Soln: From $U(s) = \dfrac{G(s)}{1-F(s)}$ and $U(s) = \dfrac{p_i}{1-s}$

we find at once $G(s) = \dfrac{p_i(1-F(s))}{1-s}$;

here $G(s) = E(s^\tau)$ and $F(s) = E(s^T)$, where T is the return-time to state i.

Hence e.g. $E(\tau) = G'(1) = \dfrac{p_i}{2}F''(1)$

$$= \frac{1}{2}\left\{\frac{E(T^2)}{E(T)} - 1\right\} \quad \text{since } p_i^{-1} = E(T).$$

11/60 Problems (Savage).

1) $\{x_n\}$ $\quad E(x_n \mid x_0, x_1, \ldots, x_{n-1}) \leq x_{n-1}$, $\quad x_n > 0$

$\Longrightarrow \quad \Pr\{\text{some } x_n - x_m,\ n > m,\ > y \mid x_0 = x\} < 1 - e^{-\frac{x}{y}}$.

2) Given n $\exists N \ni \forall$ perm. of $1, 2, \ldots, N$ (say $\pi_1, \ldots, \pi_N$) $\exists\, i_1 < i_2 < \cdots < i_n \ni$

$\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_n}$ or all $>$. $\quad N = n^2 + 1$ or $(n-1)^2 + 1$?

Pf of 2) Clearly $N = (n-1)^2 + 1$.

Beginning with π_1 find first $\pi_i > \pi_1$, first $\pi_j > \pi_i$ etc. Call these numbers of class 1. Deleting the numbers of class 1 proceed as before getting numbers of class 2. Observe that to the left of every number of class 2 there must be a larger one of class 1. Next class 3, left larger of class 2. And so on. Let c = # of classes. Then $\exists \searrow$ chain of length c, and no longer. If every class is of length $\leq r$ then max $\nearrow$ chain is of length $\leq r$.

If $r \leq n-1$ & $c \leq n-1$ then $N \leq rc \leq (n-1)^2$.

QED!

A much better proof (Savage).

Let $f : i \to (a, b)$ be defined by

a = length of longest $\nearrow$ chain from left ending at π_i

b = " " " $\searrow$ " " "

Then $i \neq j \Longrightarrow f(i) \neq f(j)$, for if $i < j$ & $\pi_i < \pi_j$ then $a_i + 1 \leq a_j$ and sim. for $\pi_i > \pi_j$.

Hence the result!!

[$(n-1)^2 + 1$ is best possible; e.g. $n = 4$, $N = 9$,

$\pi = 3, 6, 9, 2, 5, 8, 1, 4, 7$.]

12/12/60 Paul Kahn's problem.

Find $\varphi_0 \ni \int_0^\infty \varphi(x)^2 dF(x)$ is min, $0 \leq \varphi(x) \leq x$,

$$\int_0^\infty \varphi(x)\, dF(x) = \gamma.$$

<u>Soln</u> from Lemma: $\begin{cases} \min \int (\varphi(\omega) - f(\omega))^2 d\mu(\omega), & f \geq 0 \\ 0 \leq \varphi(\omega) \leq g(\omega) \end{cases}$

attained by $\varphi_0(\omega) = f(\omega) \wedge g(\omega)$. If

$\int [f(\omega) \wedge g(\omega)] f(\omega)\, d\mu(\omega) = c$ then can adjoin

condition $\int \varphi(\omega) f(\omega)\, d\mu(\omega) = c$.

12/10/60 Herman Rubin's problems.

$X_n = \pm 1$, p, q, $E(X_n) < 0$, $M = \max S_n$.

1) Find distribution of X_1 given $M > 0$; supposedly q, p (i.e. reversed)

2) Find $\Pr\{\text{return to } 0 \mid M = 0\}$; supposedly p.

12/12/60 Savage (from Thrall) problem.

X_n, ind, $N(0, I)$ in plane. What can be said about behavior of N_n = # of extreme points of convex hull of $x_1, \ldots, x_n$.

12/25/60

Jim:

Here's the best result I can get.

Let $p_j = \frac{1}{f(j)}$ $\quad j = 1, 2, \dots$ $\quad$ where $p_j > 0$, $\sum p_j = 1$,

$f(x) \nearrow \infty$ and f is differentiable and where

$f^{-1}(x) \sim x^{\alpha} L(x)$ $\quad 0 \le \alpha < 1$, $L(x)$ slowly varying, $x \to \infty$. Then

$$\sum_{1}^{\infty} \left[1 - (1 - p_j)^n\right] \sim \Gamma(1-\alpha)\, f^{-1}(n), \quad n \to \infty.$$

Maybe the conditions could be lightened.

See abstract by Bahadur in Spring 1961 (winter 1960?) AMStat

Don

5/21/61 (Leo Moser, from Bill LeVeque):

$$\prod_1^{\infty} \left\{1+\left(\frac{\sqrt{5}-1}{2}\right)^{n}\right\}^{\frac{\varphi(n)}{n}} = e$$

φ = Euler function.

6/26/61 Minty problem. Given $x_i, y_i \in \mathcal{X}$, $i = 1, 2, \dots, n$, $\ni$ $(x_i - x_j,\ y_i - y_j) \geq 0$ all i,j. Given x_{n+1} $\exists$ y_{n+1}.

Pf: Can assume $x_{n+1} = 0$. Want to solve

(*) $(x_i,\ y_i - y) \geq 0$ all i.

i.e. $Ay \leq b$, rows of A are x_i, b is (x_i, y_i).

If no solution then $\begin{cases} \eta A = 0 \\ \eta b = 1 \end{cases}$ with $\eta \leq 0$ has a solution.

i.e. $\sum \eta_i x_i = 0$, $\sum \eta_i (x_i, y_i) = 1$. So not all $\eta_i = 0$.

~~Since~~ But $(x_i, y_i) + (x_j, y_j) \geq (x_i, y_j) + (x_j, y_i)$
we mult by $\eta_i \eta_j$ and sum on i,j, getting

$$2 \sum \eta_i \geq 0$$

contradicting $\eta \leq 0$. Hence (*) has soln.

Wesler problem

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

$$\Phi(x) = \cancel{\tfrac{1}{\sqrt{2\pi}}} \int_{-\infty}^{x} f(t)\,dt$$

Well known that $x < \dfrac{f(x)}{1-\Phi(x)} \sim x$ as $x \to \infty$.

Prove $g(x) = \dfrac{f(x)}{1-\Phi(x)} - x \downarrow 0$.

12/12/61 Ergodic thm ⟹ Strong law, without explicit use of metric transitivity (or zero-one law).

$$\frac{x_1+\cdots+x_n}{n} \to y,$$

$$\frac{x_1+x_3+\cdots+x_{2n-1}}{n} \to z$$

$$\frac{x_2+x_4+\cdots+x_{2n}}{n} \to w$$

$y \sim z \sim w$ ∴ $E(y) = E(z) = E(w)$ & $E(|y|) = E(|z|) = E(|w|)$

But $\frac{z+w}{2} = y$. Hence $\underbrace{\tfrac{1}{2}\left[E(|z|)+E(|w|)\right]}_{E(|y|)} \geq E\left(\left|\frac{z+w}{2}\right|\right)$ $= E(|y|)$. So equality holds throughout. But this requires sgn z = sgn w w. prob 1. Since z & w are independent it follows that z has fixed sign w.pr. 1, so do y & w. But now applying the above to $x_n - c$ for suitable c we get $y-c$, $w-c$, $z-c$, and so contradiction unless already y, z, w <u>const.</u>

12/23/61 (Work with Savage) — nonatomic, $\{x\} \in \mathcal{A}$ for $x \in X$

$(X, \mathcal{A}, \mu)$, $\mu(X) = 1$. θ a random distribution of unit mass on X, such that for disjoint $A_1, \ldots, A_n$ the distribution of $\theta(A_1), \ldots, \theta(A_n)$ depends only on $\mu(A_1), \ldots, \mu(A_n)$.

<u>Theorem</u>: If $\theta(\{x\}) = 0$, for all x, simultaneously, w. pr. 1 then $\theta = \mu$ w. pr. 1.

<u>Proof</u>: Lemma: If $A_1, \ldots, A_n$ disjoint & $\mu(A_j) = 1/n$ then ~~…~~ $n E(\theta(A_n)^2) \to 0$. 1/4/62

(Proof of lemma: $(A_1 \times A_1) \cup \cdots \cup (A_n \times A_n)$ * tends to the diagonal D of $X \times X$ as $n \to \infty$. The

* cf. 8/22/62

product measure $\theta \times \theta$ of D is zero by $\theta(\{x\}) = 0$ all x. QED.

Now $E(\theta(A)) = f(\mu(A))$, additive, hence $= \mu(A)$. Also $E(\theta(A)\theta(B))$ = additive fcn of A for fixed B when $A \cap B = \phi$. So

$$A \cap B = \phi \Rightarrow E(\theta(A)\theta(B)) = \frac{E(\theta(B)^2)}{\mu(B)}\mu(A).$$

Then after some calculation

$$\frac{\mu(A) - \mu(A)^2}{\mu(A) - E(\theta(A)^2)} = \text{const}.$$

Let $A = A_n$ as in lemma and obtain const $= 1$. So $E(\theta(A)^2) = \mu(A)^2$ and $\operatorname{var}\theta(A) = 0$. QED

Ex. Multibeta dist of $\theta(A_1), \ldots, \theta(A_n)$, $\lambda\mu(A_1), \ldots, \lambda\mu(A_n)$. Comes from "compound Poisson Rain", $\frac{e^{-u}}{u}\,du\,dt$ (size of drop in $u, u+du$), for time $t = \lambda$. As $\lambda \to \infty$ tends to singular dist conc @ μ. const of above discussion $= \frac{\lambda}{\lambda+1} \to 1$ as $\lambda \to \infty$.

1/4/62 Prob of Machol (from Savage). $\{x_n\}$, x_{n+1} unif on $(x_n, 1)$. Find $\lim E(x_1 \cdots x_n)$. Solution: let $x_1 = 1 - u_1, \ldots, x_{n+1} = x_n + (1 - u_{n+1})x_n = 1 - u_1 \cdots u_{n+1}$ by induction. Consider $v(t) = \prod_1^\infty \{1 - t u_1 \cdots u_n\}$, a bona fide r. v. Equal

$$1 - t\sum_i u_i + t^2 \sum_{i<j} w_i w_j - t^3 \sum_{i<j<k} w_i w_j w_k + \cdots$$

$$w_i = u_1 \cdots u_i$$

$$= 1 - t\sum u_1 \cdots u_i + t^2 \sum_{i<j} u_1^2 \cdots u_i^2\, u_{i+1} \cdots u_j$$

$$- t^3 \sum_{i<j<k} u_1^3 \cdots u_i^3\, u_{i+1}^2 \cdots u_j^2\, u_{j+1} \cdots u_k + \cdots$$

& taking E gives $1 - t\sum_i \frac{1}{2^i} + t^2\sum_{i<j} \frac{1}{3^i}\frac{1}{2^{j-i}} + t^3 \sum \frac{1}{4^i}\frac{1}{3^{j-i}}\frac{1}{2^{k-j}}$

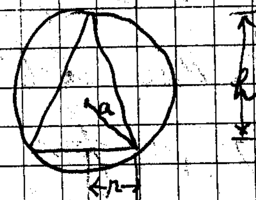
$$- \cdots = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots = e^{-t}.$$

~~Corresponds to~~ Equals $\lim_{n\to\infty} E\left(\prod_1^n (1-t+tx_j)\right)$.

November 1961 & January 1962.

Cones & spheres.

I. Find greatest cone inscribed in given sphere.



$$(h-a)^2 + r^2 = a^2$$
$$h^2 - 2ah + r^2 = 0$$
$$a = \frac{h}{2} + \frac{r^2}{2h}$$
$$= 2\,\frac{h}{4} + 1\cdot\frac{r^2}{2h}$$
$$\geq 3\sqrt[3]{\frac{h^2}{16}\,\frac{r^2}{2h}}$$
$$= 3\,\frac{1}{\sqrt[3]{32}}\sqrt[3]{r^2 h}$$
$$= 3\,\frac{1}{\sqrt[3]{\frac{32}{3}\pi}}\sqrt[3]{V}$$

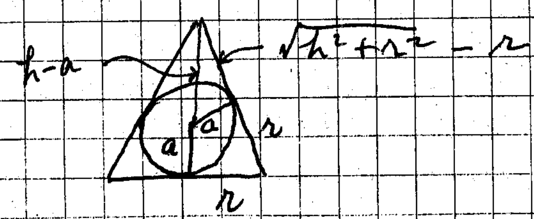
Hence

$$V \leq \frac{32}{81}\pi a^3,$$

equality only when $\frac{h}{4} = \frac{r^2}{2h}$,

i.e. $h = \sqrt{2}\,r$

II. Find smallest cone about given sphere.



$$(h-a)^2 = a^2 + \left(\sqrt{h^2+r^2}-r\right)^2$$

$\left(\frac{1}{3}\right)^{-1}$ $\frac{1}{2}x - j$

$$\cancel{h^2} - 2ah = \cancel{h^2} + 2r^2 - 2r\sqrt{h^2+r^2}$$

$$a = \frac{r}{h}\sqrt{h^2+r^2} - \frac{r^2}{h}$$

$$\frac{1}{a} = \frac{1}{r}\,\frac{h}{\sqrt{h^2+r^2} - r}$$

$$= \frac{h}{r}\,\frac{\sqrt{h^2+r^2}+r}{h^2}$$

Apr 1972 (Shapiro) square again, get

$$a^2 + \frac{2r^2a}{h} + \frac{r^4}{\cancel{h^2}} = r^2 + \frac{r^4}{h^2}$$

$\therefore$ $a^2h + 2ar^2 = r^2h$ and much easier from here on.

Hence

$$= \sqrt{\frac{1}{h^2} + \frac{1}{r^2}} + \frac{1}{h}$$

$V \geqq \frac{8}{3}\pi a^3$, equality only When $\frac{1}{h^2} = \frac{1}{8r^2}$ i.e. $h = 2\sqrt{2}r$.

$$= \sqrt{1\cdot\frac{1}{h^2} + 8\cdot\frac{1}{8r^2}} + \frac{1}{h}$$

$$\geqq \sqrt{9\sqrt[9]{\frac{1}{h^2}\,\frac{1}{8^8 r^{16}}}} + \frac{1}{h}$$

$$= 3\sqrt[9]{\frac{1}{h}\,\frac{1}{8^4 r^8}} + \frac{1}{h}$$

$$= 3\,\frac{1}{8^{4/9}\sqrt[9]{h r^8}} + 1\cdot\frac{1}{h}$$

$$\geqq 4\sqrt[4]{\frac{1}{8^{4/3}\sqrt[9]{h^3 r^{24}}\;h}}$$

$$= \frac{2}{h^{\frac{1}{12}} r^{\frac{2}{3}} h^{\frac{1}{4}}} = \frac{2\sqrt[3]{\frac{1}{3}\pi}}{\sqrt[3]{V}}$$

18 Jan 1962

Theorem: $1 > x_1 > x_2 > \cdots$, each x_n unif. on $(0, x_{n-1})$; let f be a reasonably nice function, e.g. $f(x) = [1-tx]^k$. Then $E\{\prod f(x_n)\} = \exp \int_0^1 u^{-1}[f(u)-1]\,du$.

Proof: Let $q(t)$ = Poisson process. Let $Y = \exp\int_0^\infty \log f(e^{-t})\,dq(t)$. On the one hand $Y = \exp \sum \log f(e^{-t_n}) = \prod f(e^{-t_n}) = \prod f(x_n)$, where $0<t_1<t_2<\cdots$ are the times at which q jumps. On the other hand Y is approximately ~~the~~ $\exp\sum \log f(e^{-\tau_n})\,dq_n =$ $\prod f(e^{-\tau_n})^{dq_n}$ where we have cut up $(0,\infty)$ into tiny pieces $d\tau_n$ with corresponding increments in q equal to dq_n. Now $dq = 0$ or 1 with prob. $1-d\tau$ or $d\tau$ and hence $E a^{dq} = 1 - d\tau + a\,d\tau = 1-(1-a)\,d\tau$. Since the dq_n are independent we have $EY \approx \prod\{1-[1-f(e^{-\tau_n})]\,d\tau_n\} \doteq \exp\sum\log\{1-[\sim]\,d\tau_n\}$ $\doteq \exp\sum[f(e^{-\tau_n})-1]\,d\tau_n \doteq \exp\int_0^\infty [f(e^{-\tau})-1]\,d\tau$ QED

The case $f(x)=1-x$ came to Savage from Bob Machol. Darling first got $f(x)=[1-tx]^k$ by another method — solving a differential equation.

March 1962.

Machol asks Savage for Pr{4 pts on sphere lie in some hemisphere}. Ans 7/8. Savage generalizes to $n+1$ on S^{n-1} in E^n, $1-\frac{1}{2^n}$.

Let $p_{n,k}$ = Pr{$n+k$ pts on S^{n-1} in E^n lie in some hemisphere}.

Duality Theorem (3/14/62) $p_{n,k}+p_{k,n}=1$.

Proof: We are really only dealing with random directions, $x_1, \ldots, x_n$. These lie in some half-space iff no $\lambda_j \geq 0$ ($\sum\lambda_j > 0$) exist for which $\sum\lambda_j x_j = 0$.

[redacted]

Representing each x_j by n ind $N(0,1)$ and all of them by the $n{+}k$ rows of an $(n+k)\times n$ matrix we see that $1-p_{n,k} = \Pr\{\exists \lambda \in K,\ \lambda \perp P^n\}$. The notation just written means this. K is the positive cone of E^{n+k}; P^n is a random n-dimensional subspace of E^{n+k}. (To write this I have looked at the n columns of the matrix as defining P^n.) Hence $1-p_{n,k} = \Pr\{P^k \cap K \neq \emptyset\}$ where now P^k the orthogonal complement of P^n is also random.

Let now $P \perp Q$, $P \oplus Q = E^n$. Then (except for a set of positions of probability zero in our application) exactly one of P, Q meets K nontrivially. Proof: at most 1 since two vectors interior to K cannot be $\perp$. At least one: suppose $P \cap K = (0)$. Then $\exists$ hyperplane $H = \{x \mid (x,q) = 0\}$ for some q that separates P and K. Since $0 \in P \cap H$ it follows that $P \subset H$, hence $q \perp P$, hence $q \in Q$. Without loss of generality we can take $(q,x) \geq 0$ for all $x \in K$. In particular $(q, e_i) \geq 0$, and not all zero, for the unit vectors $e_i = (0,0,\dots,1,\dots 0)$. Hence the components of q are nonnegative and $q \in K$. $Q \cap K$ is non trivial.

Applying this we obtain $1-p_{n,k} + 1-p_{k,n} = 1$, which is what had to be proved.

Evaluation of $p_{n,2}$: P^2 in E^{n+2} meets K.
Let a and b be random vectors spanning random P^2. When do λ & μ exist such that $\lambda a + \mu b \in K$? Let a_j & b_j be components of a, b and put $r_j = a_j / b_j$. It is easy to show that λ, μ exist iff the intervals determined by r_j for $b_j > 0$ and r_j for $b_j < 0$ are disjoint.

If signs of a_j & b_j are both changed then r_j is unchanged but may be moved from one group to another. Start with all $b_j > 0$. Line up r_j in order of size, $r_1 < r_2 < \dots < r_{n+2}$ without loss of generality.

Then the admissable sign-choices of the b_j are $(++\dots+)$, $(++\dots+-)$, $\dots$ $(--\dots-)$ $(--\dots-+)$, $\dots$, $2(n+2)$ in all. Hence

$$1-p_{n,2} = 1 - \frac{2(n+2)}{2^{n+2}} = 1 - \frac{n+2}{2^{n+1}},$$

I was led to the duality by getting $p_{k,2}$ and $p_{n,2}$ separately and seeing the relation.

I got $p_{2,k}$ thus.

Let $f_{n-1}(\theta)\,d\theta = Pr\{n-1$ pts at random on circle lie in an arc whose length is between θ and $\theta + d\theta$ but in no smaller arc$\}$ $0 < \theta < \pi$.

Then $f_n(\theta)\cancel{d\theta} = \frac{\theta}{2\pi} f_{n-1}(\theta)\cancel{d\theta} + 2\int_0^{\theta} f_{n-1}(\varphi)\,d\varphi \cdot \frac{\cancel{d\theta}}{2\pi}$

So $f_n' = \frac{1}{2\pi}\{\theta f_{n-1}' + 3 f_{n-1}\}$, $f_n(0) = 0$, $f_2(\theta) = 1/\pi$

$f_n = k_n \theta^{n-2}$ yields $f_n(\theta) = \frac{n(n-1)}{(2\pi)^{n-1}}\theta^{n-2}$

whose integral from 0 to π is $\frac{n}{2^{n-1}} = \frac{k+2}{2^{k+1}}$ as sought.

Don had another solution. The n points determine n arcs $x_1, \dots, x_n$ which are uniformly distributed over the simplex $x_1 + \dots + x_n = 2\pi$. Want $Pr\{\max_j x_j > \pi\}$.

31 May 1962 Letter from Coxeter gives reference to Schläfli, Collected works vol 1 pg 209-212 for number of sign-patterns

[redacted]

Delta Math.
81, 13-4 (5)

[33,000
mr Einar Hansen

23822

* cf. 12/23/61

a 8/22/62 How do I know it tends to D? I must have $\{A_{n,k}\} \ni 1 \le k \le n \Rightarrow \mu(A_{n,k}) = \frac{1}{n}$. Change notation. Want some triangular scheme to exist such that e.g. at n^{th} stage each set has measure $1/2^n$ and at the $n+1^{st}$ stage each is cut in two. Let $\{A_{n,k}\}$ be this scheme. But how can we arrange that for each x, y $\exists n \ni x \in A_{n,k_x}$ and $y \in A_{n,k_y}$ with $k_x \ne k_y$? This isn't automatic, for example we might have partitioned the unit square by vertical strips and so couldn't separate points with the same abscissa.
Probably requires some sort of separability (= countability) assumption.

