A SHOCK WAVE REFINEMENT of the FRIEDMANN-ROBERTSON-WALKER METRIC

November 28, 2003

Joel Smoller[1] *Blake Temple*[2]

1 Introduction

In the standard model of cosmology, the expanding universe of galaxies is described by a Friedmann-Robertson-Walker (FRW) metric, which in spherical coordinates has a line element given by [3, 47],

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2[d\theta^2 + sin^2\theta\ d\phi^2]\right\}. \tag{1.1}$$

In this model, which accounts for things on the largest length scale, the universe is approximated by a space of uniform density and pressure at each fixed time, and the expansion rate is determined by the cosmological scale factor $R(t)$ that evolves according to the Einstein equations. Astronomical observations show that the galaxies are uniform on a scale of about one billion lightyears, and the expansion is critical—that is, $k = 0$ in (1.1)—and so, according to (1.1), on the largest scale, the universe is infinite flat Euclidian space R^3 at each fixed time. Matching the Hubble constant to its observed values, and invoking the Einstein equations, the FRW model implies that the entire infinite universe R^3 emerged all at once from a singularity, (R=0), some 14 billion years ago, and this event is referred to as the Big Bang.

In this paper, which summarizes the work of the authors in [33, 40], we describe a two parameter family of exact solutions of the Einstein equations that refine the FRW metric by a spherical shock wave cut-off. In these exact solutions the expanding FRW metric is reduced to a region of finite extent and finite total mass

[1]Department of Mathematics, University of Michigan, Ann Arbor, MI 48109; Supported in part by NSF Applied Mathematics Grant Number DMS-010-3998, and by the Institute of Theoretical Dynamics, UC-Davis.

[2]Department of Mathematics, University of California, Davis, Davis CA 95616; Supported in part by NSF Applied Mathematics Grant Number DMS-010-2493, and by the Institute of Theoretical Dynamics, UC-Davis.

at each fixed time, and this FRW region is bounded by an entropy satisfying shock wave that emerges from the origin, (the center of the explosion), at the instant of the Big Bang $t = 0$. The shock wave, which marks the leading edge of the FRW expansion, propagates outward into a larger ambient spacetime from time $t = 0$ onward. Thus, in this refinement of the FRW metric, the Big Bang that set the galaxies in motion is an explosion of finite mass that looks more like a classical shock wave explosion than does the Big Bang of the Standard Model[3].

In order to construct a mathematically simple family of shock wave refinements of the FRW metric that meet the Einstein equations exactly, we assume $k = 0$, (critical expansion), and we restrict to the case that the sound speed in the fluid on the FRW side of the shock wave is constant. That is, we assume an FRW equation of state $p = \sigma\rho$, where σ, the square of the sound speed $\sqrt{\frac{\partial p}{\partial \rho}}$, is constant, $0 < \sigma \leq c^2$. At $\sigma = c^2/3$, this catches the important equation of state $p = \frac{c^2}{3}\rho$ which is correct at the earliest stage of Big Bang physics, [47]. Also, as σ ranges from 0 to c^2, we obtain qualitatively correct approximations to general equations of state. Taking $c = 1$, (we use the convention that $c = 1$, and Newton's constant $\mathcal{G} = 1$ when convenient), the family of solutions is then determined by two parameters, $0 < \sigma \leq 1$ and $r_* \geq 0$. The second parameter r_* is the FRW radial coordinate r of the shock in the limit $t \to 0$, the instant of the Big Bang[4]. The FRW radial coordinate r is singular with respect to radial arclength $\bar{r} = rR$ at the Big Bang $R = 0$, so setting $r_* > 0$ does not place the shock wave away from the origin at time $t = 0$. The distance from the FRW center to the shock wave tends to zero in the limit $t \to 0$ even when $r_* > 0$. In the limit $r_* \to \infty$ we recover from the family of solutions the usual (infinite) FRW metric with equation of state $p = \sigma\rho$. That is, we recover the standard FRW metric in the limit that the shock wave is infinitely far out. In this sense our family of exact solutions of the Einstein equations represents a two parameter refinement of the standard Friedmann-Robertson-Walker metric.