10/22/86 cf Wilf 1986
BAMS 15, 228-232

2/4/63 Wyman & Moser, Can J. 1957

$$\sum_0^\infty \frac{T_n}{n!} x^n = \exp\left\{x + \frac{x^2}{2}\right\}. \qquad T_n = \#\text{ of } x \in S_n \ni x^2 = e.$$

$$T_n \sim \frac{\left(\frac{n}{e}\right)^{\frac{1}{2}n} e^{\sqrt{n}}}{\sqrt{2}\; e^{\frac{1}{4}}} \left\{1 + \frac{7}{24\sqrt{n}} - \frac{119}{1152\, n} + \cdots\right\}$$

pf by $\dfrac{T_n}{n!} = \dfrac{1}{2\pi i} \oint \dfrac{e^{z + \frac{z^2}{2}}\, dz}{z^{n+1}}$

contour : $z = Re^{i\theta}$, $R^2 + R - n = 0$.

7/25/63 A problem of Chow & Robbins.
Let a coin be tossed until a stop rule R applies.
$H = +1$, $T = -1$, S_n as usual
Find
$$\max_R E\left(\frac{S_n}{n}\right) \qquad n = n(R).$$

E.g. $R \Rightarrow$

~~81~~

7/18/62 Doeges trivialyes Laha (quoted in Fisz' review article in Ann. Math. Stat.) result that $\frac{1}{1+|\lambda|^{2\alpha}}$ is cf of inf. div. process that is unimodal. Subordinate Gaussian by positive stable then by gamma

$\lambda^2 \to |\lambda|^{2\alpha} \to \log(1+|\lambda|^{2\alpha})$.

also: for $0<a<1$ and $\delta = \delta$-fncn,

(*) $\quad a\delta + (1-a)e^{-x}$

is an infinitely divisible density.

Any $\sum_0^\infty a_n (e^{-x})^{*n}$ will be so, trivially, if $\sum_0^\infty a_n x^n$ is, but this is not the case in the example (*)

cf of (*): $\quad a + \frac{(1-a)}{1-i\theta} = \frac{1-ia\theta}{1-i\theta}$

whose logarithm is $\sum_0^\infty \frac{(i\theta)^n}{n}(1-a^n)$.

$\log\frac{b}{a} = \int_0^\infty \frac{1}{t}(e^{-at}-e^{-bt})\,dt, \quad a<b$

$\log \frac{1-ia\theta}{1-i\theta} = \int_0^\infty \frac{1}{t}\left(e^{-(1-i\theta)t} - e^{-(1-ia\theta)t}\right)dt$

$\cancel{= \int_0^\infty \left(\qquad \frac{e^{-t}}{t}\,dt \right)} \quad ?$

$\cancel{a + \frac{1-a}{1-i\theta} = a\left\{1 + \frac{1-a}{a}\frac{1}{1-i\theta}\right\}}$

$\cancel{\left\{1+\frac{1-a}{a}\frac{1}{1-i\theta}\right\}^t = \sum_0^\infty \binom{t}{n}\frac{(1-a)^n}{a^n}\frac{1}{(1-i\theta)^n}} \quad ?$

$= \log a + \log\frac{\frac{1}{a} - i\theta}{1-i\theta} = \int_0^\infty (e^{i\theta u}-1)(e^{-u}-e^{-u/a})\frac{du}{u}$

with a curious relation to the Γ process, whose log cf at time 1 is $\int_0^\infty (e^{i\theta u}-1)e^{-u}\frac{du}{u}$.

Let Γ ~~...~~ be ind. Γ-processes, and let this process be called K. Then $K + a\Gamma$ is a Γ-process (ind of Γ)

7/18/62 Dinges trivializes Laha (quoted in ~~R~~ Fisz' review article in Ann. Math. Stat.), result that $\frac{1}{1+|\lambda|^{2\alpha}}$ is cf of inf. div. process that is unimodal. Subordinate Gaussian by positive stable then by gamma.

$$\lambda^2 \to |\lambda|^{2\alpha} \to \log(1+|\lambda|^{2\alpha}).$$

also: for $0<a<1$ and $\delta = \delta$-fncn,

(*) $$a\delta + (1-a)e^{-x}$$

is an infinitely divisible density.

Any $\sum_0^\infty a_n (e^{-x})^{*n}$ will be so, trivially, if $\sum_0^\infty a_n x^n$ is but this is not the case in the example (*)

cf of (*): $$a + \frac{(1-a)}{1-i\theta} = \frac{1-ia\theta}{1-i\theta}$$

whose logarithm is $\sum_0^\infty \frac{(i\theta)^n}{n}(1-a^n)$.

$$\log\frac{b}{a} = \int_0^\infty \frac{1}{t}(e^{-at}-e^{-bt})\,dt, \quad a<b$$

$$\log \frac{1-ia\theta}{1-i\theta} = \int_0^\infty \frac{1}{t}\left(e^{-(1-i\theta)t} - e^{-(1-ia\theta)t}\right)dt$$

$$\cancel{= \int_0^\infty \Big(\qquad \frac{e^{-t}}{t}\,dt} \quad ?$$

$$\cancel{a+\frac{1-a}{1-i\theta} = a\left\{1+\frac{1-a}{a}\frac{1}{1-i\theta}\right\}}$$

$$\cancel{\left\{1+\frac{1-a}{a}\frac{1}{1-i\theta}\right\}^t = \sum_0^\infty \binom{t}{n}\frac{(1-a)^n}{a^n}\frac{1}{(1-i\theta)^n}} \quad ?$$

$$= \log a + \log\frac{\frac{1}{a} - i\theta}{1-i\theta} = \int_0^\infty (e^{i\theta u}-1)(e^{-u}-e^{-u/a})\frac{du}{u}$$

with a curious relation to the Γ process, whose log cf at time 1 is $\int_0^\infty (e^{i\theta u}-1)e^{-u}\frac{du}{u}$.

Let Γ and Γ_a be ind. Γ-processes, and let this process be called K (ind of Γ_a). Then $K + a\Gamma_a$ is a Γ-process.

[redacted]

~~Delta Math.~~
~~8/13-4 (52)~~

[33,000
mr Einar Hansen

23822

* cf. 12/23/61

8/22/62 How do I know it tends to D ? I must have $\{A_{n,k}\} \ni 1 \le k \le n \Rightarrow \mu(A_{n,k}) = \frac{1}{n}$. Change notation. Want some triangular scheme to exist such that e.g. at n^{th} stage each set has measure $1/2^n$ and at the $n+1^{\text{st}}$ stage each is cut in two. Let $\{A_{n,k}\}$ be this scheme. But how can we arrange that for each x, y $\exists n \ni x \in A_{n,k_x}$ and $y \in A_{n,k_y}$ with $k_x \ne k_y$? This isn't automatic, for example we might have partitioned the unit square by vertical strips and so couldn't separate points with the same abscissa.

Probably requires some sort of separability (= countability) assumption.

10/22/86 cf Wilf 1986
BAMS 15, 228-232

2/4/63 Wyman & Moser, Can J. 1957

$$\sum_0^\infty \frac{T_n}{n!} x^n = \exp\left\{x + \frac{x^2}{2}\right\}. \qquad T_n = \# \text{ of } x \in S_n \ni x^2 = e.$$

$$T_n \sim \frac{\left(\frac{n}{e}\right)^{\frac{1}{2}n} e^{\sqrt{n}}}{\sqrt{2}\, e^{\frac{1}{4}}} \left\{1 + \frac{7}{24\sqrt{n}} - \frac{119}{1152 n} + \cdots\right\}$$

pf by $\displaystyle \frac{T_n}{n!} = \frac{1}{2\pi i} \oint \frac{e^{z + \frac{z^2}{2}}\, dz}{z^{n+1}}$

contour : $z = Re^{i\theta}$, $R^2 + R - n = 0$.

7/25/63 A problem of Chow & Robbins.
Let a coin be tossed until a stop rule R ~~applies~~.
$H = +1$, $T = -1$, S_n as usual
Find
$$\max_R E\left(\frac{S_n}{n}\right) \qquad n = n(R).$$

E.g. $R \Rightarrow$

Jan 1964.

Sparre & I refereed (& rejected) a paper of Kingman in which he proved that if $u_0=1, u_1, u_2, \dots$ & $f_1, f_2, \dots$ have the renewal relation & the u_n are completely monotonic then the $f_i \geq 0$ & $\sum f_i \leq 1$.

This is a trivial consequence of Wall Thm 69.2 which gives conclusion f_n compl. mon. too.

Conversely, if the f_n are compl. mon. & $\sum f_i \leq 1$ then $1, u_1, \dots$ is compl. mon. Use Thm 2.1 of Wall's [127].

12 March 64 Problem of Thue Poulsen

$\mathfrak{X}$ fin. dim. B-space

$\mathfrak{M}, \mathfrak{N}$ subspaces.

If the restriction to $\mathfrak{N}$ of the natural map of $\mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ has norm < 1 then

$\dim \mathfrak{N} \leq \dim \mathfrak{M}$.

~~Acta Math. Acad. Sci. Hung. V-7 (52) pg 49 B. Sz.-Nagy prod. strik. with closed [illegible] linear trans.~~
~~53. F. V. Atkinson $f + \sum \lambda^n N_n f = g$~~
~~61 B. Sz.-Nagy on Atkinson~~

10 Feb '65. Hausdorff moment problem from martingale theory.

$\{X_1, X_2, \dots\}$ exchangeable

$\mathcal{B}_n = \mathcal{B}$(symm fcns of $X_1, \dots, X_n$, arb in $X_{n+1}, \dots$)

$\mathcal{B}_n \downarrow \mathcal{B}_\infty$. $\quad E^{\mathcal{B}_n} f(X_1) = \frac{1}{n}\sum_1^n f(X_j)$

$$\downarrow$$

$$E^{\mathcal{B}_\infty} f(X_1)$$

$$E^{\mathcal{B}_\infty} \prod_{j=1}^k f_j(X_j) = \prod_{j=1}^k E^{\mathcal{B}_\infty} f_j(X_1).$$

If $X_j(\Omega) = \{0,1\}$ & $f_j(x) = I_{\{1\}}(x)$, $j \le n$

$I_{\{0\}}(x)$, $n < j \le n+s$

and $\xi = \cancel{E^{\mathcal{B}_\infty}} P^{\mathcal{B}_\infty}(X_1 = 1)$ formula becomes

$P^{\mathcal{B}_\infty}(X_1 = \dots = X_n = 1, X_{n+1} = \dots = X_{n+s} = 0) = \xi^n(1-\xi)^s$.

Then $P(X_1 = \dots = X_n = 1, X_{n+1} = \dots = X_{n+s} = 0) = \int_0^1 t^n(1-t)^s dF_\xi(t)$

and in particular $P(X_1 = \dots = X_n = 1) = \int_0^1 t^n dF_\xi(t)$.

Given $\{\mu_n\}$ with $(-\Delta)^s \mu_n \ge 0$ all s, n & $\mu_0 = 1$

just set $P(X_1 = \dots = X_n = 1, X_{n+1} = \dots = X_{n+s} = 0) = (-1)^s \Delta^s \mu_n$.

Consistency exactly right. Etc.

cf notes of lectures for similar thing for

$f(s) = \int_0^\infty e^{-st}\, dF(t)$. Here the trick is,

$P(X_j > s_j, j = 1, 2, \dots, n) = f(\sum s_j)$; $\xi(s) = \cancel{E} P^{\mathcal{B}_\infty}(X_1 > s)$;

$E(\prod \xi(s_j)) = f(\sum s_j)$. So $E(\xi(s)\xi(t)) = E(\xi(s+t))$ and

$E\,\cancel{E}(\xi(s)\xi(t) - \xi(s+t))^2 = 0$. So ($s, t$ rational) $\xi(s) = e^{-s\eta}$.

Putnam 11/66

$$x_{n+1} = x_n(1-x_n), \quad 0 < x_1 < 1$$

$$\Longrightarrow \quad n x_n \to 1$$

Pf: Put $y_n = \frac{1}{x_n}$. Then $\frac{1}{y_{n+1}} = \frac{1}{y_n}\left(1 - \frac{1}{y_n}\right)$

$$\therefore \quad y_{n+1} = \frac{y_n^2}{y_n - 1} > \frac{y_n^2 - 1}{y_n - 1} = y_n + 1 \to \infty$$

$$\therefore \quad y_{n+1} - y_n = \frac{y_n}{y_n - 1} \to 1$$

$$\therefore \quad \frac{1}{n}\sum_1^n (y_{k+1} - y_k) = \frac{y_{n+1} - y_1}{n} \to 1$$

$$\therefore \quad \frac{y_n}{n} \to 1 \quad \therefore \quad n x_n \to 1$$

Halmos

c_n = # n-tuples of 0's, 1's having no three 0's or 1's in a row.

2–1 correspondence with sequences $b_1, b_2, \dots$ of 1's & 2's summing to n.

ending in 1's $= \frac{1}{2} c_{n-1}$

" " 2's $= \frac{1}{2} c_{n-2}$

total $= \frac{1}{2} c_n$

$$\therefore \quad c_n = c_{n-1} + c_{n-2} : \text{ Fibonacci}$$

with $c_1 = 2$, $c_2 = 4$.

3/67 Bernoulli trials $\begin{cases} 0 & q \\ 1 & p \end{cases}$ $\quad \mathcal{E}: S_{3n} = 2n$

$$u = \sum_0^\infty \binom{3n}{n} p^{2n} q^n = \infty \text{ iff } p = \frac{2}{3}$$

$$= \sum_0^\infty p^{2n} q^n \frac{1}{2\pi i}\oint \frac{(1+z)^{3n}}{z^{n+1}}\,dz = \frac{1}{2\pi i}\oint \frac{dz}{z\left(1 - p^2 q \frac{(1+z)^3}{z}\right)}$$

$$= \frac{1}{2\pi i}\oint \frac{dz}{z - p^2 q (1+z)^3}$$; one pole α near 0, other two large.

$$= \operatorname{Res}_{z=\alpha}\left(z - p^2 q (1+z)^3\right) = \frac{1}{1 - 3p^2 q (1+\alpha)^2}$$

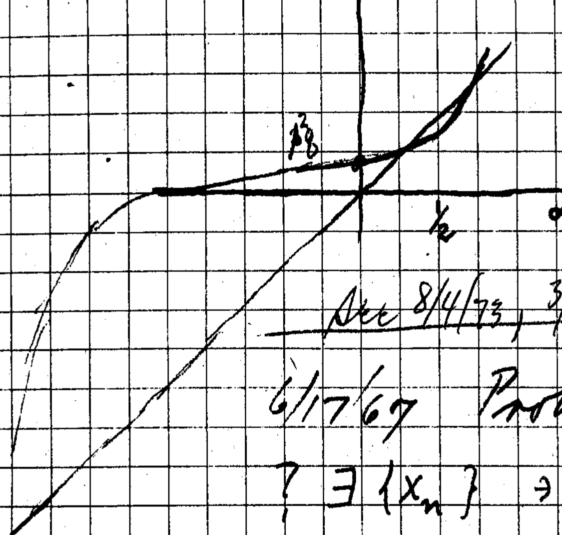
[redacted]

~~Acta Sci. Math XXII #3/1 (6/62) pg 188 Chidey - A theorem on the normality of C.C. Operators.~~

$f = P\{E \text{ at least once}\}$

$= \frac{u-1}{u} = 3p^2q(1+\alpha)^2$

where $\alpha = p^2q(1+\alpha)^3$ $\therefore$ $f = \frac{3\alpha}{1+\alpha} \leq 1, \geq 0$ iff $0 \leq \alpha \leq \frac{1}{2}$



(Double root at $\alpha = \frac{1}{2}$ iff $p = 2q = \frac{2}{3}$)

Indeed, if $\frac{27}{4}p^2q = \sin^2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$, then $f = 4\sin^2\frac{\theta}{3}$.

See 8/4/73, 3/2/77

6/17/67 Problem of Tom Cummins

? $\exists \{x_n\} \ni x_n > 0$ all $n \geq 0$ &

$$x_{n+1} = \frac{x_n - x_0}{x_{n-1}}, \quad n \geq 1, \quad ?$$

We found $x_0 = \frac{1}{4}$, $x_1 = \frac{7}{20}$, giving $x_n = \frac{Q_1(n)}{2Q_2(n)}$ with Q_1, Q_2 monic quadratic polys in n. Later Gustin (?) at Indiana found the following 1-parameter family:

$Q_1(n) = (n+1)(n+6)$
$Q_2(n) = (n+3)(n+4)$

Dec 68

Putnam problem: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx$

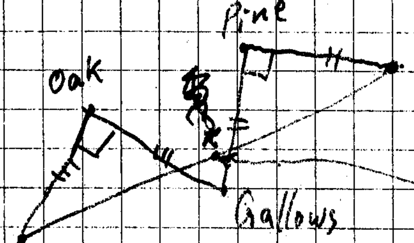
Sketch of pf:

Set $y = x - \frac{1}{x}$ $\quad x = \varphi_1(y)$ or $\varphi_2(y)$

Show $\varphi_1'(y) + \varphi_2'(y) = 1$

```
  ONE
 NINE
  SIX
 NINE
 SOON
```

NINE odd, div. by 11



Treasure at midpoint — ind. of gallows!

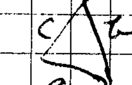
[for $\frac{x + T(x-g) + y - T(y-g)}{2}$ ind. of g]

Summer 67. Somebody's abstract in Ann Math Stat

X, Y ind, $X \sim e^{-x}$, $Y \geq 0$

$\Longrightarrow XY$ inf. div.

May 1969 Hero's formula $A = \frac{1}{2}\{s(s-a)(s-b)(s-c)\}^{1/2}$



Pf: 

$s + t = a$, get h, plug in; result is

$A \doteq \frac{1}{4}\{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\}^{1/2}$, which is Hero's formula. ~~By product: Inequality~~

~~$\frac{\alpha^2+\beta^2+\gamma^2}{2} \leq \alpha\beta + \beta\gamma + \gamma\alpha$~~

~~equality iff $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\gamma}$ or cyclic perm.~~

[redacted]

Investig Read Sci SSR
6-4 K

6/16/69

Evoked by reading Breiman, Ann Math Stat 39 (1968) 1618

Thm: if f cont. + $\nearrow$, $\varphi = f \cdot x$, $t_1 < t_2$, $\varphi(t_1) > \varphi(t_2)$, $a \in (\varphi(t_2), \varphi(t_1)) \Rightarrow \exists\, t \in (t_1, t_2) \ni \varphi(t) = a$.

Pf: Let $A = \{t \in [t_1, t_2] : \varphi(t) \geq a\}$.
$t_1 \in A$, $t_2 \notin A$.

Let $t = \sup A$.

Then $\varphi(t) = f(t)x(t) \geq f(t)x(t-\varepsilon) =$

$= f(t-\varepsilon)x(t-\varepsilon) + (f(t) - f(t-\varepsilon))x(t-\varepsilon)$

$\geq a + \cancel{O(1)}\ o(1)$, ~~unless t = t_1~~,
for we may find arb. small $\varepsilon \ni t - \varepsilon \in A$, if $t \neq t_1$, and $\varphi(t_1) \geq a$ anyhow.
Hence $t \in A$.

Also, $\varphi(t) \leq f(t)x(t+\varepsilon) = f(t+\varepsilon)x(t+\varepsilon)$

$+ (f(t) - f(t+\varepsilon))x(t+\varepsilon) < a + o(1)$ if $t \neq t_2$,
so that $\varphi(t) \leq a$; and $\varphi(t_2) < a$ anyhow.

This proves that $\varphi(t) = a$.

A thm of Lighthill (Gen'd fncs, chap V).

If $\sum_{n=1}^{\infty} x_n a_{m,n}$ is conv. for any $x_n = O(n)$,
& if $\exists \lim_{m \to \infty} \sum$ for all such x_n then $\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{m,n} =$
$= \sum_n \lim_m a_{m,n}$.

(my) pf: Set $\frac{x_n}{n} = y_n$, $n a_{m,n} = b_{m,n}$.

$\sum_n y_n b_{m,n}$ conv., all $\{y_n\} \in (m)$

$\therefore \sum_n |b_{m,n}| < \infty$. $f_m(y) = \sum y_n b_{m,n}$, $\|f_m\| = \sum |b_{m,n}|$.

$\exists \lim_m f_m(y) \Rightarrow \sup_m \|f_m\| < \infty$. (over)

Let $y_n = \{1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\}$

Then $y_n \to y = \{1, \frac{1}{2}, \dots \}$

$$f_m(y_n) \to f_m(y), \quad n \to \infty$$

$$\to f(y_n), \quad m \to \infty$$

and the first is uniform in m, for

$$|f_m(y_n) - f_m(y)| \leq \sup_m \|f_m\| \cdot \|y_n - y\| =$$

$$= \frac{1}{n+1} \sup_m \|f_m\|$$

QED

6/19/69 Woodroofe's gloss on Chung Erdös (Ann Math 1947)

C-E. For Bernoulli trials

$$P\{|\tfrac{S_n}{n} - p| < \alpha_n \text{ i.o.}\} = 0 \text{ or } 1 \text{ acc. as}$$

$\sum \sqrt{n}\,\alpha_n < \infty$ or $= \infty$; the "0" case holds for every quadratic irrational p, and for almost every p.

W. Let p be unif distd on $(0,1)$. Then $\frac{S_n}{n} - p$ has density

$$\sum_{k=0}^{n} \binom{n}{k} \left(x + \frac{k}{n}\right)^k \left(1 - x - \frac{k}{n}\right)^{n-k} \leq C\sqrt{n}$$

C an absolute const.

Hence $P\{|\frac{S_n}{n} - p| < \alpha_n\} \leq 2C\sqrt{n}\,\alpha_n$

The B.C. lemma implies that

$$P\{|\tfrac{S_n}{n} - p| < \alpha_n \text{ i.o.}\} = 0 \text{ in}$$

the space of p & the Bernoulli trials — hence for almost every p and ordinary Bernoulli trials.

[redacted]

Amer. J. Math. 74 (10/52) pp 920 G.D. Mostow ~~[illegible]~~

8/12/69 (For dealing with symmetric functions of positive integers on a computer.)

Problem (send to Monthly).

Let $S = \{s_k\}_{k=1}^{n}$ be the set of all 3-digit ~~integers~~ base-10 integers $s_k = (abc)_{10}$ which have the property $a \geq b \geq c \geq 1$; $s_k < s_{k+1}$.

?) (a) Determine n. (b) Express k as a function of a, b, c.

$$(b) \quad k = \binom{c}{1} - \binom{b}{1} + \binom{b+1}{2} - \binom{a+1}{2} + \binom{a+2}{3}$$

$$\therefore (a) \quad n = \binom{11}{3}.$$

9/3/69 $X \sim N(0,1) \Rightarrow \frac{1}{X^2}$ stable of index ½

Pf: X, Y ind. $N(0,1) \Rightarrow X \sim R\cos\Theta,\ Y \sim R\sin\Theta$ with $R^2 \sim \chi^2_{(2)}$, Θ unif on $[0, 2\pi)$. Then

$$\frac{1}{X^2} + \frac{1}{Y^2} = \frac{1}{R^2\cos^2\Theta\sin^2\Theta} = \frac{4}{R^2\sin^2 2\Theta} \sim \frac{4}{R^2\sin^2\Theta} \sim \frac{4}{Y^2}$$

QED.

It remains to identify a constant of proportionality. This is easy, since $E((1/X^2)^{-1}) = E(X^2) = 1$.

1/1970 Hypothesis testing prob. Urns $U_n = \{k_n W, k_n B\}$, draw 2, observe all. H_0: with repl; H_1: w/o repl.

N&S for $H_0 \perp H_1$: $\sum \frac{1}{k_n^2} = \infty$. (From Kakutani's thm on product measures, $\sum (p_n - q_n)^2 = \infty$ for singularity when all two-point spaces; cf Chatterji & Ma626 notes for martingale treatment. ZW 1964 3 184-192

1/23/1970 Herman Rubin says $\frac{1}{I_0(\sqrt{t})}$ is Laplace transform of ~~measure~~ prob distn on $[0,\infty)$. Easy pf by $I_0(\sqrt{t}) = \prod_{k=1}^{\infty}\left(1+\frac{t}{a_k^2}\right)$ with $J_0(a_k)=0$. But what natural r.v. did Herman have in mind?

Feb 1970 Monthly. Kesten problem: if $\{X_n\}$ iid & $E|X_n| = \infty$ then $\overline{\lim}\, |X_{n+1}/S_n| = \infty$ a.s.

Feb 1970. Reading Kesten's Memoir (No. 93) one learns incidentally that if $\nu(\mathbb{R}) = \infty$ (ν = Lévy measure for inf. div. r.v. X) then X has no atoms. References to Esseen & Döblin yield uneasy pfs as byproducts. Should be easy from $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\, dt$ = sum of squares of atoms. In inf div case we have $\varphi(t) = \exp\{iat - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty}\{e^{itx} - 1 - \frac{itx}{1+x^2}\}\,\nu(dx)$; $\sigma > 0 \Rightarrow$ no atoms, so assume $\sigma = 0$. Then $|\varphi(t)|^2 = \exp 2\mathrm{Re}\{\cdots\} = \exp 2\int_{-\infty}^{\infty}\{\cos tx - 1\}\,\nu(dx)$. So changing notation slightly we want to prove
$$\frac{1}{T}\int_0^T e^{-\int_0^\infty (1-\cos tx)\,\nu(dx)}\, dt \longrightarrow 0 \text{ as } T \to \infty, \text{ when } \nu(\mathbb{R}^+) = \infty.$$

[redacted]

Trans.

73-2. (9/52)

pg 293 JL Snell - Martingales.

321 Donoghue + Smith - Locally Convex Spaces.

Ann. Sci. L'Ecole Norm. Sup.

69-3 ✓ (7-9/52)

Annali di Matematica

33 (52)

Annals of Math Stat

23-4 (12/52)

Annals

56-3 (11/52)

pg 460 Kaplansky Algebras of Type I

494 Kadison Schwarz Inequality and Alg. Invariants for Operator Algebras.

Arch. der Math.

III-3 (10/52)

pg 198 Kneser. Konvexe Räume.

Arkiv för Mat.

2-1 (52)

pg 83 Nelson. Ideal Structure of group algebras.

How about $f(t)=\int_0^\infty (1-\cos tx)\nu(dx) \to \infty$?

Joe Ullman shows: if $V(x) := \nu([x,\infty))$ is convex then trivial. In fact

$$f(t) = -V(x)(1-\cos tx)\Big]_0^\infty + t\int_0^\infty \sin tx \; V(x)\,dx$$

$$= t\int_0^\infty \sin tx \; V(x)\,dx$$

$$= \int_0^\infty \sin x \; V\left(\frac{x}{t}\right) dx$$

$$= \int_0^\pi \sin x \; G_t(x)\,dx$$

with $G_t(x) = \sum_0^\infty (-1)^k \, V\left(\frac{x+k\pi}{t}\right)$

$$= V\left(\frac{x}{t}\right) - G_t(x+\pi)$$

Hence $\quad G_t(x) + G_t(x+\pi) = \nu(\frac{x}{t})$.

But
$$G_t(x) - G_t(x+\pi) \geq 0 \text{ follows easily}$$
from convexity of $\nu(x)$.

Therefore $\quad G_t(x) \geq \frac{1}{2}\nu(\frac{x}{t})$

$$f(t) = \int_0^\pi G_t(x) \sin x \, dx \geq \int_0^\pi \frac{1}{2}\nu(\frac{x}{t}) \sin x \, dx$$
$$\geq \frac{1}{2}\nu(\frac{\pi}{t}) \int_0^\pi \sin x \, dx = \nu(\frac{\pi}{t}) \to \infty$$

as $t \to 0$.

In the general case I can prove that for any seq. $\{t_n\} \to \infty$ there exists a subseq. (still write $\{t_n\}$) $\ni$ $A_n :=$ ave f on $[t_n, t_{n+1}] \to \infty$. In fact ave f on $[t, t'] = \int_0^\infty (1 - \frac{\sin t'x - \sin tx}{x(t'-t)}) \nu(dx)$

$$\geq \int_a^{a'} (1 - \frac{2}{x(t'-t)}) \nu(dx)$$
$$\geq \nu([a, a'])(1 - \frac{2}{a(t'-t)})$$
$$= \nu([\tfrac{3}{t'}, \tfrac{3}{t}])(1 - \frac{2t'}{3(t'-t)}) \quad \text{if } a = \frac{3}{t'}, a' = \frac{3}{t}$$

So choose subseq inductively; t_1 arb., ...
$t_{n+1} \geq 4t_n \ni \nu([\frac{3}{t_{n+1}}, \frac{3}{t_n}]) \geq 9n$.

The lower bound is then $\geq 9n(1 - \frac{24t_n}{3 \cdot 3t_n})$

Hence $A_n \geq n$. $\quad = 9n(1 - \frac{8}{9})$

$\quad = n$.

[redacted]

In fact, ave f on $[t,t']$ $\geq \int_a^\infty \left(1 - \frac{2}{x(t'-t)}\right) \nu(dx)$

$$\geq \nu(a)\left(1 - \frac{2}{a(t'-t)}\right)$$

$$= \nu\left(\frac{3}{t'}\right)\left(1 - \frac{2t'}{3(t'-t)}\right)$$

$$\geq \nu\left(\frac{3}{t'}\right)\frac{1}{9} \quad \text{if } t' \geq 4t.$$

So enough to have $\nu\left(\frac{3}{t_{n+1}}\right) \geq 9n$.

In short:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt = \infty .$$

Much simpler proof:

$$\frac{1}{T}\int_0^T f(t)\,dt = \int_0^\infty \left(1 - \frac{\sin xT}{xT}\right)\nu(dx)$$

$$\underline{\lim}\ \frac{1}{T}\int_0^T f(t)\,dt \geq \int_0^\infty \underline{\lim}_{T\to\infty}\left(1 - \frac{\sin xT}{xT}\right)\nu(dx)$$

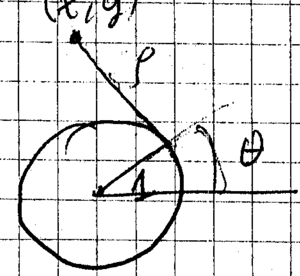
$$= \int_0^\infty 1\ \nu(dx) = \nu(\mathbb{R}) = \infty .$$

Unfortunately

$$e^{-\frac{1}{T}\int_0^T f(t)\,dt} \leq \frac{1}{T}\int_0^T e^{-f(t)}\,dt$$

LHS $\to 0$ doesn't imply RHS $\to 0$.

5/26/70 Nice coordinate system.



$$x = \cos\theta - \rho\sin\theta$$
$$y = \sin\theta + \rho\cos\theta$$

For region $x^2+y^2>1$

$$\left|J\left(\frac{x,y}{\rho,\theta}\right)\right| = \rho$$

7/25/70 Shuster in Can Math Bull $\underline{13}$, 1970

$(\Omega, \mathcal{A}, P)$, $\{A_n\}$.

$$P\{A_n \text{ i.o.}\} = 1 - \sup\{P(A): A\in\mathcal{A}, \sum_1^\infty P(A\cap A_n)<\infty\}$$

9/19/70 A new(?) pf of divergence of $\sum \frac{1}{n}$.

Suppose conv, $=S$. Then also $T=\sum\frac{1}{n^2}$ conv, $<S$.

And $\sum \frac{1}{n} = \sum_1^\infty \sum_{(k-1)^2+1}^{k^2} \frac{1}{n} > \sum_1^\infty \frac{2k-1}{k^2} = 2S-T$

$$S > 2S-T \quad \therefore \quad T > S, \text{ contra.!}$$

9/29/70 Special case of Kesten's problem: AMM April 1970? March?

(If $E|X_n| = \infty$ then $\overline{\lim} \left|\frac{X_{n+1}}{S_n}\right| = \infty$ a.s.; here $\{X_n\}$ i i d.)

Suppose X_n pos stable, index α. So $\alpha<1$.

Consider $Y(t)$, corresponding process.

[redacted]

First note, in general case, can assume $X_n > 0$; then let $A_n = \{X_{n+1} > cS_n\}$, for some fixed c. By Paul Levy's BC lemma, $\{A_n \text{ i.o.}\} = \{\sum P(A_n \mid \mathcal{A}_{n-1}) = \infty\}$ a.s. where $\mathcal{A}_{n-1} \supseteq \sigma(A_1, \dots, A_{n-1})$. Here take $\mathcal{A}_{n-1} = \sigma(X_1, \dots, X_n)$. Then $\{A_n \text{ i.o.}\} = \{\sum P(A_n \mid S_n) = \infty\}$ a.s. $= \{\sum (1 - F(cS_n)) = \infty\}$ a.s. (F = df of X_n)

Now for stables $1 - F(x) \sim x^{-\alpha}$, $x \to \infty$.

So want $\sum S_n^{-\alpha} = \infty$ a.s. Equivalent is $\int_1^\infty \frac{dt}{Y(t)^\alpha} = \infty$ a.s. By scaling property $\int_1^\infty \frac{dt}{[\lambda^{-1/\alpha} Y(\lambda t)]^\alpha} = \infty$ has same prob. But $\int_1^\infty \frac{\lambda\, dt}{Y(\lambda t)^\alpha} = \int_\lambda^\infty \frac{dt}{Y(t)^\alpha}$. So $\int_1^\infty$ and $\int_\lambda^\infty$ would need same distn, which cannot be eg even for $\lambda = 2$.

QED

$\{<\infty\}$

9/19/70 A curious inequality: $N(a_N - g_N) \uparrow$. Here a_N, g_N are arith, resp. geom., means of any ~~pos~~ nonneg seq. $\{x_n\}$. Easy to prove. A paper in J Lon MS in 67 gives NeS 42(1967) 712-718, W.N. Everitt for $\sup_N N(a_N - g_N) < \infty$: $\sum x_n < \infty$ or x_n pos & $\exists\, \xi$ pos $\ni \sum (x_n - \xi)^2 < \infty$

il 1970 arch?

[redacted]

First note, in general case, can assume $X_n > 0$; then let $A_n = \{X_{n+1} > cS_n\}$, for some fixed c. By Paul Levy's BC lemma,
$\{A_n \text{ i.o.}\} = \{\sum P(A_n \mid \mathcal{O}_{n-1}) = \infty\}$ a.s.
where $\mathcal{O}_{n-1} \supseteq \sigma(A_1, \dots, A_{n-1})$. Here take $\mathcal{O}_{n-1} = \sigma(X_1, \dots, X_n)$. Then
$\{A_n \text{ i.o.}\} = \{\sum P(A_n \mid S_n) = \infty\}$ a.s.
$= \{\sum (1 - F(cS_n)) = \infty\}$ a.s. (F = df of X_n)

Now for stables $1 - F(x) \sim x^{-\alpha}$, $x \to \infty$.

So want $\sum S_n^{-\alpha} = \infty$ a.s. Equivalent is $\int_1^\infty \frac{dt}{Y(t)^\alpha} = \infty$ a.s. By scaling property $\int_1^\infty \frac{dt}{[\lambda^{-1/\alpha} Y(\lambda t)]^\alpha} = \infty$ has

)<∞}

same prob. But $\int_1^\infty \frac{\lambda\, dt}{Y(\lambda t)^\alpha} =$

; $= \int_\lambda^\infty \frac{dt}{Y(t)^\alpha}$. So $\int_1^\infty$ and $\int_\lambda^\infty$ would need same distn, which cannot be eg even for $\lambda = 2$.

QED

---

il 1970 arch?

9/19/70 A curious inequality: $N(a_N - g_N) \uparrow$. Here a_N, g_N are arith, resp. geom., means of any ~~pos~~ nonneg seq $\{x_n\}$. Easy to prove. A paper in J Lon MS in 67 42(1967) 712-718, W.N. Everitt gives N&S

/, for $\sup_N N(a_N - g_N) < \infty$: $\sum x_n < \infty$ or x_n pos & $\exists \xi$ pos $\ni \sum (x_n - \xi)^2 < \infty$

12/17/70 $L(x)$ slowly varying if L msbl, > 0, & $L(cx)/L(x) \to 1$, $x \to \infty$, each $c > 0$.

Thm: convergence is uniform on compact subsets of $(0, \infty)$.
Cont case due to Karamata
Msbl to De Bruijn et al 1940's

New proof (J&W). Defn: $f(x) = \log L(e^x)$;

$$f(x+t) - f(x) \to 0, \ (x \to \infty), \text{ each } t.$$

Let $g_n(t) = \sup_{x \geq n} |f(x+t) - f(x)|$

Then
$$g_n(t) \to 0 \quad \text{each } t.$$

By Egorov, conv. unif. on some set $A \subset [0,1]$ of pos. Lebesgue measure.

For $t, u \leq 1, \geq 0$,
$$g_n(t-u) \leq g_n(t) + g_{n-1}(u)$$
since
$$|f(x+t-u) - f(x)| \leq |f(x+t) - f(x)| + |f(x+t-u+u) - f(x+t-u)|$$
and $x+t-u \geq n-1$ if $x \geq n$.

Hence conv. unif. on $A - A$.

By Steinhaus' Thm this contains an interval. Hence conv. unif. on some interval, hence on all.

Wrong! (By Egorov) But g_n not msbl (OK for f Borel but not Lebesgue)

Cor: $$x^a L(x) \to \begin{cases} \infty; & a > 0 \\ 0; & a < 0 \end{cases}$$

Pf: Enough to show $x + f(x) \to \infty$.

First, $n + f(n) = 1 + f(1) + \sum_{1}^{n-1} \{1 + f(k+1) - f(k)\}$
$\to \infty$ since $f(k+1) - f(k) \to 0$.

Then $x + f(x) = n + f(n) + x - n + f(x) - f(n)$;

for $n \leq x < n+1$, $x - n$ is bdd.

and $f(x) - f(n) \to 0$ as $x \to \infty$ by uniformity! Hence $x + f(x) \to \infty$.

[redacted]

Solution of February problem, thanks to hint from Rosenblatt & Blum, Pac J. 9 (1959) 1–7.

Let $N(x) = \nu([x, \infty))$. We have

$$J \equiv \int_0^\infty (1-\cos tx)\,\nu(dx) \geq \int_\alpha^\infty (1-\cos tx)\,\nu(dx)$$

$$= N(\alpha) \int_\alpha^\infty (1-\cos tx)\,\frac{\nu(dx)}{N(\alpha)}. \quad \text{Then}$$

$$e^{-J} \leq e^{-N(\alpha)\int_\alpha^\infty (1-\cos tx)\frac{\nu(dx)}{N(\alpha)}} \leq$$

$$\leq \int_\alpha^\infty \frac{\nu(dx)}{N(\alpha)}\, e^{-N(\alpha)(1-\cos tx)}, \quad \text{by Jensen}$$

Now average w/r t over $[0, T]$. Letting $T \to \infty$ get upper bound

$$\int_\alpha^\infty \frac{\nu(dx)}{N(\alpha)}\, \frac{1}{2\pi}\int_0^{2\pi} e^{-N(\alpha)(1-\cos\theta)}\,d\theta$$

$$\leq \int_\alpha^\infty \frac{\nu(dx)}{N(\alpha)} \left\{ 2\varphi + e^{-N(\alpha)(1-\cos\varphi)} \right\}$$

$$= 2\varphi + e^{-N(\alpha)(1-\cos\varphi)}, \quad \text{since}$$

$1-\cos\theta \geq 0$ on $[0,\varphi] \cup [2\pi-\varphi, 2\pi]$

" $\geq 1-\cos\varphi$ on $[\varphi, 2\pi-\varphi]$

Let $\alpha \downarrow 0$. $N(\alpha) \uparrow \infty$. $\therefore$ $\overline{\lim} \leq 2\varphi$.

Since φ arb, $\overline{\lim} = 0$. QED.

From Dec 1970 Sci Am (Martin Gardner)

Examples due to Bradley Efron of ind

$X_i \ (i=1,2,3,4) \ni P(X_{i+1} < X_i) = 2/3 \quad (i+1 \bmod 4)$

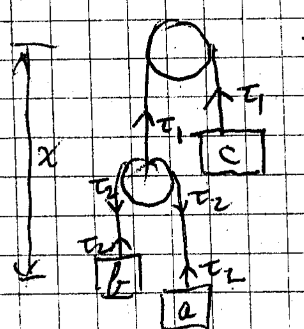
best possible

	X_1	p	X_2	p	X_3	p	X_4	p
Ex 1.	0	1/3	3	1	2	2/3	1	1/2
	4	2/3			6	1/3	5	1/2
Ex 2.	2	1/6	0	1/6	5	1/3	4	2/3
	3	1/3	1	1/6	6	2/3	12	1/3
	9	1/6	7	1/6				
	10	1/6	8	1/2				
	11	1/6						

Examples said to exist for case 3 with .618 in place of 2/3. For n asymptotic to 3/4.

Apr 1971

On Pulleys



$$V = -g(bx + cy) + ga(x+2y)$$

$$T = \frac{a}{2}(\dot{x}+2\dot{y})^2 + \frac{b}{2}\dot{x}^2 + \frac{c}{2}\dot{y}^2$$

$$L = T - V$$

Lengths l_1, l_2. Coord. of a is $l_1 - y + l_2 - (x - [l_1 - y])$
$= \text{const} - x - 2y$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0 \\[2ex] \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{y}} - \dfrac{\partial L}{\partial y} = 0 \end{cases}$$

gives
$$\begin{cases} (a+b)\ddot{x} + 2a\ddot{y} = -(a-b)g \\ 2a\ddot{x} + (4a+c)\ddot{y} = -(2a-c)g \end{cases}$$

so that
$$\ddot{x} = -\frac{3ac + 4ab + bc}{\Delta} g$$

$$\ddot{y} = \frac{ac - 4ab + bc}{\Delta} g$$

where $\Delta = 4ab + ac + bc$

Via tensions instead: $\tau_1 = 2\tau_2$

(4) $\tau_2 - ga = F_a = a(\ddot{x} + 2\ddot{y})$

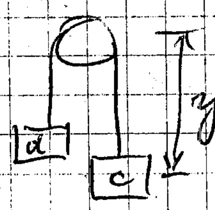
$\tau_2 - gb = F_b = -b\ddot{x}$

$\tau_1 - gc = F_c = -c\ddot{y}$

Eliminating τ_i gives same equations. Also

p 2/3, 3/4. can evaluate $\tau_2 = \dfrac{4abc}{\Delta}$.

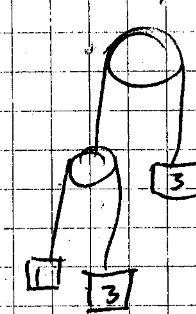
Also note that the motion of c is equivalent to that in the system



with $d = \dfrac{4ab}{a+b}$

A special case: $\ddot{y} = 0$ iff $c = \dfrac{4ab}{a+b}$.

Ex: $(a, b, c) = (1, 3, 3)$!



← doesn't move!

30.6.71 An easy lemma drawn from Kesten's solution, AMM, Mar. 1971, p 306, "(4)"

Three colors, probs p, q, r $(p+q+r=1)$

n trials.

$m \le n$. A: color 1 $\ge$ once in first m trials

B: color 2 $\ge$ " .

$$\Rightarrow P(A \cap B) \le P(A)P(B)$$

Pf: $P(A \cap B) = 1 - (1-p)^m - (1-q)^n + (1-p-q)^m (1-q)^{n-m}$

$\le 1 - (1-p)^m - (1-q)^n + (1-p)^m (1-q)^n$

Because $1 - p - q \le (1-p)(1-q)$.

QED

2.7.71 See nice "Note on Order statistics" by A.G. Konheim, p. 524 AMM 78 (1971, May).

$\{X_n\}$ ind., distd F; $E(|X|) < \infty$; $X_{n,n} = \max_{1 \le i \le n} X_i$

$\Rightarrow \{E(X_{n,n})\}$ determines F.

5 Sept 71 In August Murali Rao asked:

if $X_n \downarrow 0$ a.s., $X_n \in L_1$, $\mathcal{F}_n$ σ-fields (arb.), must $E(X_n \mid \mathcal{F}_n) \to 0$ a.s.?