The exact solutions for the case $r_* = 0$ were first constructed in [33], and are

[3]The fact that the entire infinite space R^3 emerges at the instant of the Big Bang, is, loosely speaking, a consequence of the *Copernican Principle*, the principle that the earth is not in a special place in the universe *on the largest scale of things*. With a shock wave present, the *Copernican Principle* is violated in the sense that the earth then has a special position relative to the shock wave. But of course, in these shock wave refinements of the FRW metric, there is a spacetime on the other side of the shock wave, beyond the galaxies, and so the scale of uniformity of the FRW metric, the scale on which the density of the galaxies is uniform, is no longer the largest length scale.

[4]Since when $k = 0$, the FRW metric is invariant under the rescaling $r \to \alpha r$ and $R \to \alpha^{-1}R$, we fix the radial coordinate r by fixing the scale factor α with the condition that $R(t_0) = 1$ for some time t_0, say present time.

qualitatively different from the solutions when $r_* > 0$, which were constructed later in [40]. The difference is that when $r_* = 0$, the shock wave lies closer than one Hubble length from the center of the FRW spacetime throughout its motion, but when $r_* > 0$, the shock wave emerges at the Big Bang at a distance beyond one Hubble length. (The Hubble length depends on time, and tends to zero as $t \to 0$.) We show in [40] that one Hubble length, equal to $\frac{c}{H}$ where $H = \frac{\dot{R}}{R}$, is a critical length scale in a $k = 0$ FRW metric because the total mass inside one Hubble length has a Schwarzschild radius equal exactly to one Hubble length[5]. That is, one Hubble length marks precisely the distance at which the Schwarzschild radius $\bar{r}_s \equiv 2M$ of the mass M inside a radial shock wave at distance $\bar{r}$ from the FRW center, crosses from inside ($\bar{r}_s < \bar{r}$) to outside ($\bar{r}_s > \bar{r}$) the shock wave. If the shock wave is at a distance closer than one Hubble length from the FRW center, then $2M < \bar{r}$ and we say that the solution lies *outside the Black Hole*, but if the shock wave is at a distance greater than one Hubble length, then $2M > \bar{r}$ at the shock, and we say the solution lies *inside the Black Hole.* Since M increases like $\bar{r}^3$, it follows that $2M < \bar{r}$ for $\bar{r}$ sufficiently small, and $2M > \bar{r}$ for $\bar{r}$ sufficiently large, so there must be a critical radius at which $2M = \bar{r}$, and in Section 2, (taken from [40]), we show that when $k = 0$, this critical radius is exactly the Hubble length. When the parameter $r_* = 0$, the family of solutions for $0 < \sigma \leq 1$ starts at the Big Bang, and evolves thereafter *outside* the Black Hole, satisfying $\frac{2M}{\bar{r}} < 1$ everywhere from $t = 0$ onward. But when $r_* > 0$, the shock wave is further out than one Hubble length at the instant of the Big Bang, and the solution begins with $\frac{2M}{\bar{r}} > 1$ at the shock wave. From this time onward, the spacetime expands until eventually the Hubble length catches up to the shock wave at $\frac{2M}{\bar{r}} = 1$, and then passes the shock wave, making $\frac{2M}{\bar{r}} < 1$ thereafter. Thus when $r_* > 0$, the whole spacetime begins *inside the Black Hole*, (with $\frac{2M}{\bar{r}} > 1$ for sufficiently large $\bar{r}$), but eventually evolves to a solution *outside the Black Hole.* The time when $\bar{r} = 2M$ actually marks the event horizon of a *White Hole*, (the time reversal of a Black Hole), in the ambient spacetime beyond the shock wave. We show that when $r_* > 0$, the time when the Hubble length catches up to the shock wave comes after the time when the shock wave comes into view at the FRW center, and when $2M = \bar{r}$, (assuming t is so large that we can neglect the pressure from this time onward), the whole solution emerges from the White Hole as a finite ball of mass expanding into empty space, satisfying $\frac{2M}{\bar{r}} < 1$ everywhere thereafter. In fact, when $r_* > 0$, the zero pressure Oppenheimer-Snyder solution *outside the Black*

[5]Since c/H is a good estimate for the age of the universe, it follows that the Hubble length c/H is approximately the distance of light travel starting at the Big Bang up until present time. In this sense, the Hubble length is a rough estimate for the distance to the further most objects visible in the universe.

Hole gives the large time asymptotics of the solution, (c.f. [27, 41, 35] and the comments after Theorems 6-8 below).