No – Example (partially inspired by conversation with Art Pittenger)

Let $\{Y_n\}$ ind $\{0,1\}$, $P\{Y_n = 1\} = 1/(n+1)$, $n = 1, \dots$; $C_n = \{Y_n = 1\}$, $A_n = C_1^c C_2^c \cdots C_n^c$, $B_n = A_n \cup C_n$.

Then $P(A_n) = \frac{1}{2}\frac{2}{3}\cdots\frac{n}{n+1} = P(C_n)$,

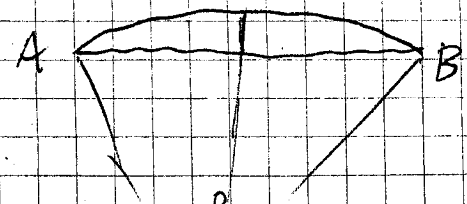
so $P(A_n \mid B_n) = 1/2$. Since $P\{C_n \text{ i.o.}\} = 1$ and $B_n \supset C_n$, also $P\{B_n \text{ i.o.}\} = 1$.

Let $X_n = I_{A_n}$, $\mathcal{F}_n = \{\phi, B_n, B_n^c, \Omega\}$.

Then $E(X_n \mid \mathcal{F}_n) = \frac{1}{2} I_{B_n}$,

so $\overline{\lim}\, E(X_n \mid \mathcal{F}_n) = \frac{1}{2}$ on $\{B_n \text{ i.o.}\}$, i.e. a.s.

10/1/71 Cleve Moler: 1 ft added to 1 mi RR track, forms circular arc bow. How high in middle?



$OA = OB = r$

$\angle BOA = 2\theta$

$\overline{AB} = 2r\sin\theta = 1$

$\widehat{AB} = 2r\theta = 1 + \varepsilon$

Find $r(1 - \cos\theta)$

[redacted]

$$\frac{\theta}{\sin\theta} = 1+\varepsilon \qquad r = \frac{1+\varepsilon}{2\theta}$$

Find $\frac{1+\varepsilon}{2\theta}(1-\cos\theta)$ given $\frac{\theta}{\sin\theta} = 1+\varepsilon$

Find $\frac{1-\cos\theta}{2\sin\theta}$ if $\frac{\theta}{\sin\theta} = 1+\varepsilon$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$1-\cos\theta = 2(1-\cos^2\theta/2) = 2\sin^2\theta/2$

$2\sin\theta = 4\sin\theta/2\,\cos\theta/2$

So: Find $\frac{\tan\theta/2}{2}$ if $\theta/\sin\theta = 1+c$

$$1 = (1+\varepsilon)\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots\right)$$

$$\cancel{=1+\theta} \quad \frac{1}{1+\varepsilon} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots$$

$$1 - \varepsilon + \varepsilon^2 - \cdots$$

$$\theta^2 = 6(\varepsilon - \varepsilon^2 + \cdots)$$

$$\theta = \sqrt{6\varepsilon}\left(1 - \frac{\varepsilon}{6} + \cdots\right)^{1/2}$$

$$\frac{\tan\theta/2}{\cancel{\theta}2} = \frac{\theta}{4} + \cdots \quad ; \quad \frac{\sqrt{6}}{4} = .6^{+}$$

$\varepsilon = \frac{1}{5280}$ $\quad \sqrt{\varepsilon} = 73^{1/2}$

12/5/71 A neat Putnam problem.

Show: If n^c is integer for all n then c is positive integer or 0.

Pf: Suppose $k < c < k+1$ $\quad k$ integer.

Use $\Delta^{\ell} f(x) = \int_0^1 \cdots \int_0^1 f(x+t_1+\cdots+t_\ell)\,dt_1\cdots dt_\ell = f^{(\ell)}(x+\theta\ell)$

To conclude $\Delta^{k+1} n^c = \frac{c(c-1)\cdots(c-k)}{(n+\theta(k+1))^{k+1-c}}$, $0<\theta<1$

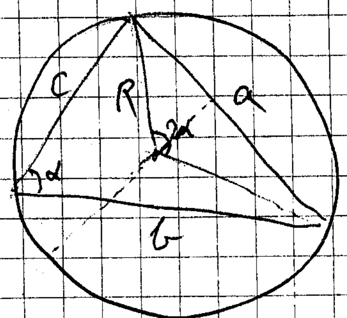
which is an integer, yet a positive no. <1 for large n.

Contra!

Another Putnam Problem.

If a, b, c are the sides of a triangle having lattice points as vertices ~~then~~ and R is the radius of the circumscribing circle then $abc \geq 2R$

Pf:



$$R \sin\alpha = a/2$$

$$a = 2R\sin\alpha$$
$$b = 2R\sin\beta$$
$$c = 2R\sin\gamma$$

$$\text{Area} = \tfrac{1}{2} b\, c \sin\alpha$$

$$R\cdot\text{Area} = \tfrac{1}{2} bc\, a/2$$

$$\therefore\ abc = 4R\cdot\text{area}$$

But $\text{area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\,\text{integer}$ $(= 0 \iff R = \infty)$ $\therefore\ \underline{abc \geq 2R}$

16 Dec 1971. Holsztynski says he proved this at age 17.

Let $0 \leq x \leq 1$, $x = .x_1 x_2 \ldots$ $x_n = 0,1$ its binary representation, $s(x)$ = limiting frequency of 1's in x if such exists, undefined otherwise.

Let f be any pointwise limit of cont fncns on $[0,1]$ to $[0,1]$.

Consider the equation $s(x) = f(x)$.

The claim is that in any

[redacted]

interval of [0,1] the equation has a continuum number of solutions!

d $\quad$ <u>30 Dec 71.</u> An elementary monthly problem.
(E2328 pg 1138 Dec 71)

$$G \text{ a semigp } \ni \forall a\, \exists!\, a^* \ni a a^* a = a.$$
$$\Longrightarrow$$
$$G \text{ is a gp.}$$

Pf: (Jack McLaughlin).

Clearly aa^* is idempotent. We let e be any idempotent and show first that eGe is a gp with e as identity and $*$ as inverse. Namely, if $x = exe$ then $x = \cancel{x^* x x^*} x \cdot x x^* x = x e x^* e x$
$= xe = xe$ and so $ex^*e = x^*$; then $xx^* = xx^*xx^*xx^*$ shows that $(xx^*)^* = xx^*$, and $xx^* = xx^*xx^*$
$= x^*x^*exx^*$ shows that $xx^* = e$, so $x^* = x^{-1}$. $= xx^*$

Next, let f be another idempotent, and let $(fef)^{-1} = fxf$. Then

$$feffxf = f \quad ; \quad f^3 = f \therefore f = f^*$$
$$fefxf = f$$
$$f^* efxf^* = f^* \quad \therefore \quad efx = f \therefore f = ef$$

0,1 $\quad$ By symmetry then $\cancel{e}fe = e$. ~~But also~~ $\cancel{fxffef = f, \ xfe = f}$, So $efe = e \therefore f = e^* = e^{-1} = e$.

So there is only one idempotent, 1.

Hence $a = \cancel{a1e}\ aa^*a = a1 = 1a$ since a^*a and aa^* are idempotent.

Thus 1 is identity, $1G1 = G$ is gp.

1/10/72. Conversation with Shields.

$$0 \le f(x) \le 1, \quad \text{ess sup } f = 1$$

$$\Rightarrow \quad r_n = \frac{\int_0^1 f(x)^{n+1}\,dx}{\int_0^1 f(x)^n\,dx} \to 1, \quad n \to \infty.$$

Pf: We show $r_n \nearrow$. Easy, for

$$r_n - r_{n-1} = \frac{\int_0^1\int_0^1 \left(f(x)^{n+1} f(y)^{n-1} - f(x)^n f(y)^n\right) dy\,dx}{\int_0^1\int_0^1 f(x)^n f(y)^{n-1}\,dy\,dx}$$

Numerator equals

$$\frac{1}{2}\int_0^1\int_0^1 (f(x)f(y))^{n-1}(f(x)-f(y))^2\,dy\,dx \ \ge 0$$

and clearly denominator is 0.

Clearly $\lim r_n \le 1$. If $\lim r_n < 1$ then $\sum_0^\infty \int_0^1 t^n f^n(x)\,dx$ converges for some $t > 1$. But it is $\int_0^1 \frac{dx}{1 - tf(x)}$, which cannot do so by ess sup condition.

$A = (a_{ij})$, $1 \le i, j < \infty$ $\quad \sum_i |a_{ij}|^2, \sum_j |a_{ij}|^2 < \infty$

Each j, i resp. For what $f \in (l_2)$ is it true that $(Af, f) = (f, A^T f)$?

($f = 1, \frac{1}{2}, \frac{1}{3}, \dots$

$$A = \begin{pmatrix} 1^2 & 1\cdot 2 & 0 & \cdots \\ 0 & 2^2 & 2\cdot 3 & \cdots \\ 0 & 0 & 3^2 & \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

no!)

[redacted]

1/19/72 Renyi problem (from Lukacs)

X $N(0,1)$ $\Rightarrow$ X^2 is ID. & X symmetric & X ID

Is the converse true?

(Role of symmetry not clear. Conceivably unsymmetric X could be ID but $\underset{50-50}{\pm}|X|$ not. Even for const it is so!)

Probably useless remark: $E(e^{-sY^2}) = \psi(s)$
$E(e^{i\theta Y}) = \varphi(\theta)$

$$\Longrightarrow \psi(s) = \int_0^\infty \mathrm{Re}\,\varphi(\theta)\, \frac{e^{-\theta^2/4s}}{\sqrt{\pi s}}\, d\theta$$

Feb 72 Two problems from Don Darling.

1) Find characteristic roots of matrix

$$a_{ij} = \frac{\sqrt{2j}}{i\sqrt{j}}, \quad i,j = 1,2,\cdots n$$

ans: reciprocals of zeros of n^{th} Laguerre poly.

I get determinant $\det(I-\lambda A)$ down to

$$\begin{vmatrix} \lambda-1 & \lambda & \lambda & \cdots & \lambda \\ 1 & \lambda-2 & \lambda & \cdots & \lambda \\ 0 & 2 & \lambda-3 & \cdots & \lambda \\ 0 & 0 & 3 & \cdots & \lambda-n \end{vmatrix}$$

which fairly easily is seen to satisfy recurrence $L_n(\lambda) = (\lambda-n+1)L_{n-1}(\lambda) - n^2 L_{n-2}(\lambda)$ (approximate recollection).

2) Find k minimizing $\sup\{e^{-x} + e^{-k/x} : x \geq 0\}$

Soln seems to be $k = (\log 2)^2$, and one must show $2^{-x} + 2^{-1/x} \leq 1$, $x > 0$.

Comes down to $\log(1+x^2) \leq x \log 2$ on $[0,1]$, nice convexity result.

21 Mar 72

a 3 min talk in math club.

Barnes

A triangle has integer sides

$\Longleftrightarrow$ ~~$\Leftarrow$~~

Its vertices are integer lattice points (an arrangement is possible)

$\Longleftrightarrow$

rational lattice points

$\Longleftrightarrow$ area is rational

Consecutive integers which make triangles:

3 4 5

13 14 15

51 52 53

193 194 195

$3n^2(n^2-4) = m^2$

put $m = 3l$

& $n = 3r+2$

get $r(3r+4) = l^2$

Isomorphism of tic-tac-toe with game: select 3 integers from 1, 2, …, 9, ~~10~~ which add to 15

— isomorphism ~~with~~ via magic square

~~1 6 8~~   ~~1 8 6~~

~~5~~   ~~5~~

~~3~~   ~~4 2 9~~

Where does next letter go in

A   E F   H I

   B C D   G   J   ?

[redacted]

ans supposed to be <u>above</u>, since K is formed with segments. But my answer was <u>below</u>, because initial letter of <u>name</u> of K (kay) is a <u>consonant</u>!

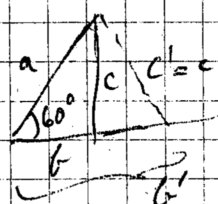
---

9 Apr 72 $\quad a^2 - ab + b^2 = c^2 \quad (a,b)=1, \quad a>b$

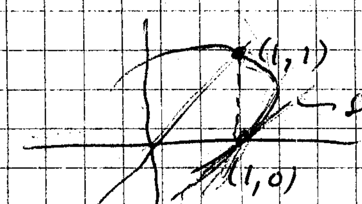
$x^2 - xy + y^2 = 1$

$y = m(x-1) \qquad m = p/q \qquad (p,q)=1.$

Solution:



$$x = \frac{m^2-1}{m^2-m+1}$$



$$y = \frac{m^2-2m}{m^2-m+1}$$

Hence $(p,q)=1$, $p>2q$,

$$\left.\begin{array}{l} a \; = p^2 - q^2 \\ b \; = p^2 - 2pq \\ c \; = p^2 - pq + q^2 \\ b' = 2pq - q^2 \end{array}\right\} \text{divide by 3 iff } 3 \mid (p+q)$$

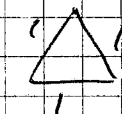
from reasoning thus:

$a = (p-q)(p+q)$

$b = p(p-2q)$

~~Extra~~ So if $\pi \mid a$, $\pi \mid b$ then $\pi \mid (p+q)$, $\pi \mid (p-2q)$ $\therefore \pi = 3$

Note: $p=1$, $q=0$ gives



<u>Examples</u>

p	q	a	b	c	b'
5	1	8	5	7	3
4	1	15	8	13	7
3	1	8	3	7	5
5	2	21	5	19	16
7	2	15	7	13	8
7	3	40	7	37	33

$b \to b'$ accounted for by

$P = 2p - q$

$Q = p - 2q$

$\therefore P^2 - Q^2 = 3(p^2 - q^2)$

$2PQ - Q^2 = 3(p^2 - 2pq)$

13 Apr 1972 (cf Nov 1961)

Another max-min w/o calculus example.

Find $\min\{2x^{-2}+(1-x)^{-2} : 0<x<1\}$

Solution We have $\left(\dfrac{a^{1/3}+wb^{+1/3}}{1+w}\right)^{+3} \geq$
$\left(\dfrac{a^{-2/3}+wb^{-2/3}}{1+w}\right)^{-3/2}$ with equality iff $a=b$.

Take $a=(1-x)^{+3}$, $b=x^{+3}/2$, $w=2^{1/3}$

$RHS = (1+w)^{-3} \geq LHS = \left(\dfrac{(1-x)^{-2}+2x^{-2}}{1+w}\right)^{-3/2}$

Hence $(1-x)^{-2}+2x^{-2} \geq (1+w)^3$

strict save when $x=\frac{w}{1+w}$.

22 Apr 72. Shapiro finds that Parseval's equation for $I_{[0,1]}$ via Legendre expansion is

$$\frac{1}{3}\left(\frac{3}{2}\right)^2+\frac{1}{7}\left(\frac{1}{2}\cdot\frac{7}{4}\right)^2+\frac{1}{11}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{11}{6}\right)^2+\frac{1}{15}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{15}{8}\right)^2$$
$$+\cdots=1.$$

If this has an easy direct proof then the completeness of Legendre polys follows at once (he says.)

Here is such a proof (JSW). The series is $\sum_0^\infty \binom{-1/2}{n}^2 \frac{4n+3}{4(n+1)^2}$. But we identify the n^{th} term as $\binom{-1/2}{n}^2-\binom{-1/2}{n+1}^2$! QED.

[redacted]

9/6/72

A lemma of Burkholder — Ann Math Stat 1962

$$a_n \downarrow 0,\ \sum a_n = \infty \implies \exists\, n_k \ni \sum a_{n_k} = \infty$$
and $n_k \div k \to \infty$.

see Putnam comments 11/16/77

Sketch Pf: $a_1 + \dots + a_{n_1} \geq 1$

$a_{2n_1+2} + \dots + a_{2n_2} \geq 1$

$a_{3n_2+3} + \dots + a_{3n_3} \geq 1$

$\vdots$

renumber $1, \dots, n_1, 2n_1+2, \dots, 2n_2, 3n_2+3, \dots, 3n_3, \dots$

as $1, \dots, n_1, n_1+1, \dots, n_2, n_2+1, \dots, n_3, \dots$

I.e. for any integer r, choose $l = l_r \ni$

$$n_{l-1} < r \leq n_l$$

Then the r^{th} term of the new series, a_{n_r}, is

$$a_{l r}$$

and $n_r / r = l_r \to \infty$.

Note: not all divergent series of positive terms have monotone divergent subseries.

E.g. $1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, \dots$

rearrange to

$\underbrace{1/3, 1/2, 1}, \underbrace{1/8, 1/7, 1/6, 1/5, 1/4}, \underbrace{1/15, \dots, 1/9}, \underbrace{1/24, \dots, 1/16}, \dots$

$\frac{1}{k^2+2k}, \dots, \frac{1}{k^2}$

monotone subseries cannot have more than one term from any block, so converges at least as fast as $\sum \frac{1}{k^2}$.

9/27/72 A problem of Stam.

For $\alpha \in (0,1)$ let $1 - e^{-t_1} = \alpha$,

$$(1-e^{-2t_1})(1-e^{-t_2}) = \alpha$$

$$\vdots$$

$$(1-e^{-nt_1})\cdots(1-e^{-t_n}) = \alpha$$

How do $\{t_n\}$ behave.

Conjecture $t_n \uparrow \infty$ is false. In fact computation shows that $t_2 > t_1$, $t_2 > t_3 > \cdots > t_1$. This was done via quantities $\beta_n \equiv e^{-t_n}$ satisfying (of course)

$$1 - \beta_1 = \alpha$$

$$(1-\beta_1^2)(1-\beta_2) = \alpha$$

$$\vdots$$

$$(1-\beta_1^n)\cdots(1-\beta_n) = \alpha$$

trying $\beta_1 = .001, .01, .05, .1, .2, .5$.

Convergence very fast, e.g. even for .5 $n = 20$ gives 7 sig figs. Program *BASIC.

Presumed limit β_∞ is solution of

$$\prod_1^\infty (1-\beta^n) = \alpha.$$

$\dfrac{1}{\prod_1^\infty(1-\beta^n)}$ is some classical function, for it counts partitions of n into nondecreasing parts.

Michael Taylor pointed to paper by Edward Nelson in 1959 Annals of Math. on measures & stoch procs

9/28/72 Michael Taylor on stochastic integral.

Suppose $W(t) \in H$ for each $t \geq 0$ with orthogonal increments and

$$\| W(t+h) - W(t) \| = f(h) \quad (\doteq M\sqrt{h})$$

Suppose $T(t) : H \to H$, bdd linear, mble [?] in t, and with $\int_0^\infty \|T(t)\|^2 dt < \infty$; let P_t = projection on span of $W(s)$, $s \leq t$, and suppose that $T(t)$ commutes with $P_{t'}$, all $t \leq t'$.

If $T(t)$ is a step function we put

$$\int_0^\infty T(t)\, dW(t) = \sum T(t_j)[W(t_{j+1}) - W(t_j)].$$

The summands are orthogonal, for if

$$t_1 < t_2 \leq t_3 < t_4$$

then $P_{t_3}(W(t_4) - W(t_3)) = 0$. So

$P_{t_3}(W(t_2) - W(t_1)) = W(t_2) - W(t_1)$.

$$\big(T(t_1)(W(t_2) - W(t_1)),\ T(t_3)(W(t_4) - W(t_3))\big)$$

$$= \big(T(t_1) P_{t_3}(W(t_2) - W(t_1)),\ \text{"}\ \big)$$

$$= \big(P_{t_3} T(t_1)(W(t_2) - W(t_1)),\ \quad \big)$$

$$= \big(T(t_1)(W(t_2) - W(t_1)),\ P_{t_3} T(t_3)(W(t_4) - W(t_3))\big)$$

$$= \big(\ \text{"}\ ,\ \text{"}\ \big)$$

$$= \big(\ \text{"}\ ,\ 0\ \big)$$

$$\therefore \left\| \int_0^\infty T(t)\, dW(t) \right\|^2 = \sum \|T(t_j)\|^2 (t_{j+1} - t_j) = \int_0^\infty \|T(t)\|^2 dt$$

So the B-space of fncns $[0,\infty) \to L(H,H)$ commuting as above is mapped linearly & cont. into H, since step fncns make up dense set.

Some 1972 Putnam problems

A-1: Show $\nexists\, n, k \ni \binom{n}{k}, \dots, \binom{n}{k+3}$ in AP.

Soln. We ask when $\exists\, n, k \ni \binom{n}{k}, \dots, \binom{n}{k+2}$ in AP. Writing the differences equal, $\binom{n}{k+1} - \binom{n}{k} = \binom{n}{k+2} - \binom{n}{k+1}$ and simplifying we arrive at

$$n^2 - (4k+5)n + 4k^2 + 8k + 2 = 0$$

So $(4k+5)^2 - 4(4k^2+8k+2)$ is $\square$

So $8k+17$ is $\square$

$$k = 1, 4, 8, 13, \dots$$

$$k = \frac{x^2}{2} + \frac{5x}{2} + 1, \quad x = 1, 2, \dots$$

Then $n = \dfrac{4k+5 \pm \sqrt{8k+17}}{2}$, "−" duplicates from $k \leftrightarrow n-k$

$$= x^2 + 6x + 7, \quad x = 1, 2, \dots$$

Examples.

$x =$	1	2	3	4	...
$k =$	1	4	8	13	...
$n =$	7	14	23	34	...

(note how choice of minus sign works. For $x = 2$ it gives $n = 7$, $k = 4$ $\therefore \binom{7}{4}, \binom{7}{5}, \binom{7}{6}$ same as $\binom{7}{1}, \binom{7}{2}, \binom{7}{3}$ found at previous line.)

Note also that $\dfrac{k - \frac{n}{2}}{\sqrt{n \frac{1}{2} \frac{1}{2}}} = \dfrac{-\frac{x}{2} - O(1)}{\frac{x}{2} + O(1)} = -1 + o(1)$

corresponding to inflection point on $y = e^{-x^2/2}$ at $x = -1$. !!

For prob. as stated clearly no such for n & both $k, k+1$.

[redacted]

A-2 $\quad *$ on $S \times S$ & S $\ni$

$$x * (x * y) \equiv y \equiv (y * x) * x$$

Prove: $*$ is commutative.

Pf: rewrite w/o $*$:

$$x(xy) = y = (yx)x.$$

Then

$$(xy)((xy)y) = y$$
$$= (xy)x$$

and hence $\quad yx = ((xy)x)x = xy.$

QED

A-3. Defn: $x_n \xrightarrow[c]{} x$ iff $\frac{1}{n}\sum x_i \to x$.

Defn: f supercontinuous iff $x_n \xrightarrow[c]{} x \Rightarrow$
$f(x_n) \xrightarrow[c]{} f(x)$.

Theorem: f supercont iff $f(x) = ax + b$ for some const a, b.

Pf: Given x_0 ~~and~~ x_1 and x_2, w/ $x_0 < x_1 < x_2$ choose a sequence $\{\varepsilon_n\}$ of 0's & 1's such that

$$\frac{1}{n}\sum_1^m \{\varepsilon_k x_0 + (1-\varepsilon_k)x_2\} \to x_1.$$

Let $\tilde{x}_n = \begin{cases} x_0 & \text{if } \varepsilon_n = 1 \\ x_2 & \text{if } \varepsilon_n = 0 \end{cases}$ $\quad$ (*) W/o loss of gen. assume $f(x_0) = x_0$, $f(x_2) = x_2$.

Then $\tilde{x}_n \xrightarrow[c]{} x_1$. So $f(\tilde{x}_n) \xrightarrow[c]{} f(x_1)$

But $f(\tilde{x}_n) = \begin{cases} f(x_0) & \text{if } \varepsilon_n = 1 \\ f(x_2) & \text{if } \varepsilon_n = 0 \end{cases}, = \begin{cases} x_0 \\ x_2 \end{cases}$

so $\quad f(\tilde{x}_n) \xrightarrow[c]{} x_1$, ~~where~~

$\therefore f(x_1) = x_1$ and f is linear.

(*) f supercont iff $\alpha f + \beta$ supercont.

A-4. Show that circle maximizes perimeter of ellipse inscribed in square.

Pf: For $\alpha x^2+2\beta xy+\gamma y^2=1$ to be tangent to the four sides of the square $|x|\vee|y|=1$ we must have $\alpha x^2+2\beta x+\gamma-1=0$ has double root, so $\beta^2-\alpha(\gamma-1)=0$,

so $\alpha=\alpha\gamma-\beta^2=\gamma$ by sym.

Then the squares of the semiaxes satisfy $\left(\frac{1}{\lambda}-\alpha\right)\left(\frac{1}{\lambda}-\gamma\right)-\beta^2=0$

$$(\alpha\gamma-\beta^2)\lambda^2-\lambda(\alpha+\gamma)+1=0$$

$$\alpha\lambda^2-2\alpha\lambda+1=0,\quad \lambda=\frac{\alpha\pm\sqrt{\alpha^2-\alpha}}{\alpha}$$

$$=\frac{\alpha\pm\beta}{\alpha}=1\pm\frac{\beta}{\alpha}.$$

Hence quarterperimeter is

$$\int_0^{\pi/2}\sqrt{\alpha\frac{\cos^2t}{\alpha-\beta}+\alpha\frac{\sin^2t}{\alpha+\beta}}\,dt$$

had extra inversion here $=\int_0^{\pi/2}\sqrt{\frac{\alpha}{\alpha^2-\beta^2}\left((\alpha+\beta)\cos^2t+(\alpha-\beta)\sin^2t\right)}\,dt$

$$=\int_0^{\pi/2}\sqrt{\alpha+\beta(\cos^2t-\sin^2t)}\,dt$$

$$=\int_0^{\pi/2}\sqrt{\alpha+\beta\cos2t}\,dt$$

$$=\frac{1}{2}\int_0^{\pi/2}\left(\sqrt{\alpha+\beta\cos t}+\sqrt{\alpha-\beta\cos t}\right)dt$$

[redacted]

~~$(\sqrt{x}+\sqrt{y})^2 = x + 2\sqrt{xy} + y$~~

~~$\sqrt{x}+\sqrt{y} \geq \sqrt{x+y}$~~

~~$\sqrt{} > \sqrt{x-y}$~~

$(\sqrt{x+y} + \sqrt{x-y})^2$

$= 2x + 2\sqrt{x^2-y^2}$

$\leq 4x$

$\sqrt{x+y} + \sqrt{x-y} \leq 2\sqrt{x}$

~~Integral $\leq \int_0^{\pi/2} \sqrt{\alpha}\, dt$~~

~~$= \sqrt{\alpha}\,\frac{\pi}{2}$~~

~~Too crude.~~

~~$\alpha^2 - \alpha - \beta^2 = 0$~~

~~$\alpha = \frac{1+\sqrt{1+\beta^2}}{2} > 1.$~~

$$\int_0^{\pi/2} \sqrt{\left(1+\frac{\beta}{\alpha}\right)\cos^2 t + \left(1-\frac{\beta}{\alpha}\right)\sin^2 t}\; dt$$

$$= \int_0^{\pi/2} \sqrt{1+\frac{\beta}{\alpha}\cos 2t}\; dt$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sqrt{1+\frac{\beta}{\alpha}\cos t} + \sqrt{1-\frac{\beta}{\alpha}\cos t}\right) dt$$

$\leq \int_0^{\pi/2} \sqrt{[illegible]}\; dt = \frac{\pi}{2}\,[illegible]$ = circle's quarter-perimeter.

A-5 $n \nmid 2^n - 1$

Can restrict to n odd.

dt

4/11/73 Montgomery's math club talk on Conway's games.

One example only.

Move as in peg game: $\boxed{x\ x\ \ }\rightarrow\boxed{\ \ \ \ x}$ in any direction. Can you locate x's in $Rl<0$ so that you can achieve $Rl\ x=k$?

k	min # x	
1	2	
2	4	
3	8	
4	20	!
5	∞	!!

Pf of !! "Pagoda fcn"

	⋮	⋮					
	·	·	·	–	·	ω^3	ω^2
···	ω^7	ω^6	–	–	·	ω^2	ω
···	ω^6	ω^5	·	·	ω^3	ω	1
···	ω^7	ω^6	1	2	3	ω^4	ω
	·	·	–	–	·	ω^3	ω^2
	⋮	⋮					

$\omega^2+\omega=1.$

observe that no legal move can increase sum of numbers in occupied squares. Sum of <u>all</u> in left half plane $=1$.

$\therefore$ no finite set in LHP can yield 1, i.e. can end with x in square labelled 1.

8/4/73 (See 3/67)

$$\alpha=p^2q(1+\alpha)^3$$

$$=\sum_1^\infty\frac{(p^2q)^n}{n!}\frac{(3n)!}{(2n+1)!}$$

by Lagrange expansion

and, ditto, $f=\sum_1^\infty\frac{(p^2q)^n}{n!}\frac{3(3n-2)!}{(2n-1)!}$

$$\therefore\quad 1=\sum_1^\infty\left(\frac{4}{27}\right)^n\frac{3(3n-2)!}{(2n-1)!\,n!}$$

[redacted]

Abel: $\int \frac{3x-7}{\sqrt{x(x-1)(x-4)(x-9)}}\,dx$ is elementary

cf 11/15/76

A monthly problem:

Evaluate $\sum_{1\le i<j\le n} |\omega_i-\omega_j|^{-2}$ for $\omega_1,\dots,\omega_n$ the n^{th} roots of unity.

Generalization: exponent $-2 \to -2r$. Sum $= S_r^{(n)}$

For fixed n let $f(x) = \sum_0^\infty S_{r+1}^{(n)}(2x)^r$.

Get

$$f(x) = \sum_{1\le i<j\le n} \frac{|\omega_i-\omega_j|^{-2}}{1-(2x)|\omega_i-\omega_j|^{-2}}$$

$$= \sum_{1\le i<j\le n} \frac{1}{|\omega_i-\omega_j|^{+2}-2x}$$

$$= \frac{n}{2}\,\frac{1}{2}\sum_{1}^{n-1} \frac{1}{1-x-\cos\frac{2k\pi}{n}}$$

$\frac{1}{2}$ from fact that given sum $= \frac{1}{2}\sum_{i\neq j}$.

since if $\omega_j = e^{\frac{2\pi i}{n}j}$, $|\omega_h-\omega_j|^{+2} = |1-\omega_{j-h}|^2$
$= 2-2\cos\frac{2k\pi}{n}$ $\quad k=j-h$.

So
$$f(x) = \frac{n}{4}\sum_{0}^{n-1}{}' \frac{1}{1-x-\cos\frac{2k\pi}{n}} + \frac{n}{4x}$$

$$= -\frac{n}{4}\frac{d}{dx}\log\prod_{0}^{n-1}\left(1-x-\cos\frac{2k\pi}{n}\right) + \frac{n}{4x}$$

$$= -\frac{n}{4}\frac{d}{dx}\log\{1-\cos[n\cos^{-1}(1-x)]\} + \frac{n}{4x}$$

since $\cos n\cos^{-1}t$ is a poly in t of degree n taking value 1 with correct multiplicities.

$$= -\frac{n}{4}\,2\,\frac{d}{dx}\log\sin\frac{n}{2}\cos^{-1}(1-x) + \frac{n}{4x}$$

$$= -\frac{n}{2}\cot\left(\frac{n}{2}\cos^{-1}(1-x)\right)\cdot\frac{n}{2}\,\frac{1}{\sqrt{2x-x^2}} + \frac{n}{4x}$$

$=$

$$\cot\frac{x}{2} = \frac{2}{x} - \frac{x}{6} - \frac{x^3}{360} - \cdots$$

so result is

$$-\frac{n}{2}\left\{\frac{2}{n\cos^{-1}(1-x)} - \frac{n}{6}\cos^{-1}(1-x) - \frac{n^3}{360}\left(\cos^{-1}(1-x)\right)^3 - \frac{n^5}{\cancel{3024}\ 15120}\{\cos^{-1}(1-x)\}^5\right.$$

$$\left.\cdots\right\}\cdot\frac{n}{2}\cdot\frac{1}{\sqrt{2x-x^2}} + \frac{n}{4x}$$

$$= \frac{-n}{2\cos^{-1}(1-x)\sqrt{2x-x^2}} + \frac{n^3}{24}\frac{\cos^{-1}(1-x)}{\sqrt{2x-x^2}} + \frac{n^5}{1440}\frac{\cos^{-1}(1-x)^3}{\sqrt{2x-x^2}}$$

$$+\frac{n}{4x}$$

$$\cos^{-1}(1-x) = \sin^{-1}\sqrt{2x-x^2}$$

$$= \sqrt{2x-x^2}\left\{1 + \frac{2x-x^2}{6} + \frac{3}{40}(2x-x^2)^2\right.$$

$$\left. + \frac{5}{112}(2x-x^2)^3 + \cdots\right\}$$

$$= \sqrt{2x-x^2}\left(1 + \frac{x}{3} + \frac{2}{15}x^2 + \frac{2}{35}x^3 + \cdots\right)$$

$\therefore$

$$f(x) = \frac{-n}{2(2x-x^2)\left(1+\frac{x}{3}+\frac{2}{15}x^2+\frac{2}{35}x^3+\cdots\right)} + \frac{n}{4x}$$

$$+\frac{n^3}{24}\left(1+\frac{x}{3}+\frac{2}{15}x^2+\frac{2}{35}x^3+\cdots\right)$$

$$+\frac{n^5}{1440}(2x-x^2)\left(1+\frac{x}{3}+\frac{2}{15}x^2+\cdots\right)^3$$

$$+\frac{n^7}{\cancel{211680}\ \cancel{120960}\ 60480}(2x-x^2)^2(1+\cdots)^3$$

Big denominator.

$$= 4x\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}+\frac{2}{15}x^2+\frac{2}{35}x^3+\cdots\right)$$

$$= 4x\left\{1 - \frac{x}{6} - \frac{x^2}{30} - \frac{x^3}{105} - \cdots\right\}$$

$$\therefore f(x) = -\frac{n}{4x}\left\{1 + \frac{x}{6} + \frac{11x^2}{180} + \frac{191x^3}{216} + \cdots\right\}$$

$$+\frac{n}{4x} + \frac{n^3}{24}\left\{1+\frac{x}{3}+\frac{2}{15}x^2+\cdots\right\} + \frac{n^5}{1440}\left(2x+x^2+\cdots\right)$$

[redacted]

$$= -\frac{n}{24} - \frac{11nx}{720} - \frac{191nx^2}{864\cdot 5\cdot 7} + \cdots$$

$$+\frac{n^3}{24} + \frac{n^3 x}{72} + \frac{n^3 x^2}{180} + \cdots$$

$$-1 \quad + \frac{n^5 x}{720} + \frac{n^5 x^2}{1440} + \cdots \longrightarrow + \frac{n^7}{\cancel{50420}\ \cancel{30240}\ 15120} x^2$$

Const term: $\frac{n^3 - n}{24} = \frac{(n-1)(n+1)n}{24}$

Coeff x : $\frac{-11n + 10n^3 + n^5}{1440} = \frac{n(n^2-1)(n^2+11)}{1440 = 2^5 3^2 5}$

Coeff x^2: $\cancel{\frac{-1685n + 1176n^3 + 147n^5 + 4n^7}{4\cdot 52,920 = 2^3 3^3 5\, 7^2}}$ ~~46725 1337~~

$$\frac{-191n + 168n^3 + 21n^5 + 2n^7}{15120\cdot 2}$$

$$= \frac{n(n^2-1)(2n^4 + 23n^2 + 191)}{15120\cdot 2}$$

$$\therefore \sum |w_i - w_j|^{-6} = \frac{n(n^2-1)(2n^4+23n^2+191)}{60480\cdot 2 = 120960 = 2^7 3^3 5\cdot 7}$$

This gives $\frac{1}{9}$ @ $n = 3$ which is correct, being $3\,\frac{1}{(\sqrt{3})^6}$.

12/23

Dubins' Theorem: $X_i = \begin{cases} 0, & p \\ 1, & 1-p \end{cases}$ ind.

$Y = \sum 2^{-i} X_i \qquad F = \text{df of } Y$

$\Longrightarrow F(ab) \leq F(a)F(b)$

His result is a byproduct of gambling-theoretic considerations.

Is there an elementary proof?!

A nice example of a Markov Chain.
(Double-or-nothing on statespace $E = \{0, 1, \cdots, 5\}$.)

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 & 0 \\ 2 & q & 0 & 0 & 0 & p & 0 \\ 3 & q & q & 0 & 0 & 0 & p \\ 4 & 0 & 0 & \cancel{q} & q & 0 & p \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$Px = x$ gives

x_0, x_5 arb. Take $x_0 = 0, x_5 = 1$.

$$x_1 = \frac{p^3(q+1)}{1-p^2q^2} = .173 \text{ @ } p = 9/19 \qquad .2 \text{ @ } p = \tfrac{1}{2}$$

$$x_2 = \frac{p^2(q+1)}{1-p^2q^2}$$

$$x_3 = \frac{p(p^2q+1)}{1-p^2q^2}$$

$$x_4 = \frac{p(q+1)}{1-p^2q^2}$$

12-20-73

Suit distributions in five card hands

	Deck: 24-card (A-9)	32-card (A-7)	52	∞
5	.06%	.11%	.20%	.39%
4,1	2.54	3.34	4.29	5.86
3,1,1	20.33	21.36	22.32	23.44
3,2	8.47	9.34	10.30	11.72
2,2,1	38.11	37.37	36.52	35.16
2,1,1,1	30.49	28.48	26.37	23.44
	100.00	100.00	100.00	100.01

$P\{\geq 3 \text{ cards in one suit, poker}\} = 37.11\%$

$P\{\geq 3 \mid \geq 2 \text{ spades}\} = 34.62\%$

a slight paradox

[redacted]

Don's problem (4/13/75 Probably wrong – from Dutko)

Show $\sum_{2}^{\infty} \frac{\log k}{k} (1 - e^{-i\lambda})^k \quad |\lambda| < \frac{\pi}{3}$

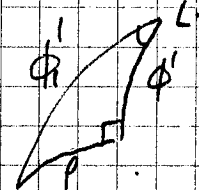
has analytic extension which is log of an infinitely divisible cf ___ and find its Levy-Khinchin representation

(a student (Dutko) arrived at this through some limit distribution question.)

ϕ = lat. L = long.

Highest point between (ϕ_1, L_1), (ϕ_2, L_2) at

(ϕ, L) , $L = L_1 + x$, $\tan x = \frac{\tan\phi_2}{\tan\phi_1 \sin(L_2 - L_1)} - \cot(L_2 - L_1)$



$$\cos\rho = \sin\phi_1 \sin\phi + \cos(L - L_1)\cos\phi_1 \cos\phi$$

$$\sin\phi_1 = \sin\phi \cos\rho$$

$$\therefore \sin\phi_1 = \sin^2\phi \sin\phi_1 + \cdots$$

$$\sin\phi_1 \cos^2\phi = \cos(L - L_1)\cos\phi \cos\phi_1 \sin\phi$$

$$\boxed{\tan\phi_1 = \cos(L - L_1)\tan\phi}$$

$$\phi_1 = 45^\circ$$

$L - L_1$	ϕ
0°	45°
10	45° 26′
20	46° 47′
30	49° 6′
40	52° 33′
50	57° 16′
60	63° 26′
70	71° 7′
75	75° 29′
80	80° 8′
85	85° 1′

1/6/74

Some powers of 2 — courtesy multiple-precision HP-45 work

n	2^n
10	1024
20	1,048,576
40	1,099,511,627,776
80	1,208,925,819,614,629,174,706,176
160	1,461,501,637,330,902,918,203,684,832,716,283,019,655,932,542,976

Check on last: It is $\equiv 562 \bmod 1001$

$\equiv 2 \bmod 7$ $\quad 2^6 \equiv 1,\ 2^{160} \equiv 2^4 \equiv 2$

$\equiv 1 \bmod 11$ $\quad 2^{10} \equiv 1,\ 2^{160} \equiv 1$

$\equiv 3 \bmod 13$ $\quad 2^{12} \equiv 1,\ 2^{160} \equiv 2^4 \equiv 3.$

and $\equiv 601 \bmod 999$

$\equiv 7 \bmod 27$ $\quad 2^9 \equiv -1,\ 2^7 \equiv -7,\ 160 = 17\cdot 9 + 7$

$\equiv 9 \bmod 37$ $\quad 2^{36} \equiv 1,\ 2^{16} \equiv 9,\ 160 = 4\cdot 36 + 16$

2/17/74 A nice Markov chain — from a problem of Jonathan's!

4 dice are tossed until two or more show the same face. If not all 4 do then the odd ones are tossed until they too agree. Want distn of N = number of tosses.

MC with states 4, 2, 1, 0 (number of dice remaining)

$$\begin{array}{c|cccc} & 4 & 2 & 1 & 0 \\ \hline 4 & 5/18 & 5/8 & 5/54 & 1/216 \\ 2 & 0 & 25/36 & 5/18 & 1/36 \\ 1 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1 \end{array} = \left[\begin{array}{c|c} \underset{3\times 3}{A} & b \\ \hline 0 & 1 \end{array}\right]$$

To get $E_x(s^N)$ compute matrix $(I-sA)^{-1}$, apply to vector sb, result is vector $\{E_x(s^N): x=4,2\}$

We only want $x=4$ component.

Result comes to

$$s \cdot \frac{\frac{50}{6^5}s^2 + \frac{200}{6^5}s + \frac{1}{6^3}}{(1-\frac{5}{18}s)(1-\frac{25}{36}s)(1-\frac{5}{6}s)}$$

$$= s \cdot \left[13 \cdot 6^{-3}(1-\tfrac{5}{18}s)^{-1} - 99 \cdot 6^{-3}(1-\tfrac{25}{36}s)^{-1} + 87 \cdot 6^{-3}(1-\tfrac{5}{6}s)^{-1}\right]$$

so that

$$P\{N=n\} = p_n = \frac{1}{12} g_n\left(\frac{5}{18}\right) - \frac{3}{2} g_n\left(\frac{25}{36}\right) + \frac{29}{12} g_n\left(\frac{5}{6}\right)$$

$$g_n(q) = q^{n-1} p$$

and

$$E(N) = \frac{1}{12} \cdot \frac{18}{13} - \frac{3}{2} \cdot \frac{36}{11} + \frac{29}{12} \cdot 6 = 9.71$$

Some values

n	p_n	$P\{N > n\}$
1	.00463	
2	.03408	
3	.06332	
4	.08088	
5	~~.08796~~ .08801	
6	.08796	.64114
7	.08351	.55763
8		.48091
9		.41203
17		.10588
18		.08866
30		ca .01
42		ca. .001

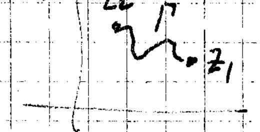
Feb 74

Hugh Montgomery.

Prove: sum of digits of $a^n \to \infty$

for $a = 1974, 1975$. Not known for 1973!

Joe Ullman:



To minimize

$$\int_\Gamma \frac{|dz|}{|z|^2}$$

take Γ as arc of circle through $0, z_1, z_2$ not containing 0.

An oldie from Mike Taylor

Two pts at random from unit square, X = dist betw them; find $P\{X < 1\}$.

Ans: $\pi - 8/3 + 1/2 = 97.49\%$. Details:

For U_i ind. unif $(0,1)$, $\{U_1 - U_2\}$ has density $\{1 - |u|\}^+$, $-1 \le u \le 1$. So

$|U_1 - U_2| \sim \rho(u) = 2(1-u)\, I_{[0,1]}$.

So $|U_1 - U_2|^2 \sim \left(\frac{1}{\sqrt{x}} - 1\right) I_{[0,1]}$

and convoluting this with itself gives density of X^2, viz.

$$\pi - 4\sqrt{x} + x \quad \text{on } (0,1)$$

$$4\left\{\tan^{-1}\frac{1}{\sqrt{x-1}} + \sqrt{x-1}\right\} - x - \pi - 2 \quad \text{on } (1,2)$$

Then $\int_0^1 (\pi - 4\sqrt{x} + x)\, dx = \pi - \frac{8}{3} + \frac{1}{2}$.