The exact solutions in the case $r_* = 0$ give a general relativistic version of an explosion into a static, singular, isothermal sphere of gas, qualitatively similar to the corresponding classical explosion outside the Black Hole, [33]. The main difference physically between the cases $r_* > 0$ and $r_* = 0$ is that when $r_* > 0$, (the case when the shock wave emerges from the Big Bang at a distance beyond one Hubble length), a large region of uniform expansion is created behind the shock wave at the instant of the Big Bang. Thus, when $r_* > 0$, lightlike information about the shock wave propagates *inward from the wave*, rather than *outward from the center*, as is the case when $r_* = 0$ and the shock lies inside one Hubble length[6]. It follows that when $r_* > 0$, an observer positioned in the FRW spacetime *inside* the shock wave, will see exactly what the standard model of cosmology predicts, up until the time when the shock wave comes into view in the far field. In this sense, the case $r_* > 0$ gives a Black Hole cosmology that refines the standard FRW model of cosmology to the case of finite mass. One of the surprising differences between the case $r_* = 0$ and the case $r_* > 0$ is that, when $r_* > 0$, the important equation of state $p = \frac{1}{3}\rho$ comes out of the analysis as special at the Big Bang. When $r_* > 0$, the shock wave emerges at the instant of the Big Bang at a finite non-zero speed, (the speed of light), *only* for the special value $\sigma = 1/3$. In this case, the equation of state on both sides of the shock wave tends to the correct relation $p = \frac{1}{3}\rho$ as $t \to 0$, and the shock wave decelerates to subliminous speed for all positive times thereafter, (see [40] and Theorem 8 below).

In all cases $0 < \sigma \leq 1$, $r_* \geq 0$, the spacetime metric that lies beyond the shock wave is taken to be a metric of Tolmann-Oppenheimer-Volkoff (TOV) form,

$$ds^2 = -B(\bar{r})d\bar{t}^2 + A^{-1}(\bar{r})d\bar{r}^2 + \bar{r}^2[d\theta^2 + sin^2\theta\ d\phi^2]. \tag{1.2}$$

The metric (1.3) is in standard Schwarzschild coordinates, (diagonal with radial coordinate equal to the area of the spheres of symmetry), and the metric components depend only on the radial coordinate $\bar{r}$. Barred coordinates are used to distinguish TOV coordinates from unbarred FRW coordinates for shock matching. The mass function $M(\bar{r})$ enters as a metric component through the relation,

[6] One can imagine that when $r_* > 0$, the shock wave can get out through a great deal of matter early on when everything is dense and compressed, and still not violate the speed of light bound. Thus when $r_* > 0$, the shock wave "thermalizes", or more accurately "makes uniform", a large region at the center, early on in the explosion.

$$A = 1 - \frac{2M(\bar{r})}{\bar{r}}. \tag{1.3}$$

The TOV metric (1.3) has a very different character depending on whether $A > 0$ or $A < 0$; that is, depending on whether the solution lies *outside the Black Hole* or *inside the Black Hole.* In the case $A > 0$, $\bar{r}$ is a spacelike coordinate, and the TOV metric describes a static fluid sphere in general relativity.[7] When $A < 0$, $\bar{r}$ is the timelike coordinate, and (1.3) is a dynamical metric that evolves in time. The exact shock wave solutions are obtained by taking $\bar{r} = R(t)r$ to match the spheres of symmetry, and then matching the metrics (1.1) and (1.3) at an interface $\bar{r} = \bar{r}(t)$ across which the metrics are Lipschitz continuous. This can be done in general. In order for the interface to be a physically meaningful shock surface, we use the result in Theorem 4 below, (see [32]), that a single additional conservation constraint is sufficient to rule out delta function sources at the shock, (the Einstein equations $G = \kappa T$ are second order in the metric, and so delta function sources will in general be present at a Lipschitz continuous matching of metrics), and guarantee that the matched metric solves the Einstein equations in the weak sense. The Lipschitz matching of the metrics, together with the conservation constraint, leads to a system of ordinary differential equations (ODE's) that determine the shock position, together with the TOV density and pressure at the shock. Since the TOV metric depends only on $\bar{r}$, the equations thus determine the TOV spacetime beyond the shock wave. To obtain a physically meaningful outgoing shock wave, we impose the constriant $\bar{p} \leq \bar{\rho}$ to ensure that the equation of state on the TOV side of the shock is qualitatively reasonable, and as the entropy condition we impose the condition that the shock be compressive. For an outgoing shock wave, this is the condition $\rho > \bar{\rho}$, $p > \bar{p}$, that the pressure and density be larger on the side of the shock the receives the mass flux—the FRW side when the shock wave is propagating away from the FRW center. This condition breaks the time reversal symmetry of the equations, and is sufficient to rule out rarefaction shocks in classical gas dynamics, [30, 40]. The ODE's, together with the equation of state bound and the conservation and entropy constraints, determine a *unique* solution of the ODE's for every $0 < \sigma \leq 1$ and $\bar{r}_* \geq 0$, and this provides the two parameter family of solutions discussed here, [33, 40]. The Lipschitz matching of the metrics implies that the total mass M is continuous across the interface, and so when $r_* > 0$, the total mass of the entire solution, inside and outside the shock wave, is finite at each time $t > 0$, and both the FRW and TOV spacetimes emerge at