Ullman

Much more than areal proportion $\frac{\pi}{4} = 78.54\%$

Thm for 525: if $s_n p^{(n)}_{k_n} \to f(x)$ whenever $(k_n - m_n)/s_n \to x$, for a.a. x, with $f(x)$ a prob. density, then $\sum_{a_n}^{b_n} p_k^{(n)} \to$

73! $\int_\alpha^\beta f(x)\,dx$ whenever $(a_n - m_n)/s_n \to \alpha$, etc β.

Only needed for $k_n = [s_n x + m_n]$ in fact. Does this imply the more general hypothesis? Mike Taylor gives counterexample elegantly as follows.

Choose $A \subset [0,1] \ni A$ & A^c both dense & $P(A) = 1/2$. Let $f(x) = 1/2$ on A, $3/2$ on A^c. Let $\mathcal{F}_n$ be the 2^{-n} partition of $[0,1]$ (σ-field generated) and $f_n = E(f|\mathcal{F}_n)$. Then $f_n \to f$ a.e. Let $p_k^{(n)} = 2^{-n} f_n\left(\frac{k}{2^n}\right)$, $1 \le k \le 2^n$.

(I choose f_n const on $\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, say.) Then $2^n p^{(n)}_{[2^n x]} \to f(x)$ a.e, for $2^n p^{(n)}_{[2^n x]} = f_n(x)$ a.e.

Given x, can assume $x \in A$, choose $k_n \ni \frac{k_n}{2^n} \to x$ so that alternately $\frac{k_n}{2^n} \in A, A^c$. Then

$\overline{\lim}\ 2^n p^{(n)}_{k_n} = 3/2$, $\underline{\lim} = 1/2$, no $\lim 2^n p^{(n)}_{k_n}$ exists.

In other words, it is not the case that for all x outside a fixed null-set we have $2^{-n} k_n \to x \Rightarrow 2^n p^{(n)}_{k_n} \to f(x)$.

Details in choice of k_n. (Above not stated quite right.)

$y_1 \in A^c \ni |y_1 - x| < 1/2$. $\quad n_1 \ni n \ge n_1 \Rightarrow |f_n(y_1) - f(y_1)| < 1/2$

$y_2 \in A \ni |y_2 - x| < 1/4$. $\quad n_2 \ge n_1 \ni n \ge n_2 \Rightarrow |f_n(y_2) - f(y_2)| < 1/4$

$\vdots$

$y_k \to x$, $|y_k - x| < 2^{-k}$, $\quad n_k \ni n \ge n_k$ ~~&~~ $\Rightarrow |f_n(y_k) - f(y_k)| < 2^{-k}$
$\quad n_k \ge n_1, \dots, n_{k-1}$

$y_{\text{even}} \in A$, $y_{\text{odd}} \in A^c$.

4/12/74 Chow - Robbins *Great Expectations* p 131 gives

$$\prod_{j=1}^{\infty}\left(\frac{j+2}{j}\right)^{1/(j+1)} \approx 3.8695$$

Confirmation:

$$\sum_1^{\infty} \frac{1}{j+1} \log\frac{j+2}{j}$$

$$= \sum_1^{\infty} \frac{1}{j+1}\,\frac{2}{j} + \sum_1^{\infty} \frac{1}{j+1}\left(\log\left(1+\frac{2}{j}\right) - \frac{2}{j}\right)$$

$$= 2 + \sum_2^{\infty} \frac{1}{j+1}\left(\log\left(1+\frac{2}{j}\right) - \frac{2}{j} + \frac{2}{j(j+1)}\right)$$

$$+ \frac{1}{2}(\log 3 - 2) \qquad - \sum_2^{\infty} \frac{2}{(j-1)j(j+1)}$$

$$= \frac{1}{2}(\log 3 + 1) + \sum_2^{\infty} \frac{1}{j+1}\left(\log\left(1+\frac{2}{j}\right) - \frac{4}{j} + \frac{2}{j-1}\right)$$

terms are $O(j^{-4})$ (coeff = (4/3))
Sum to 50^{th} term, HP-65

End result = 3.86947..

Joe Ullman. $F(0)=0$, $F(1)=1$, $F \uparrow$ cont.
$\ell(x)$ = arclength of $y=F(x)$ up to x.
$F = F_{ac} + F_s$

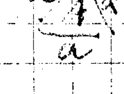
Thm: $\ell(x) = \int_0^x \sqrt{1+F'(t)^2}\,dt + F_s(x). \le 2$

Cor: $\ell(1) = 2$ iff $F_{ac} \equiv 0$.

Charlie Grumfiel: Hero's formula for $\triangle$ A

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{a+b+c}{2}.$$

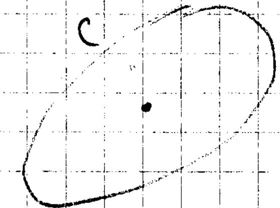
Pf: A^2 deg 4, $=0$ whenever $a+b=c$ or $a+c=b$ or $b+c=a$.
This gives factors $s-a$, $s-b$, $s-c$.
Symmetric in a, b, c — gives factor s.

 $A = \frac{1}{2}\, a\, \frac{\sqrt{3}}{2}\, a = \frac{\sqrt{3}}{4} a^2$ gives const.

[redacted]

[redacted]

Titus problem:

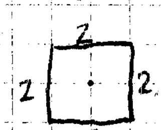


C central-symmetric convex. metric δ determined by C, $\delta(x,y) = r$ if $\frac{x-y}{r} \in C$.

$\pi_C = \frac{1}{2}\,\delta$-arclength of C.

Prove: $3 \le \pi \le 4$

C square



$\pi_C = 4$

C circle $\pi_C = \pi$.

C hexagon

$\pi_C = 3$

Andy Blass gave neat pf at math club 3/12/74.

4-7/74 A Conway problem.

In tossing a fair coin let A be any ordered triple of outcomes. Then $\exists$ B another such, with $P\{B < A\} \ge 2/3$ "<" means "occurs before".

There is an algorithm for computing these probabilities.

Let $\sigma(A, B)$ = overlap score of A, B = $\sum_0^2 2^i \varepsilon_i$, with $\varepsilon_i = 1$ or 0 according as last ~~3/4~~ $i+1$ positions of A agree with first ~~3/4~~ $i+1$ of B.

Example: $A = HTH$, $B = HHH$

$$\sigma(A,A) = 5 \quad , \quad \sigma(B,B) = 7$$
$$\sigma(A,B) = 1 \quad , \quad \sigma(B,A) = 1$$

Then
$$\frac{P(B < A)}{P(A < B)} = \frac{\sigma(A,A) - \sigma(A,B)}{\sigma(B,B) - \sigma(B,A)}$$

$$\left(= \frac{5-1}{7-1} = \frac{2}{3} \text{ in ex.} \right)$$

$\sigma(A,B)$

$A \backslash B$	H_3	H_2T	HTH	HT_2	TH_2	THT	T_2H	T_3
HHH	7	3	1	1	0	0	0	0
HHT	0	4	2	2	1	1	1	1
HTH	1	1	5	1	2	2	0	0
HTT	0	0	0	4	1	1	3	3
THH	3	3	1	1	4	0	0	0
THT	0	0	2	2	1	5	1	1
TTH	1	1	1	1	2	2	4	0
TTT	0	0	0	0	1	1	3	7

$P(B<A)/P(A<B)$

								max
—	1	3/2	3/2	7*	7/5	7/3*	1	7
1	—	1/2	1/2	3*	3/5	1	3/7	3
2/3	2*	—	1	1	1	5/3	5/7	2
2/3	2*	1	—	1	1	1/3	1/7	2
1/7	1/3	1	1	—	1	2*	2/3	2
5/7	5/3	1	1	1	—	2*	2/3	2
3/7	1	3/5	3*	1/2	1/2	—	1	3
1	7/3*	7/5	7*	3/2	3/2	1	—	7

*: Ratio ≥ 2, prob $\geq 2/3$.

This generalizes to patterns of length other than 3 and to unfair coins.

¶ Here is a pf by Andy Blass.

Consider a doubly infinite seq. of H's & T's, $\{\omega_i\}_{-\infty}^{\infty}$ with p, q product measure.

Let $T_n(A)$ = time $\geq n$ of end of first A (after n or at n)

$T_n(B)$ sim.

$$W_n = \{T_n(A) < T_n(B)\}$$

Observe: (*) $W_n = W_{n+1} - \{T_n(B) = n \;\&\; T_{n+1}(A) < T_{n+1}(B)\}$

$\cup \; \{T_n(A) = n \;\&\; \quad > \quad \}$

Then $p = P(W_n) = P(W_{n+1}) - \underbrace{p_B\,\alpha_B + p_A\,\beta_A}_{=}$

$= \; p \; -$

where $p_B = P\{T_n(B) = n\}$, p_A sim.

$\alpha_B = P\{T_{n+1}(A) < T_{n+1}(B) \mid T_n(B) = n\}$, β_A sim.

[redacted]

Hence $p_B \alpha_B = p_A \beta_A$.

Next let $\alpha_n = P\{T_n(A) < T_n(B) \mid T_0(B) = 0\}$

and take ~~[illegible]~~ cond. prob. of $(*)$ given $T_0(B)=0$.

Then $\alpha_n = \alpha_{n+1} - P\{T_n(B) = n \mid T_0(B) = 0\} \cdot$

$$P\{T_{n+1}(A) < T_{n+1}(B) \mid T_n(B) = n\}$$

$+$ sim.

Hence

$$\alpha_{n+1} = \alpha_n + \rho_n(B,B)\alpha_B - \rho_n(B,A)\beta_A$$

$$\rho_n(A,B) = \begin{cases} 0 & \text{if last } l+n \text{ } A \neq \text{first } l+n \text{ } B \\ P\{\text{last } l-n-1 \text{ of } B\} & \text{if } = \end{cases}$$

~~$p^i q^j$~~ ~~& i H's j T's ($i+j=n$)~~

Of course $\alpha_0 = 0$ and $\alpha_l = \alpha$.

Hence

$$\alpha = \sum_0^{l-1} \rho_n(B,B)\alpha_B - \sum_0^{l-1} \rho_n(B,A)\beta_A$$

Thus

$$\frac{1-\alpha}{\alpha} = \frac{\sum \rho_n(A,A)\beta_A - \sum \rho_n(A,B)\alpha_B}{\sum \rho_n(B,B)\alpha_B - \sum \rho_n(B,A)\beta_A}$$

$$= \frac{\sum \rho_n(A,A)/p_A - \sum \rho_n(A,B)/p_B}{\sum \rho_n(B,B)/p_B - \sum \rho_n(B,A)/p_A}$$

$$\left(\text{all } \sum = \sum_0^{l-1}\right)$$

which is tantamount to our stated formula when $p = q = 1/2$.

¶ Here is a pf based on general MC theory.

From Chung MC 2nd Edn p. 65, if $i, j \neq k$ all in one positive class,

$$(*) \quad {}_k f_{ij}^* = (m_{ik} + m_{kj} - m_{ij}) / (m_{jk} + m_{kj}).$$

See 3 pgs later for easy pf

$$\| \quad P_i\{T_j < T_k\}, \quad m_{ij} = E_i(T_j) \text{ assumed} < \infty.$$

pg 28, period 1 $\Rightarrow$ $p_{ii}^{(n)} \to \frac{1}{m_{ii}}$, stationary distn in positive case

pg 66 $\quad \sum_{1}^{\infty} \{p_{ik}^{(n)} - p_{kk}^{(n)}\} = 1 - \frac{m_{ik}}{m_{kk}}$

or $\quad m_{ik} = m_{kk} - m_{kk} \sum_{1}^{\infty} \{p_{ik}^{(n)} - p_{kk}^{(n)}\}$

Therefore for the stationary distn $\{1/m_{ii}\}$

$$P\{T_j < T_k\} = \sum_i \frac{{}_kf_{ij}^*}{m_{ii}}$$

$$= \sum_i \frac{m_{ik} + m_{kj} - m_{ij}}{m_{ii}(m_{kj} + m_{jk})}$$

$$= \frac{m_{kj} + m_{kk} - m_{jj}}{m_{kj} + m_{jk}} - \frac{m_{kk}}{m_{kj} + m_{jk}} \sum_{n=1}^{\infty} \sum_i \frac{p_{ik}^{(n)} - p_{kk}^{(n)}}{m_{ii}} + \frac{m_{jj}}{m_{kj} + m_{jk}} \sum_n \sum_i \frac{p_{ij}^{(n)} - p_{jj}^{(n)}}{m_{ii}}$$

$$= \frac{m_{kj} + m_{kk} - m_{jj} - \sum_{n=1}^{\infty} (p_{jj}^{(n)} m_{jj} - p_{kk}^{(n)} m_{kk})}{m_{kj} + m_{jk}}$$

using $\quad \sum \frac{p_{ij}^{(n)}}{m_{ii}} = \frac{1}{m_{jj}}$ by stationarity

$$= \frac{\cancel{m_{jj}} - m_{jj} \sum_{1}^{\infty} (p_{kj}^{(n)} - \cancel{p_{jj}^{(n)}}) + m_{kk} - \cancel{m_{jj}} - \sum_{1}^{\infty} (\cancel{p_{jj}^{(n)} m_{jj}} - p_{kk}^{(n)} m_{kk})}{m_{kj} + m_{jk}}$$

$$= \frac{\cancel{m} \sum_{0}^{\infty} (p_{kk}^{(n)} m_{kk} - p_{kj}^{(n)} m_{jj})}{m_{kj} + m_{jk}}$$

In our case $m_{kk} = \frac{1}{p_k}$ $(\forall k)$ (k an l-tuple)
and $\forall k, j$ $p_{kj}^{(n)} = \cancel{m_{jj}}\ \cancel{\ }\ p_j$ for $n \geq l$.

Hence $$P\{T_j < T_k\} = \frac{\sum_0^{l-1} \frac{p_{kk}^{(n)}}{p_k} - \sum_0^{l-1} \frac{p_{kj}^{(n)}}{p_j}}{m_{kj} + m_{jk}}$$

and the odds are as before.

7/31/74 An inequality of Montgomery. X, Y iid
k an integer $\Rightarrow$
$$P\{-k \leq X - Y \leq k\} \leq (2k+1)\, P\{-1 \leq X - Y \leq 1\}.$$

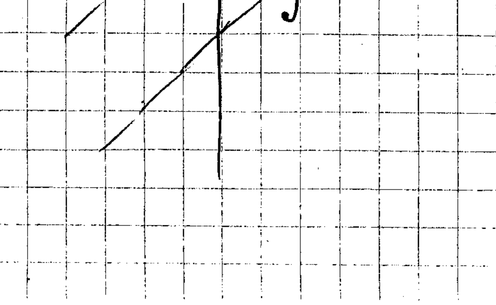
Pf: (gew) $$P \underset{①}{\leq} \sum_{j=-\infty}^{\infty} \sum_{r=j-k}^{j+k} p_j p_r \qquad p_j = P\{j \leq Y < j+1\}$$

$$\leq \sqrt{\sum\sum p_j^2}\ \sqrt{\sum\sum p_r^2}$$

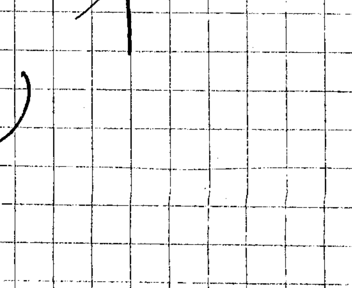
$$= (2k+1) \sum p_j^2$$

$$\underset{②}{\leq} (2k+1)\, P\{-1 \leq X - Y \leq 1\}$$

①



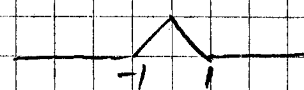
②



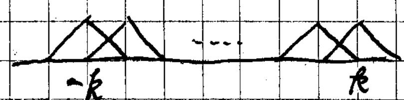
(cont'd)

Hugh generalizes, using Machinery.

Lemma: (QBW) $\quad E\left((1-|X|)^+\right) = \int_{-\infty}^{\infty} \underbrace{\varphi(t)}_{\text{cf of } X} \frac{1-\cos t}{\pi t^2}\, dt$



$$E \sum_{-k}^{k} \{1 - |X - j|\}^+ = \int_{-\infty}^{\infty} \varphi(t) \sum_{-k}^{k} e^{ijt} \frac{1-\cos t}{\pi t^2}\, dt$$



$\geq \quad P\{|X| \leq k\}$

If $\varphi(t) \geq 0$ (as holds for X of form $X_1 - X_2$, X_i iid) then the integral has modulus $\leq$

$$(2k+1) \int_{-\infty}^{\infty} \varphi(t) \frac{1-\cos t}{\pi t^2}\, dt \text{ --}$$

which in turn $\leq (2k+1)\, P(|X| \leq 1)$.

→ Easy pf for (*) three pgs back.

We have $\quad E_i(T_j) = E_i(T_j\,;\, T_j < T_k) + E_i(T_k\,;\, T_k < T_j)$

$\qquad\qquad + E_i(T_j - T_k\,;\, T_k < T_j)$

By strong Markov 3^{rd} term is $E_k(T_j)\, P_i(T_k < T_j)$

$\qquad = E_k(T_j)\; {}_j f^*_{ik}$

Hence $\quad m_{ij} - m_{kj}\; {}_j f^*_{ik} = m_{ik} - m_{jk}\; {}_k f^*_{ij}$

$\qquad m_{ij} - m_{kj}\,(1 - {}_k f^*_{ij}) = m_{ik} - m_{jk}\; {}_k f^*_{ij}$

Whose solution is (*).

Cf also pg 636 of Darling-Siegert $\underset{\text{Annals Stat}}{\text{AMS}}$ 24 (1953)

[redacted]

Dubins (Vancouver, 23 Aug 74)

X_n unif on unit circle, iid.
S_n = sum of first n.

Then $P(|S_n| < 1) = \frac{1}{n+1}$!!

Known pf is analytic, but should find geometric pf.

(Kluyver, Kon. Akad. van Weten. te Amsterdam, XIV, 1, 1905)
Lord Rayleigh's coll^d works vol 6 pg 613 QC3.R266 s

$$P(|S_n| < a) = a\int_0^\infty J_1(ax)\, J_0(x)^n\, dx$$

cf also Kingman Acta Math v.107 (1963) p 11–

For X_n unif on unit sphere Bessel fncns of order 3/2, 1/2 come in. These reduce to trig fncns and (Rayleigh)

$$P(|S_n| < a) = \frac{2}{\pi}\int_0^\infty dx\ \frac{\sin ax - ax\cos ax}{x}\left(\frac{\sin x}{x}\right)^n dx$$

For $a = 1$, $\frac{2}{\pi(n+1)}\int_0^\infty dx\ \left(\frac{\sin x}{x}\right)^{n+1} dx$, for which cf Whittaker / Watson 4^th Edn p. 123 prob 13. The result is

$$\frac{1}{2^n (n+1)!}\left\{ (n+1)^n - \binom{n+1}{1}(n-1)^n + \binom{n+1}{2}(n-3)^n - \ldots \right.$$
$$\left. \text{as long as } (n - \ldots) > 0 \right\}$$

Some values are

n	$P(\lvert S_n\rvert < 1)$		
2	1/4 =	.25000	(geom^ly easy)
3	1/6	.16667	
4	23/192	.11979	
5	11/120	.09167	
6	841/11520	.07300	
7	151/2520	.05992	
8	259,723/5,160,960	.05032	
9	15,619/362,880	.04304	

9/24/74

Conway's Random Number generator, HP-65

LBL A, 10, RCL 6, 9x, −, f LOG, EEX 5, ×, f^{-1} INT,
g LST x, f INT, EEX 10, ÷, +, STO 6, RTN. 25 steps

Some tests $X_1 = g\,\pi\ f^{-1}\,\mathrm{INT}$

I. $\sum_{1}^{40000} X_i = 20002.36918$ (40000 ← 4×10^4)

II.

		$\lvert\Delta\rvert$	sign
#([0, .2))	8109	109	+
" .4	7894	106	−
" .6	7928	72	−
" .8	8018	18	+
" 1.0	8051	51	+
	40,000		

$\sqrt{npq} = 80$ ($p = 1/5$, $q = 4/5$) $\qquad \chi^2_4 = 3.9$

III $1 \le i \le 16100$, same X_1, $\sum_{i=1}^{16100} X_i^{\nu}$ $\quad \nu = 2, \dots, 6$

ν	Σ	Expⁿ	Δ/σ	sign
2	5391..	5367	.64	+
3	4044..	4025	.53	+
4	3234	3220	.41	+
5	2693	2683	.30	+
6	2305	2300	.17	+

$\sigma^2 = \left(\frac{1}{2\nu+1} - \frac{1}{(\nu+1)^2}\right) 16{,}100$

$\text{Exp}^n = \frac{16{,}100}{\nu+1}$

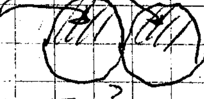
cf 11/15/76 !

IV. $1 \le i \le 10000$ $\qquad \sum_{1}^{10000} X_{2i} X_{2i-1} = 2517.57858$

$\text{Exp}^n = 2500$

17.57858

$\sigma = \frac{100\sqrt{7}}{12} = 22.05$

V. $1 \le i \le 919$ Pairs at random from 

mean dist = 2.04 theory = ?
mean dist² = 4.64 theory = 5 I think

11/6/74 Example for Jeff Rauch: $f \in L_1(0,1)$, $a_n \downarrow 0$

$\ni$ $f(x+a_n)$ converges almost nowhere on a set!

$f = I_A$, $A = \{x : x$ has no bad blocks$\}$

$x = .\underbrace{w_1}_{1}\, \underbrace{w_2\, w_3}_{11}\, \underbrace{w_4\, w_5\, w_6}_{111} \ldots$ binary

bad blocks $\longrightarrow$

$\mu(A) = \frac{1}{2}\,\frac{3}{4}\,\frac{7}{8}\,\frac{15}{16}\ldots > 0.$ $.288788$ (HP-65, 31 steps).

$\{a_n\} = .1, .011, .010, .001, .000111, .000110, .000101, .000100,$
$.000011, .000010, .000001, \ldots$

Note: $\forall x \in A$, $x + a_n \notin A$ i.o., for bad blocks are repeatedly created.

Hence $\{I_A(x+a_n)\}$ does not converge anywhere on A.

4/10/75 March Monthly problem (elementary)

$$\int_0^1 \log(1-x)\log(1+x)\,dx$$

Soln: $(1-\log 2)^2 + 1 - \frac{\pi^2}{6} = .5507754127$

Sketch of method. $x \leftrightarrow 1-x$, expand $\log(2\mp x)$.
Get $-\log 2 + \sum_1^\infty \frac{1}{2^n n(n+1)^2}$. Series breaks up (partial fractions), interesting part is $\sum \frac{1}{2^n n^2}$, which, amazingly, is elementary

$$\frac{1}{2}(\log 2)^2 - \frac{\pi^2}{12}$$

cf J. Edwards Integral Calculus Vol II Chelsea 1954
QA308

$\sum \frac{x^n}{n^2}$ Elementary for $x = \pm 1, \frac{1}{2}, \sin\frac{\pi}{10}$
one more, make $\sin^2\frac{\pi}{10}$.

4/13/75 Martin Gardner Mar & Apr Sci Am

Game, II pays I a_{ij} $\quad i,j \in N$

$$a_{ij} = \begin{cases} 1 & i < j-1 \\ -2 & i = j-1 \\ 0 & i = j \\ 2 & i = j+1 \\ -1 & i > j+1 \end{cases}$$

Unique optimal strategy: $\left(\frac{1}{16}, \frac{5}{16}, \frac{4}{16}, \frac{5}{16}, \frac{1}{16}, 0, 0, 0 \ldots\right)$

Due to NS Mendelsohn AMM 53 (1946) 86–88

See also Herstein-Kaplansky Matters Mathematical Harper & Row 1974 pp 212–5

6/19/75 Convergence Improvement (cf 4/12/74)

$$S = \sum_1^\infty \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1)$$ from Fourier series for $\cosh x$ on $[-\pi, \pi]$ @ $x = \pi$.

$$\sum_2^N \left(\frac{1}{n^2-1} - \frac{1}{n^2+1}\right) = \underbrace{\sum_2^N \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)}_{= \frac{3}{4} + o(1)} + \frac{1}{2} - S + o(1)$$

$$= 2 \sum_2^N \frac{1}{n^4-1} = \underbrace{2\left(\frac{1}{2^4-1} + \frac{1}{3^4-1}\right)}_{c} + 2 \sum_4^N \left(\frac{1}{n^4-1} - \frac{1}{(n^2-1)(n^2-9)}\right)$$

$$+ 2 \sum_4^N \frac{1}{(n^2-1)(n^2-9)}$$

$$= c + 2 \sum_4^N \frac{-10}{(n^4-1)(n^2-9)} + \frac{2}{48} \sum_4^N \left\{\frac{1}{n-3} - \frac{3}{n-1} + \frac{3}{n+1} - \frac{1}{n+3}\right\}$$

$$= c - 20 \sum_4^N \text{''} + \frac{1}{24}\left\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{3}{3} - \frac{3}{4}\right\} + o(1)$$

So $\frac{3}{4} + \frac{1}{2} - S = c - 20 \sum_4^\infty + \frac{7}{240}$

$$S = \frac{17}{16} + 20 \sum_4^\infty \qquad \text{terms in } \Sigma \text{ are } O(n^{-6})$$

16 terms give $20\Sigma = .01417$

$S = 1.07687$

We can get $O(n^{-8})$ by

$$\frac{1}{(n^2-25)(n^2-9)(n^2-1)} - \frac{1}{(n^2-9)(n^4-1)}$$

$$= \frac{26}{(n^2-25)(n^2-9)(n^4-1)}$$, while the 1st term is

$$\frac{1}{3840}\left\{\frac{1}{n-5} - \frac{5}{n-3} + \frac{10}{n-1} - \cdots - \frac{1}{n+5}\right\}$$

whose $\sum_{6}^{\infty}$ is $\frac{1}{3840}\left\{H_{10} - 5(H_8 - H_2) + 10\left(\frac{1}{5}+\frac{1}{6}\right)\right\}$

$$H_n = \sum_{1}^{n} \frac{1}{k}$$

$$= \frac{1}{3840} \cdot \frac{319}{630}$$

This leads to

$$S = \frac{17}{16} + 20\left\{\frac{1}{(4^4-1)(4^2-9)} + \frac{1}{(5^4-1)(5^2-9)}\right\} + \frac{319}{192 \cdot 630}$$

$$- 520 \sum_{6}^{\infty} \frac{1}{(n^2-25)(n^2-9)(n^4-1)}$$

6/19/75 Steph Horner's Markov Chain

$$p_{ij} = b(j; i+1, q) \qquad i,j \geq 0$$

It turns out to have stationary & limiting distr $\{\pi_j\}$ with

$$\sum_0^\infty \pi_j t^j = \prod_1^\infty \left(1-(1-t)q^k\right)$$

(Note value $\pi_0 = \prod_1^\infty (1-q^k)$; (cf 9/27/72)

$$= (\text{Euler})\ 1+\sum_1^\infty (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2}\right))$$

Nontransitivity

1	2	3	4	5	6	7	8	9
A					A		A	
		B		B		B		
	C		C					C
A					A		A	

3 3-man teams, A, B, C, strengths as shown.

A vs B 5 to 4
B vs C "
C vs A " !!

Sept 75 A problem of Hugh Montgomery. (Done by a student.)

$$a_0 = 1,\ a_{2n} = a_n,\ a_{2n+1} = (-1)^n a_n$$

$$\Longrightarrow \sum_1^\infty \frac{a_{2n+1}}{n} = 0$$

Soln (HLM). By a complicated argument $\sum \frac{a_n}{n}$ & $\sum \frac{(-1)^n a_n}{n}$ converge.

Then their sum is $2\sum \frac{a_{2n}}{2n} = \sum \frac{a_{2n}}{n}$

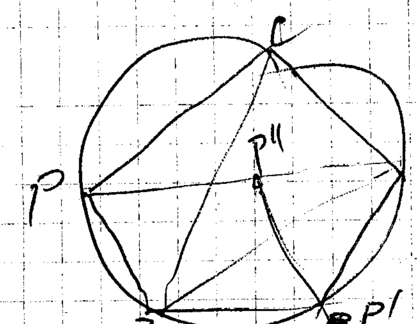
$= \sum \frac{a_n}{n}$; hence $\sum \frac{(-1)^n a_n}{n} = 0$,

which is simply $\sum \frac{a_{2n+1}}{n} = 0$!

From Tech Rev (MIT) June 75

$2 \mid (n-m)\ ,\ 3 \mid (n+m)$

$$\Longrightarrow \quad \frac{n^2-m^2}{\frac{2}{3}(n+m)} = \text{int.}$$



$\angle A = \angle B = \angle C = 60^\circ$

$\Longrightarrow AP = BP + CP$

Pf: $\angle BPA = \angle BCA = 60^\circ$

Draw P' on $\odot \ni AP' = BP$

Then $\angle P'AP = \angle BPA = 60^\circ$

$BP' = PC$

$BP \parallel AP$

Choose P'' on $AP \ni P'P'' \parallel BP$

Then

$PP'' = BP' = PC$

$PB = P'P'' = AP'$

$\therefore AP'P''$ Equilateral

$\therefore AP'' = PB$

$\therefore AP = AP'' + P''P = PB + PC$!

A bad Gilles problem again (cf Jan 74)

#6044 Monthly Aug & Sept 1975 p. 766

Show $\prod_{\substack{\alpha^{49}=1 \\ \alpha \neq 1}} (\alpha^4 + \alpha + 1) = 83^3 \ (= 571{,}787)$

False! HP-65 gives $5{,}039{,}063 \ (\approx 171.44^3)$

We note also that $x^{49} \equiv \underbrace{-664x^3 + 622x^2 + 1835x + 1133}_{=:\, f(x)}$ mod $x^4 + x + 1$

(HP-65: 49, STO 8, 1, STO 1, 0, STO 2, STO 3, STO 4, RCL 4, STO 5, RCL 3, STO 4,

RCL 2, STO 3, RCL 1, ~~STO 5~~ RCL 5, −, STO 2, g LSTx, CHS,
STO 1, g DSZ, GTO D, R/S)

So with $x^4+x+1 = \prod_1^4 (x-a_i)$ we have $\prod(\alpha^4+\alpha+1) = \prod\prod_1^4(\alpha-a_i)$

With $\alpha^{49}=1$ we have $(\alpha-a+a)^{49}-1=0$

$$(\alpha-a)^{49}+\dots+a^{49}-1=0$$

Hence $\prod_{\alpha^{49}=1}(\alpha-a) = a^{49}-1$

$$\prod_{\substack{\alpha^{49}=1\\ \alpha\neq 1}}(\alpha-a) = -\frac{a^{49}-1}{a-1}$$

$$\prod\prod = \frac{\prod_1^4(a_i^{49}-1)}{\underbrace{\prod_1^4(a_i-1)}_{3}}$$

$$= \frac{1}{3}\prod_1^4\left[f(a_i)-1\right]$$

for which further simplification is needed.

By HP, $\{a_i\} = \{1.18375^{\pm1};\ \cos\theta_+ = .61426$

$\cos\theta_- = -.86025\}$ 11/15/76

Then $\prod(x-a_i^7) = x^4-7x^3+14x^2-6x+1$

$$\prod(x-a_i^{49}) = x^4-6524x^3+15,119,748x^2+3963x+1$$

giving $\prod(a_i^{49}-1) = 1-6524+15119748+3963+1$

$$= 15117189$$

$$= 3\cdot 5,039,063$$

$$= 3\cdot 197\cdot 25579$$

9/28/76 To Feller's birthday problem

$$\log q = \sum_{k=1}^{n-1} \log\left(1-\frac{k}{y}\right) \qquad \text{E.g. } y=365,\ n=23$$

lies between $-\sum_1^{n-1}\frac{k}{y} = -\frac{n(n-1)}{2y}$

and
$$-\sum_1^{n-1}\left(\frac{k}{y}+\frac{1}{2}\frac{k^2}{y(y-k)}\right)$$
$$> -\frac{n(n-1)}{2y} - \frac{1}{2y(y-n+1)}\sum_1^{n-1}k^2$$
$$= -\frac{n(n-1)}{2y}\left(1+\frac{2n-1}{6(y-n+1)}\right)$$

already tight enough in the above case.

11/15/76 A curiousity (Cf. 9/24/74)

U uniform on $[0,1]$.

$\mathrm{Var}(U^t) = \max$ on $[0,\infty)$ for $t = \frac{\sqrt5+1}{2} = 1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}$

value of max $= \frac{5\sqrt5-11}{2}$

! $= \cfrac{1}{11+\cfrac{1}{11+\cfrac{1}{11+\cdots}}}$!

11/15/76 Abel $\int \frac{3x-7}{\sqrt{x(x-1)(x-4)(x-9)}}\,dx$ elementary. | 4/25/77 Cf AMM 79(1972)1148 for another ex.

(Cf. after 8/4/73).

Today I got ans. $\log \frac{x^2-7x+\sqrt{x(x-1)(x-4)(x-9)}}{x^2-7x-\sqrt{x(x-1)(x-4)(x-9)}} + C$.

Method: assume $\log\frac{P+\sqrt{Q}}{P-\sqrt{Q}}$, $Q = x(x-1)(x-4)(x-9)$

Then $\frac{d}{dx} = \frac{Q'P-2QP'}{(P^2-Q)\sqrt{Q}}$, so $(P^2-Q)L \overset{?}{=} Q'P-2QP'$

where $L = 3x-7$. We now try P quadratic such that P^2-Q is also quadratic and find that indeed $P^2-Q = 36x$ if $P = \cancel{x^2+}\ x^2-7x$. Then $Q'P-2QP' = 108x^2-252x$
$= (3x-7)(P^2-Q)$
$= L(P^2-Q)$

and we

11/20/76 HP-67 program for calendar uses $\left[\frac{a}{30.6001}\right]$ at a certain point; a is an integer. ? $\left[\frac{a}{30.6}\right] > \left[\frac{a}{30.6001}\right]$

For what pairs a, n is

$$\frac{a}{30.6001} < n \leq \frac{a}{30.6}$$

$$306000 \leq \frac{10^4 a}{n} < 306001$$

$$306000\, n \leq 10^4 a < 306001\, n$$

$$306000\, n + x = 10^4 a, \quad 0 \leq x < 1$$

$$306000\, n + y = 10^4 a, \quad 0 \leq y < n$$

$$306\, n + z = 10\, a, \quad 0 \leq z < \frac{n}{1000}$$

$$153\, n + w = 5\, a, \quad 0 \leq w < \frac{n}{2000}$$

$$5a - 153n = w$$

For $w = 0$, $a = 153t$, $n = 5t$ $\quad t = 1, 2$ significant for calendar

For $w = 1$, $a = 153t - 61$, $n = 5t - 2$

general $w \neq 0$, $a = (153t - 61)w$, $n = (5t-2)w > 2000w$

$$5t > 2002$$
$$t > 400$$

rewrite as $a = (61139 + 153t)w$, $t \geq 1$

or $\left.\begin{array}{l} a = (61392 + 153t)w \\ n = (2003 + 5t)w \end{array}\right\}$, $t \geq 0$

11/25/76 A theorem on projection of regular tetrahedron.

Let $P_i : (x_i, y_i, z_i)$ $i = 1, 2, 3, 4$ be vertices of a regular tetrahedron with center $O : (0, 0, 0)$ and $OP_i = 1$. Then $\sum x_i^2 = \sum y_i^2 = \sum z_i^2 = 4/3$,

$$\sum x_i = \sum y_i = \sum z_i = \sum x_i y_i = \cdots = 0.$$

Pf: Clearly $\sum x_i^2 + \sum y_i^2 + \sum z_i^2 = 4$. ~~The three sums are equal since cyclic permutation $x \to y \to z \to x$ preserves the properties.~~

~~Similarly $(\sum x_i, \sum y_i, \sum z_i) = \vec{0}$, $\therefore \sum x_i = \sum y_i = \sum z_i = 0$.~~

We have a standard tetrahedron with vertices $\Pi_i : (\xi_i, \eta_i, \zeta_i)$

$$= \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{6}}{3}, -\frac{1}{3}\right), \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right), (0, 0, 1)$$

and can rotate ~~any~~ this one into any other.

Thus $x_i = \xi_i \cos\alpha_1 + \eta_i \cos\beta_1 + \zeta_i \cos\gamma_1$

$y_i = \qquad {}_2 \qquad\qquad {}_2 \qquad\qquad {}_2$

$z_i = \qquad {}_3 \qquad\qquad {}_3 \qquad\qquad {}_3$

a / 30.6001

So $\sum x_i^2 = \sum(\xi_i \cos\alpha_1 + \dots)^2$

$= \sum \xi_i^2 \cos^2\alpha_1 + \sum \eta_i^2 \cos^2\beta_1 + \sum \zeta_i^2 \cos^2\gamma_1$

$= \frac{4}{3}(\cos^2\alpha_1 + \dots)$

$= 4/3$

and similarly for $\sum y_i^2$, $\sum z_i^2$.

$\sum x_i = \sum y_i = \sum z_i = 0$ express centroid at O.

$\sum x_i y_i = \sum(\xi_i \cos\alpha_1 + \dots)(\xi_i \cos\alpha_2 + \dots)$

$= \frac{4}{3}(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2)$

int $+ \left(\sum \xi_i \eta_i\right)(\cos\alpha_1\cos\beta_2 + \cos\alpha_2\cos\beta_1)$

ses $+$ two more terms

$= 0 + 0(\cos\alpha_1\cos\beta_2 + \cos\alpha_2\cos\beta_1) + 0 + 0$

$= 0$

which must be an unnecessarily complicated proof.

<u>Converse</u>: If $\sum x_i^2 = \sum y_i^2 = \frac{4}{3}$, $\sum x_i = \sum y_i = \sum x_i y_i = 0$ then $\exists z_i \ni P_i : (x_i, y_i, z_i)$ $(i = 1 \dots 4)$ are vertices of a regular tetrahedron with center at O and $OP_i = 1$.

<u>Pf</u>: In R^4 we have $u = (1,1,1,1)$, $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$. These are mutually $\perp$. We choose $z \perp \langle u, x, y \rangle$ w/ $\|z\|^2 = 4/3$. Then for $v = (1,0,0,0)$ we have

$v = \frac{(v,u)u}{\|u\|^2} + \frac{(v,x)x}{\|x\|^2} + \dots$

so $1 = \|v\|^2 = \frac{1}{4} + \frac{3}{4}(x_1^2 + y_1^2 + z_1^2)$

$\therefore x_1^2 + y_1^2 + z_1^2 = 1$ and similarly $x_2^2 + \dots, \dots$.

For $w = (0,1,0,0)$ we have $\qquad \cos\angle OP_1, OP_2 = -1/3$.

$0 = (v, w) = \frac{1}{4}u + \frac{3}{4}\cdot\frac{4}{\ }(x_1x_2 + y_1y_2 + z_1z_2) \therefore$

12/15 The foregoing can be viewed as characterizing $\perp$ projections of a regular unit (RUT) tetrahedron in space onto a plane.

We can also characterize oblique (parallel) projections.

For (ξ, η, ζ) in space oblique proj^n is defined by

$$(*) \quad \begin{aligned} x &= \xi - \alpha\zeta \\ y &= \eta - \beta\zeta \end{aligned}$$

for suitable α, β. Note: this forces the origin into the origin.

If (ξ_i, η_i, ζ_i) $i = 1,2,3,4$ are vertices of a RUT we have

$$(1ab) \quad \sum x_i = \sum y_i = 0$$

$$(**) \left\{ \begin{aligned} \sum x_i^2 &= \tfrac{4}{3}(1+\alpha^2) \\ \sum y_i^2 &= \tfrac{4}{3}(1+\beta^2) \\ \sum x_i y_i &= \tfrac{4}{3}\alpha\beta \end{aligned} \right.$$

and so

$$(2) \quad \left(\sum x_i y_i\right)^2 = \underbrace{\left(\sum x_i^2 - \tfrac{4}{3}\right)}_{\geq 0}\underbrace{\left(\sum y_i^2 - \tfrac{4}{3}\right)}_{\geq 0}$$

Thus, (1ab) and (2) are necessary condi^{n}s,

They are also sufficient,

Given vectors x, y in R^4, $\perp 1$,

we can find vectors ξ, η, ζ in $R^4 \ominus \langle 1 \rangle$ such that $(*)$ holds and $\|\xi\|^2 = \cdots = \frac{4}{3}$, (ξ, η, ζ) mutually $\perp$. α, β of $*$ are defined from $(**)$.

We first choose $u, v \in \langle x, y \rangle$ and biortho thereto. Then choose a unit vector $w \perp 1, x, y$.

Define $\zeta = -\frac{4}{3}\alpha u - \frac{4}{3}\beta v + (1+\alpha^2+\beta^2)^{-1/2}\gamma$

A well-known fact is that

2UT)

$$\begin{bmatrix} \|x\|^2 & (x,y) \\ (x,y) & \|y\|^2 \end{bmatrix} \begin{bmatrix} \|u\|^2 & (u,v) \\ (u,v) & \|v\|^2 \end{bmatrix} = I$$

From this a calculation shows that indeed $\|\zeta\|^2 = \frac{4}{3}$.

Define $\xi = x + \alpha\zeta$, $\eta = y + \beta\zeta$. Then

$$\|\xi\|^2 = \frac{4}{3}(1+\alpha^2) - 2\frac{4}{3}\alpha^2 + \alpha^2\frac{4}{3} = \frac{4}{3} = \|\eta\|^2,$$

$$(\xi,\eta) = (x,y) + \alpha(y,\zeta) + \beta(x,\zeta) + \alpha\beta\|\zeta\|^2$$
$$= \frac{4}{3}\alpha\beta + -\frac{4}{3}\alpha\beta - \frac{4}{3}\alpha\beta + \frac{4}{3}\alpha\beta = 0$$

$(\xi,\zeta) = (\eta,\zeta) = 0$ and since $\xi, \eta, \zeta \perp 1$ we are back to the argument of the previous page.

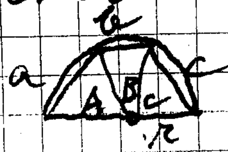
A final note. If (2) is replaced by

(3) $$\left(\sum x_i y_i\right)^2 = \left(\sum x_i^2 - \frac{4}{3}r^2\right)\left(\sum y_i^2 - \frac{4}{3}r^2\right)$$

then we get to a regular tetrahedron of radius r, by applying the above to $\frac{1}{r}x, \frac{1}{r}y$. Now $(\sum x_i y_i)^2 \leq \sum x_i^2 \sum y_i^2$, with equality iff $x_i \propto y_i$. Thus, if the four points (x_i, y_i) in the plane (with $\sum x_i = \sum y_i = 0$) are not collinear we can surely find $r \ni \frac{4}{3}r^2$ makes (3) true, $\frac{4}{3}r^2 \leq (\min \sum x_i^2, \sum y_i^2)$. *Hence* any four noncollinear points are *some* oblique projection of *some* regular tetrahedron.

12/15 O'Neill (#338) problem, HP V3N8P7

Find radius r of semicircle circumscribed about chords a, b, c.



From $2r \sin(A/2) = a$ etc get

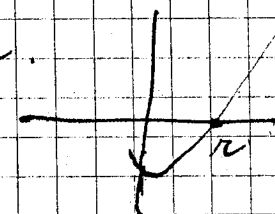
$$f(r) := 4r^3 - (a^2+b^2+c^2)r - abc = 0$$

This has unique + soln, which is $\le \max(a, b, c) =: \rho$, for from

$$4r^3 = Lr + M \qquad L := a^2+b^2+c^2$$
$$\le 3\rho^2 r + \rho^3 \qquad M := abc$$

we get $0 \geq (r-\rho)(2r+\rho)^2 \therefore r \le \rho$,

and $f(0) < 0$, f cubic.
$f'(0) < 0$



Newton iteration gives

$$r_{n+1} = \frac{8r_n^3 + abc}{12r_n^2 - a^2 - b^2 - c^2}$$

$r_0 = \rho$

Example: $a, b, c = 9, 11, 15$

$\Longrightarrow r = 11.76081$

Also, it's not hard to show

$$0 \le r_{n+1} - r \le (r_n - r)^2 \frac{12\rho^2}{2L\rho + 3M}$$

3/2/77 (cf 3/67, 8/4/73)

Replace p^2q by p^2qs, get

$$F(s) = \sum_1^\infty \frac{(p^2q)^n}{n!}\,\frac{3(3n-2)!}{(2n-1)!}\,s^n \qquad \leftarrow = \frac{2}{3n-1}\binom{3n}{n}$$

$$= 3p^2qs + 6p^4q^2s^2 + 21p^6q^3s^3 + \cdots$$

the gen fncn for $\frac{1}{3}$ the first return to 0 in $-1, +2$ (p,q) random walk.