[7] The metric (1.3) is, for example, the starting point for the stability limits of Buchdahl and Chandresekhar for stars, [47, 36, 37].

the Big Bang. The total mass M on the FRW side of the shock has the meaning of total mass inside radius $\bar{r}$ at fixed time, but on the TOV side of the shock, M does not evolve according to equations that give it the interpretation as a total mass because the metric is *inside the Black Hole*. Nevertheless, after the spacetime emerges from the Black Hole, the total mass takes on its usual meaning outside the Black Hole, and time asymptotically the Big Bang ends with an expansion of finite total mass in the usual sense. Thus, when $r_* > 0$, our shock wave refinement of the FRW metric leads to a Big Bang of *finite total mass*.

A final comment is in order regarding our overall philosophy. The family of exact shock wave solutions described here are rough models in the sense that the equation of state on the FRW side satisfies $\sigma = const.$, and the equation of state on the TOV side is determined by the equations, and therefore cannot be imposed. Nevertheless, the bounds on the equations of state imply that the equations of state are qualitatively reasonable, and we expect that this family of solutions will capture the gross dynamics of solutions when more general equations of state are imposed. For more general equations of state, other waves, such as rarefaction waves and entropy waves, would need to be present to meet the conservation constraint, and thereby mediate the transition across the shock wave. Such transitional waves would be pretty much impossible to model in an exact solution. But the fact that we can find global solutions that meet our physical bounds, and that are qualitatively the same for all values of $\sigma \in (0, 1]$ and all initial shock positions, strongly suggests that such a shock wave would be the dominant wave in a large class of problems.

In Section 2 we derive the FRW solution when $\sigma = const.$, and discuss the Hubble length as a critical length scale. In Section 3 we state the general theorems in [32] for matching gravitational metrics across shock waves. In Section 4 we discuss the construction of the family of solutions in the case $r_* = 0$, and in Section 5 we discuss the case $r_* > 0$. (See [33, 40, 41] for details.)

2 The FRW Metric

According to Einstein's Theory of General Relativity, all properties of the gravitational field are determined by a Lorentzian spacetime metric tensor g, whose line element in a given coordinate system $x = (x^0, ..., x^3)$ is given by

$$ds^2 = g_{ij}dx^i dx^j. \tag{2.1}$$

(We use the Einstein summation convention whereby repeated up-down indices are assumed summed from 0 to 3.) The components g_{ij} of the gravitational metric

g satisfy the Einstein equations,

$$G^{ij} = \kappa T^{ij}, \quad T^{ij} = (\rho c^2 + p)w^i w^j + p g^{ij}, \tag{2.2}$$

where we assume the stress-energy tensor T of a perfect fluid. Here G is the Einstein curvature tensor,

$$\kappa = \frac{8\pi\mathcal{G}}{c^4} \tag{2.3}$$

is the coupling constant, $\mathcal{G}$ is Newton's gravitational constant, c is the speed of light, ρc^2 is the energy density, p is the pressure, and $\mathbf{w} = (w^0, ..., w^3)$ are the components of the 4-velocity of the fluid, c.f. [47], and again we use the convention that $c = 1$ and $\mathcal{G} = 1$ when convenient.

Putting the metric ansatz (1.1) into the Einstein equations (2.2) gives the equations for the FRW metric, [47],

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{\kappa}{3}\rho - \frac{k}{R^2}, \tag{2.4}$$

and

$$\dot{\rho} = -3(p + \rho)H. \tag{2.5}$$

The unknowns R, ρ and p are assumed to be functions of the FRW coordinate time t alone, and "dot" denotes differentiation with respect to t.