Since $P(s) = \dfrac{1}{1-F(s)} = \sum_0^\infty \binom{3n}{n}(p^2qs)^n$

we get the remarkable identity

$$\binom{3n}{n} = \sum_1^n \frac{2}{3k-1}\binom{3k}{k}\binom{3n-3k}{n-k}$$

which HP-67 checks numerically for $n=9$.

3/23/77 (For Query 116 by FSlater in Feb notices)

Sum of two fncns, each with rt & left limits everywhere in $[0,1]$ and with range a Lebesgue null-set, need not have $m(\text{range}) = 0$.

$$x = \sum_1^\infty 2^{-i} b_i(x) \qquad b_i(\cdot) \text{ rt cont } 0 \text{ or } 1$$

$$f(x) = \sum_{i \text{ odd}} \qquad g(x) = \sum_{i \text{ even}}$$

both rt cont & have left limits, by unif conv

$f(x) = S_n(x) + R_n(x)$, S_n = n^{th} partial sum, taking 2^n values; values of $R_n(x)$ in an interval of length $\le c/4^n$ ∴ $m(\text{Range}) \le c/2^n \to 0$.
For $g(x)$, rescale.

4/5/787 (from Joe Ullman)

$$\sum_1^\infty \frac{(-1)^{n-1}x^{2n}}{(2n)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)$$

$$= \int_0^x \frac{\sin u}{u}\,du \cdot \sin x$$

$$- \int_0^x \frac{1-\cos u}{u}\,du \cdot \cos x$$

$$= o(1) + \alpha \sin x + (\beta - \log x)\cos x$$

where $\alpha = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

$$\beta = -\int_0^1 \frac{1-\cos u}{u}\,du + \int_1^\infty \frac{\cos u}{u}\,du$$

method: $1+\frac{1}{2}+\cdots+\frac{1}{2n} = \int_0^1 \frac{1-t^{2n}}{1-t}\,dt$

leading to $\Sigma = \int_0^1 \frac{\cos tx - \cos x}{1-t}\,dt$

$$= \int_0^1 \int_{tx}^x \frac{\sin u}{1-t}\,du\,dt$$

4/6/787 Question arising in prob. sem.

Z = zero-set of Brn Mtn $X(t)$.
$Y(t)$ = ind. Brn Mtn.

Is $Y(Z)$ dense in R?

Ans: Yes. A proof: let $t_0 \equiv 0$, $t_{n+1} = \inf\{t: t \in Z, t \geq t_n + 1\}$

Then $\{t_{n+1} - t_n\}$ iid $\therefore$ $\{Y(t_{n+1}) - Y(t_n)\}$ iid, non lattice,

Since t_1 has density $1/\pi t\sqrt{t-1}$ on $(1,\infty)$, $E|Y(t_1)|$ fails to exist. But cf $Y(t_1) = \frac{1}{\pi}\int_1^\infty \frac{e^{-\frac{1}{2}t\theta^2}}{t\sqrt{t-1}}\,dt = f(\theta) =$

= (very remarkably)* $P(|X| \geq |\theta|)$ where X is $N(0,1)$. Since $f(\theta) = 1 - c|\theta| + o(|\theta|)$, $|\theta| \to 0$, the sufficient condition $\exists\, \delta > 0 \ni \varlimsup_{r\nearrow 1} \int_{-\delta}^{\delta} (1 - rf(\theta))^{-1}\,d\theta = \infty$ applies

and we conclude that the random walk $S_n = Y(t_n)$ is recurrent.

Oberhettinger (Fourier transforms of distributions) shows that our $f(\theta)$ corresponds to the density

$$\left(\frac{2}{\pi}\right)^{3/2} e^{-y^2/2}\int_0^1 e^{y^2u^2/2}\,du\,, \quad -\infty<y<\infty.$$

Ref QA273.6 formula (281).

I confirm this by direct calculation of the density of $Y(t_1)$ as

$$\frac{1}{\pi}\int_1^\infty \frac{e^{-y^2/2t}}{\sqrt{2\pi t}\; t\sqrt{t-1}}\,dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{e^{-\frac{y^2}{2}t}}{\sqrt{2\pi}\sqrt{\frac{1}{t}}\,t\sqrt{\frac{1}{t}-1}}\,dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{e^{-\frac{y^2}{2}(1-t)}}{\sqrt{2\pi}\;\sqrt{t}}\,dt$$

$$= \frac{1}{\pi}\, e^{-\frac{y^2}{2}}\frac{1}{\sqrt{2\pi}}\int_0^1 \frac{e^{+y^2t/2}}{\sqrt{t}}\,dt$$

$$= \frac{\sqrt{2}}{\pi^{3/2}}\, e^{-y^2/2}\int_0^1 e^{+y^2t^2/2}\,dt$$

factor 2? maybe symmetry.

* not so remarkable. Pólya's thm shows that $\int_{|\theta|}^\infty f(x)\,dx$ will be a cf if $f \searrow$ and $\int_0^\infty f(x)\,dx=1$

2/16/78 from Martin Gardner coln Feb Sci Am

Dice $\{1,2,2,3,3,4\}$ $\{1,3,4,5,6,8\}$ have same sum-distn as conventional ones.

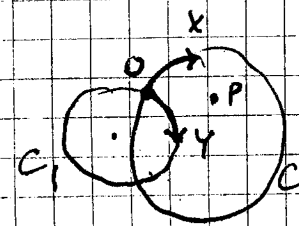
Zeros of gen fns $\quad \omega^6=1$, ~~with~~ ω primitive

$\cancel{0,\omega,\omega^2,\omega^3}$ $\quad \cancel{0,\omega,\omega^2,\omega^3,\omega^4,\omega^4,\omega^5,\omega^5}$

$0,\omega^2,\omega^3,\omega^4$ $\quad 0,\omega,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^5$

"Sicherman's Dice"

3/12/80 Probs from XXI International Olympiad
Feb. '80 Monthly

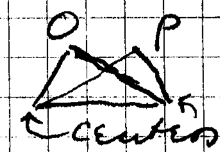


$\exists P$ s.t.
$\angle OX = \angle OY$
$\Rightarrow \overline{PX} = \overline{PY}$

solution: In $\mathbb{C}$ write $c_2 = a + ze^{i\theta}$ so $c_1 = (a+z)e^{i\theta}$ with $\underline{a}$ real, and take $P = -\bar{z}$. Verify $|c_1 - P| = |c_2 - P|$.

(Joe U observed that you can reverse OY and take $P = -z$ with same conclusion)

Location of P:



Show $1979 \mid 1319! \sum_1^{1319} \frac{(-1)^{n-1}}{n}$

Start

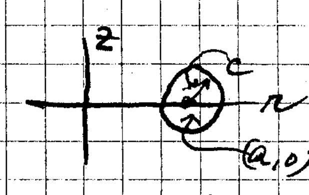
Case of $\frac{3x+1}{2} \mid x! \cdot \sum_1^{x}$ " where $x, \frac{3x+1}{2}$ both prime. Smaller examples are (3,5), (7,11), (11,17) for which the result holds.

Ullman Problem: apart from the two obvious ones is there another plane that cuts a torus in two circles, e.g. the internally tangent one?

Joe
ma

Ans: yes.

Torus (cyl. coords.) $(r-a)^2 + z^2 = c^2$



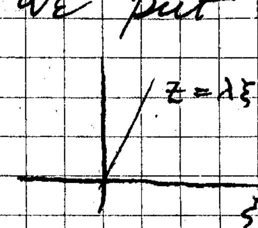
$$r^2 = \xi^2 + y^2$$

$$(r^2 + \cancel{z}^2 + a^2 - c^2)^2 = 4a^2(\xi^2 + y^2)$$

$z = \lambda\xi$, cutting plane

$$((1+\lambda^2)\xi^2 + y^2 + a^2 - c^2)^2 = 4a^2(\xi^2+y^2)$$

In the cutting plane a generic point is $(\xi, y, \lambda\xi)$. A change $\Delta\xi$ of ξ moves the point a distance $\sqrt{1+\lambda^2}\,\Delta\xi$. We put $\xi = \dfrac{x}{\sqrt{1+\lambda^2}}$, so $\sqrt{1+\lambda^2}\,\Delta\xi = \Delta x$



Eqn becomes

$$(x^2+y^2+a^2-c^2)^2 = 4a^2\left(\frac{x^2}{1+\lambda^2} + y^2\right)$$

$\lambda \to \infty$ gives $\quad x^2+y^2+a^2-c^2 = \pm 2ay$

$$x^2 + (y \mp a)^2 = c^2$$

$\lambda = 0$ gives

$$\begin{aligned} &x^4+y^4+a^4+c^4 \\ &+2x^2y^2 - 2x^2c^2 - 2y^2c^2 \\ &-2a^2c^2 - 2x^2a^2 - 2y^2a^2 = 0 \end{aligned}$$

$$(x^2+y^2-a^2-c^2)^2 - 4a^2c^2 = 0$$

$$x^2+y^2 = (a \pm c)^2$$

$\lambda^2 = \dfrac{c^2}{a^2-c^2}$ is the tangency condition. It gives eventually $\quad x^2 + (y \pm c)^2 = a^2 \quad$!!

Joe says
maybe Polya.

not (10/22/86)

9/2/80 (from 8/16) A difficult integral:

$$\int \frac{\sqrt{x+\sqrt{x^2+a^2}}}{x}\,dx$$

(Computer algebra short course, UM)

9/22/80 (See also Putnam Practice notes, 1977 esk. prob. 6)

$$a^x = x$$

			Iteration cvgs
$a > e^{1/e}$	:	no solution	
$a = e^{1/e}$	:	one solution, e	for $x_0 \le e$
$1 < a < e^{1/e}$	:	two solutions	for $x_0 <$ larger
$a = 1$	:	one soln, 1	for x_0 ~~...~~ arb.
$e^{-e} < a < 1$	:	one soln	for all x_0
1) $a = e^{-e}$	:	one soln	"
2) $0 < a < e^{-e}$	:	" "	for no x

To 1): Soln is $x = \frac{1}{e}$. Let $x_n = \frac{1}{e}(\varepsilon_n + 1)$.
Iteration is

$$x_{n+1} = e^{-e x_n}$$

$$\frac{1}{e}(\varepsilon_{n+1} + 1) = e^{-\varepsilon_n - 1}$$

$$\varepsilon_{n+1} + 1 = e^{-\varepsilon_n}$$

$$\varepsilon_{n+1} = e^{-\varepsilon_n} - 1$$

$$\varepsilon_{n+2} = e^{-e^{-\varepsilon_n} + 1} - 1$$

And see that $e^{1-e^{-x}} - 1 < x$ for $x > 0$
$> x$ for $x < 1$

by integrating $e^{-x} \le \frac{1}{1+x}$ (from $1 + x \le e^x$)
and exponentiating.

2) Here a^x has slope < -1 at $a^x = x$.
This forces

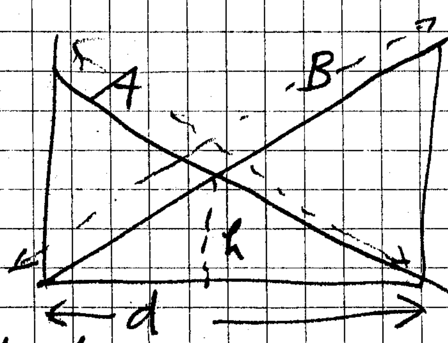
$$|x_{n+2} - x| > |x_n - x|, \text{ for } x \text{ the root.}$$

And in this case. $x = a^{a^x}$ has 3

[redacted]

roots, the middle one being root of $a^+ = x$, unstable for iteration, the others being alternately limit points for the original iteration.

Ladder problem.



given A, B, h find d.

Soln: $1 = \dfrac{h}{\sqrt{B^2-d^2}} + \dfrac{h}{\sqrt{A^2-d^2}}$ by similar triangles. Solve for d.

$$(A^2-d^2)(B^2-d^2) = h^2(A^2-d^2+B^2-d^2+2\sqrt{\ }\sqrt{\ })$$

$$[(d^2-A^2)(d^2-B^2) - h^2(A^2+B^2-2d^2)]^2 = 4h^4(A^2-d^2)(B^2-d^2)$$

$$d^2 = x, \quad h^2 = H, \quad c = A^2+B^2+2H, \quad d = A^2B^2 - H(A^2+B^2)$$

$$(x^2 - cx + d)^2 = H^2(x^2 - ex + f) \qquad e = A^2+B^2, \quad f = A^2B^2$$

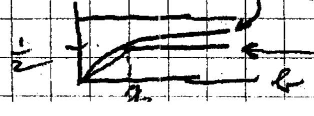
$$x^4 - 2cx^3 + (c^2 - H^2 + 2d)x^2 + (-2cd + H^2 e)x + d^2 - H^2 f = 0.$$

3/12/81 Joe Ullman problem: Prove $\sum a_n = \infty$, $a_n \downarrow$ $\Rightarrow \sum a_n(1+na_n)^{-1} = \infty$.

Pf: clearly $a_n(1+na_n)^{-1} \downarrow$. ∴ Cauchy condensation $\Rightarrow$ look at

$$\sum \frac{2^n a_{2^n}}{1+2^n a_{2^n}} = \sum \frac{b_n}{1+b_n}, \quad \sum b_n = \infty$$

∴ $\sum \dfrac{b_n}{1+b_n} = \infty$ since $\dfrac{b}{1+b} \geq \frac{1}{2} I_{b\geq 1} + \frac{1}{2} b\, I_{b<1}$



3/28/81 Hoppe problem — from "Simpson's Paradox" (Lindley? QA273. [746])

$$\begin{bmatrix} 20 & 20 \\ 24 & 16 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 21 & 9 \end{bmatrix} + \begin{bmatrix} 12 & 18 \\ 3 & 7 \end{bmatrix}$$

det < 0 det > 0 det > 0

always positive with non-neg entries? Ans: yes.

Pf: $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\rho}{4}a & \frac{b}{2} \\ \frac{c}{2} & (1-\frac{\rho}{4})d \end{bmatrix} + \begin{bmatrix} (1-\frac{\rho}{4})a & \frac{b}{2} \\ \frac{c}{2} & \frac{\rho}{4}d \end{bmatrix}$$

with $\rho = \frac{bc}{ad}$.

11/19/81 A query of Hoppe. If Y has a density g on $\mathbb{R}$ does

$$((10^n Y)) \equiv 10^n Y - [10^n Y] \xrightarrow[\mathcal{D}]{} \text{unif}(0,1)?$$

Ans: Yes & in more generality.

Thm $((tY)) \to$ unif, $t \to \infty$.

Pf: We have to show

$$E\left(\exp 2\pi i k ((tY))\right) \to \delta_{k,0}, \quad k = 0, \pm 1, \dots$$

This is $$\int_{-\infty}^{\infty} \exp 2\pi i k ((ty))\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} \exp(2\pi i k t y)\, g(y)\, dy \longrightarrow 0 \text{ by}$$

the Riemann-Lebesgue lemma.

[redacted]

12/8/81 By integration, $\min_{1 \le i \ne j \le n} |X_i - X_j|$ is exponential with parameter $\frac{n(n-1)}{2}$, $n \geq 2$, X_i ind. exponential para 1. (Question of Hugh Montgomery on a 425 Bluebook.)

9/9/82 (Mont, from Hinman)

$$a_{n+1} = |a_n| - a_{n-1}$$

has period 9 !

9/21/82 (Hugh) X_i unif $0, 1, \cdots, n-1$ $i = 1, \ldots, 3$

$S = X_1 + \cdots + X_3$

$P(3 | S) \geq 1/4$ (divides)

Sol'n: $P(3|S) = \frac{1}{3} \pm \begin{cases} \frac{2}{3n^3}, & \begin{matrix} + \text{ for } n \equiv 1 \\ - \text{ for } n \equiv -1 \end{matrix} \\ 0, & \text{for } 3|n \end{cases}$

4/3/83 Is $\frac{d^2}{dx^2} \frac{x^{a+1} - x}{x - 1} \geq 0$ on $(0,1)$?

for $\underline{a}$ an integer, it is $\frac{d^2}{dx^2} \underbrace{(x^n + \cdots + x)}_{n}$

so yes.

It works out as

$$[a(a-1)x^{a+1} - 2(a^2-1)x^a + (a+1)a\,x^{a-1} - 2](x-1)^{-3}$$

$a > 1$: True in general. Integrate twice from x to 1 the inequality $\cancel{1 \le \frac{1}{t^{a+1}}}$

$2a(a-1) \le \frac{2a(a-1)}{t^{a+1}}$. For $0 < a < 1$ it is reversed.

Math Comp 40 (1983) 561–563 7/13/83
M.L. Glasser

If $\{a_j\}$ pos consts ~~et~~ and

$$u = x - \sum \frac{a_j}{x - c_j}$$

then $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(u)\,dx$$

Problem from Mel Hochster 8/17/83

Given $z_1, \dots, z_5 \in \mathbb{C}$ with $|z_j| = 1$,

$$\sum_j z_j = \sum_j z_j z_{j+1} = 0 \quad (j+1 \bmod 5)$$

$\Longrightarrow \forall j,\ z_j = \alpha\omega^j$ for some α, $|\alpha| = 1$,

where $\omega^5 = 1$. (? Generalization to 7, 9, 11, … ?)

Mel gives someone's (Lovasz) elementary solution.

$$0 = \left(\sum z_j\right)\left(\sum \bar{z}_k\right) = 5 + \sum_{j \neq k} z_j \bar{z}_k = 5 + 2\,\mathrm{Re}\left(\sum z_j \bar{z}_{j+1} + \sum z_j \bar{z}_{j+2}\right)$$

$$\cancel{\left(= 5 + 2\sum z_j z_{j+2} \text{ without } -\right)}$$

$$0 = \sum_j z_j z_{j+1} \cdot \sum_k \bar{z}_k \bar{z}_{k+1}$$

$$\cancel{= 5 + 2\,\mathrm{Re}\left(\sum_j z_j \bar{z}_{j+2} + \sum_j z_j z_{j+1} \bar{z}_{j+2} \bar{z}_{j+3}\right)}$$

$$\cancel{\mathrm{Re} \sum z_j \bar{z}_{j+1} = \mathrm{Re} \sum z_j z_{j+1} \bar{z}_{j+2} \bar{z}_{j+3}}$$

Hoppe: N&S for $p \neq 0$ to have a $\frac{8/26/83}{\mathrm{cf}\sqrt{\ }}$?

10/14/84

Shanks' swindle, perhaps unconscious.
An oldie

Show that $A = B$, with

$$A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$$

$$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 + 10\sqrt{29}}}$$

His mimeographed sheet entitled
INCREDIBLE IDENTITIES
argues from Galois theory applied to the quartic [sic!] that A satisfies. It turns out that the roots are

$$\pm\sqrt{5} \pm \sqrt{11 - 2\sqrt{29}} \pm \sqrt{11 + 2\sqrt{29}}$$

with 1 or 3 plus signs, and the clue is that $\sqrt{5} = \sqrt{11 - 2\sqrt{29}}\,\sqrt{11 + 2\sqrt{29}}$. Without using quartics or Galois theory one finds that with $\alpha = \sqrt{11 - 2\sqrt{29}}$, $\beta = \sqrt{11 + 2\sqrt{29}}$

$$A = \alpha\beta + \sqrt{\alpha^2 + 2\alpha\beta + \beta^2}$$
$$B = \beta + \sqrt{\alpha^2(1 + \beta^2) + 2\alpha\beta \cdot \alpha}$$

which plainly are disguises of each other.

$\sum z_j \bar{z}_{j+2}$

Apr / May 1985 – correspondence with R. Bumby about this. He discovered independently.

6/21/86

This integral came from Computer algebra conf. summer 1980 in Ann Arbor. It was supposed to be hard and is so via trig sub. I told Alan Adler about it and he cracked it in an afternoon via trigh sub. Viz.:

$$\int \frac{\sqrt{x + \sqrt{x^2 + a^2}}}{x}\,dx = \sqrt{a}\int \frac{e^{t/2}}{\sinh t}\cosh t\,dt$$

$$= 2\sqrt{a}\int \frac{\bar{x}^2 + \bar{x}^{-2}}{\bar{x}^2 - \bar{x}^{-2}}\,d\bar{x} = 2\sqrt{a}\int \frac{u^4 + 1}{u^4 - 1}\,du \quad \text{etc}$$

6/21/85

Bob Young's adv. problem in the Feb. monthly. Show that $\sum_{n=1}^{\infty} \lambda_n^{-2} = \frac{1}{10}$, where λ_n runs through the positive zeros of $\tan x - x = 0$.

Solution. Using Cauchy's partial fraction expansion $f(\zeta) = \sum \operatorname{Res} \frac{f(z)}{\zeta - z}$ over the poles of f. (See Copson)

This gives $\tan \zeta = \sum_{n=1}^{\infty} \frac{1}{\zeta^2 - (2n-1)^2 \frac{\pi^2}{4}}$

And $\frac{\tan^2 \zeta}{\tan \zeta - \zeta} = \frac{3}{\zeta} + 2\zeta \sum \frac{1}{\zeta^2 - \lambda_n^2} + \tan \zeta$,

the last term coming from the series above. Etc.

10/26/88

A thing I learned from Hugh Montgomery sometime in the late 70's. Due to Wm Gosper.

$$\zeta(3) = \frac{5}{2} \sum_{1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}$$

See Apery paper in Math Intell. v. 1.

3/2/91 see 3/28/81 Chung's Elementary book gives

A	1000	50	95	5000
D	9000	950	5	5000
	City Res.		Rural res.	
	T	U	T	U

Alive Dead Treated Untreated