To verify that the Hubble length $\bar{r}_{crit} = 1/H$ is the limit for FRW-TOV shock matching outside a Black Hole, write the FRW metric (1.1) in standard Schwarzschild coordinates $\bar{\mathbf{x}} = (\bar{r}, \bar{t})$ where the metric takes the form

$$ds^2 = -B(\bar{r}, \bar{t})d\bar{t}^2 + A(\bar{r}, \bar{t})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{2.6}$$

and the mass function $M(\bar{r}, \bar{t})$ is defined through the relation

$$A = 1 - \frac{2M}{\bar{r}}. \tag{2.7}$$

It is well known that a general spherically symmetric metric can be transformed to the form (2.6) by coordinate transformation, [47, 12]. Substituting $\bar{r} = Rr$ into (1.1) and diagonalizing the resulting metric we obtain, (see [41] for details),

$$ds^2 = -\frac{1}{\psi^2}\left\{\frac{1 - kr^2}{1 - kr^2 - H^2\bar{r}^2}\right\}d\bar{t}^2 + \left\{\frac{1}{1 - kr^2 - H^2\bar{r}^2}\right\}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{2.8}$$

where ψ is an integrating factor that solves the equation

$$\frac{\partial}{\partial \bar{r}}\left(\psi\frac{1-kr^2-H^2\bar{r}^2}{1-kr^2}\right)-\frac{\partial}{\partial t}\left(\psi\frac{H\bar{r}}{1-kr^2}\right)=0, \tag{2.9}$$

and the time coordinate $\bar{t}=\bar{t}(t,\bar{r})$ is defined by the exact differential

$$d\bar{t}=\left(\psi\frac{1-kr^2-H^2\bar{r}^2}{1-kr^2}\right)dt+\left(\psi\frac{H\bar{r}}{1-kr^2}\right)d\bar{r}. \tag{2.10}$$

Now using (2.7) in (2.4), it follows that

$$M(t,\bar{r})=\frac{\kappa}{2}\int_0^{\bar{r}}\rho(t)s^2ds=\frac{1}{3}\frac{\kappa}{2}\rho\bar{r}^3. \tag{2.11}$$

Since in the FRW metric $\bar{r}=Rr$ measures arclength along radial geodesics at fixed time, we see from (2.11) that $M(t,\bar{r})$ has the physical interpretation as the total mass inside radius $\bar{r}$ at time t in the FRW metric. Restricting to the case of critical expansion $k=0$, we see from (2.4), (2.11) and (2.10) that $\bar{r}=H^{-1}$ is equivalent to $\frac{2M}{\bar{r}}=1$, and so at fixed time t, the following equivalences are valid:

$$\bar{r}=H^{-1}\ \text{ iff }\ \frac{2M}{\bar{r}}=1\ \text{ iff }\ A=0. \tag{2.12}$$

We conclude that $\bar{r}=H^{-1}$ is the critical length scale for the FRW metric at fixed time t in the sense that $A=1-\frac{2M}{\bar{r}}$ changes sign at $\bar{r}=H^{-1}$, and so the universe lies *inside a Black Hole* beyond $\bar{r}=H^{-1}$, as claimed above. Now we proved in [36] that the standard TOV metric outside the Black Hole cannot be continued into $A=0$ except in the very special case $\rho=0$. (It takes an infinite pressure to hold up a static configuration at the event horizon of a Black Hole.) Thus to do shock matching beyond one Hubble length requires a metric of a different character, and for this purpose, in [41] we introduce the TOV metric *inside the Black Hole*—a metric of TOV form, with $A<0$, whose fluid is co-moving with the timelike radial coordinate $\bar{r}$.

The Hubble length $\bar{r}_{crit}=\frac{c}{H}$ is also the critical distance at which the outward expansion of the FRW metric exactly cancels the inward advance of a radial light ray impinging on an observer positioned at the origin of a $k=0$ FRW metric. Indeed, by (1.1), a light ray traveling radially inward toward the center of an FRW coordinate system satisfies,

$$c^2dt^2=R^2dr^2, \tag{2.13}$$

so that

$$\frac{d\bar{r}}{dt} = \dot{R}r + R\dot{r} = H\bar{r} - c = H(\bar{r} - \frac{c}{H}) > 0, \tag{2.14}$$

if and only if

$$\bar{r} > \frac{c}{H}.$$

Thus the arclength distance from the origin to an inward moving light ray at fixed time t in a $k = 0$ FRW metric will actually *increase* as long as the light ray lies beyond the Hubble length. An inward moving light ray will, however, eventually cross the Hubble length and reach the origin in finite proper time, due to the increase in the Hubble length with time.

We now calculate the infinite redshift limit in terms of the Hubble length. It is well known that light emitted at (t_e, r_e) at wavelength λ_e in an FRW spacetime will be observed at (t_0, r_0) at wavelength λ_0 if

$$\frac{R_0}{R_e} = \frac{\lambda_0}{\lambda_e}.$$

Moreover, the redshift factor z is defined by

$$z = \frac{\lambda_0}{\lambda_e} - 1.$$

Thus, infinite redshifting occurs in the limit $R_e \to 0$, where $R = 0$, $t = 0$ is the Big Bang. Consider now a light ray emitted at the instant of the Big Bang, and observed at the FRW origin at present time $t = t_0$. Let r_∞ denote the FRW coordinate at time $t \to 0$ of the furthestmost objects that can be observed at the FRW origin before time $t = t_0$. Then r_∞ marks the position of objects at time $t = 0$ whose radiation would be observed as infinitly redshifted, (assuming no scattering). Note then that a shock wave emanating from $\bar{r} = 0$ at the instant of the Big Bang, will be observed at the FRW origin before present time $t = t_0$ only if its position r at the instant of the Big Bang satisfies $r < r_\infty$. To estimate r_∞, note first that from (2.13) it follows that an incoming radial light ray in an FRW metric follows a lightlike trajectory $r = r(t)$ if

$$r - r_e = -\int_{t_e}^{t} \frac{d\tau}{R(\tau)},$$

and thus

$$r_\infty = \int_0^{t_0} \frac{d\tau}{R(\tau)}. \tag{2.15}$$

Using this, the following theorem is proved in [41].

Theorem 1 *If the pressure p satisfies the bounds*

$$0 \le p \le \frac{1}{3}\rho, \tag{2.16}$$

then for any equation of state, the age of the universe t_0 and the infinite red shift limit r_∞ are bounded in terms of the Hubble length by

$$\frac{1}{2H_0} \le t_0 \le \frac{2}{3H_0}, \tag{2.17}$$

$$\frac{1}{H_0} \le r_\infty \le \frac{2}{H_0}. \tag{2.18}$$

(We have assumed that $R = 0$ when $t = 0$ and $R = 1$ when $t = t_0$, $H = H_0$.)

The next theorem gives closed form solutions of the FRW equations (2.4), (2.5) in the case when $\sigma = const$. As a special case we recover the bounds in (2.17) and (2.18) from the cases $\sigma = 0$ and $1/3$.

Theorem 2 *Assume $k = 0$ and the equation of state*

$$p = \sigma\rho, \tag{2.19}$$

where σ is taken to be constant,

$$0 \le \sigma \le 1.$$

Then, (assuming an expanding universe $\dot{R} > 0$), the solution of system (2.4), (2.5) satisfying $R = 0$ at $t = 0$ and $R = 1$ at $t = t_0$ is given by,

$$\rho = \tfrac{4}{3\kappa(1+\sigma)^2}\tfrac{1}{t^2}, \tag{2.20}$$

$$R = \left(\tfrac{t}{t_0}\right)^{\frac{2}{3(1+\sigma)}}, \tag{2.21}$$

$$\tfrac{H}{H_0} = \tfrac{t_0}{t}. \tag{2.22}$$

Moreover, the age of the universe t_0 and the infinite red shift limit r_∞ are given exactly in terms of the Hubble length by

$$t_0 = \frac{2}{3(1+\sigma)}\frac{1}{H_0}, \tag{2.23}$$

$$r_\infty = \frac{2}{1+3\sigma}\frac{1}{H_0}. \tag{2.24}$$

From (2.24) we conclude that a shock wave will be observed at the FRW origin before present time $t = t_0$ only if its position r at the instant of the Big Bang satisfies $r < \frac{2}{1+3\sigma}\frac{1}{H_0}$. Note that r_∞ ranges from one half to two Hubble lengths as σ ranges from 1 to 0, taking the intermediate value of one Hubble length at $\sigma = 1/3$, c.f. (2.18).

Note that using (2.20)-(2.21) in (2.11), it follows that

$$M = \frac{\kappa}{2}\int_0^{\bar{r}} \rho(t)s^2 ds = \frac{2\bar{r}^3}{9(1+\sigma)^2 t_0^{\frac{2}{1+\sigma}}} t^{\frac{-2\sigma}{1+\sigma}}, \tag{2.25}$$

so $\dot{M} < 0$ if $\sigma > 0$. It follows that if $p = \sigma\rho$, $\sigma = const. > 0$, then the total mass inside radius $r = const.$ decreases in time.

3 The General Theory of Shock Matching

The matching of the FRW and TOV metrics in the next two sections is based on the following theorems that were derived in [32][8] .

Theorem 3 *Let Σ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector* $\mathbf{n}$ *relative to the spacetime metric g, let K denote the second fundamental form on Σ, and let G denote the Einstein curvature tensor. Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are equivalent:*
(i) $[K] = 0$ at each point of Σ.
(ii) The curvature tensors R^i_{jkl} and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.

[8]Theorems 3 and 4 apply to non-lightlike shock surfaces. The lightlike case was done in [29].

(iii) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates.
(iv) For each $P \in \Sigma$, there exists a coordinate frame that is locally Lorentzian at P, and can be reached within the class of $C^{1,1}$ coordinate transformations.
Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions, $[G]^\sigma_i n_\sigma = 0$, (which express the weak form of conservation of energy and momentum across Σ when $G = \kappa T$), hold at each point on Σ.

Here $[f]$ denotes the jump in the quantity f across Σ, (this being determined by the metric separately on each side of Σ because g_{ij} is only Lipschitz continuous across Σ), and by $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous.

In the case of spherical symmetry, the following stronger result holds. In this case, the jump conditions $[G^{ij}]n_i = 0$, that express the weak form of conservation across a shock surface, are implied by a single condition $[G^{ij}]n_i n_j = 0$, so long as the shock is non–null, and the areas of the spheres of symmetry match *smoothly* at the shock and change monotonically as the shock evolves. Note that in general, assuming that the angular variables are identified across the shock, we expect conservation to entail two condtions, one for the time and one for the radial components. The fact that the smooth matching of the spheres of symmetry reduces conservation to one condition can be interpreted as an instance of the general principle that directions of smoothness in the metric imply directions of conservation of the sources.

Theorem 4 *Assume that g and $\bar{g}$ are two spherically symmetric metrics that match Lipschitz continuously across a three dimensional shock interface Σ to form the matched metric $g \cup \bar{g}$. That is, assume that g and $\bar{g}$ are Lorentzian metrics given by*

$$ds^2 = -a(t,r)dt^2 + b(t,r)dr^2 + c(t,r)d\Omega^2, \tag{3.1}$$

and

$$d\bar{s}^2 = -\bar{a}(\bar{t},\bar{r})d\bar{t}^2 + \bar{b}(\bar{t},\bar{r})d\bar{r}^2 + \bar{c}(\bar{t},\bar{r})d\Omega^2, \tag{3.2}$$

and that there exists a smooth coordinate transformation $\Psi : (t,r) \to (\bar{t},\bar{r})$, defined in a neighborhood of a shock surface Σ given by $r = r(t)$, such that the metrics agree on Σ. (We implicitly assume that θ and φ are continuous across the surface.) Assume that

$$c(t,r) = \bar{c}(\Psi(t,r)), \tag{3.3}$$

in an open neighborhood of the shock surface Σ, so that, in particular, the areas of the 2-spheres of symmetry in the barred and unbarred metrics agree on the shock surface. Assume also that the shock surface $r = r(t)$ in unbarred coordinates is mapped to the surface $\bar{r} = \bar{r}(\bar{t})$ by $(\bar{t}, \bar{r}(\bar{t})) = \Psi(t, r(t))$. Assume, finally, that the normal $\mathbf{n}$ to Σ is non–null, and that

$$\mathbf{n}(c) \neq 0 \tag{3.4}$$

where $\mathbf{n}(c)$ denotes the derivative of the function c in the direction of the vector $\mathbf{n}$.[9] *Then the following are equivalent to the statement that the components of the metric $g \cup \bar{g}$ in any Gaussian normal coordinate system are $C^{1,1}$ functions of these coordinates across the surface Σ :*

$$[G^i_j] n_i = 0, \tag{3.5}$$

$$[G^{ij}] n_i n_j = 0, \tag{3.6}$$

$$[K] = 0. \tag{3.7}$$

Here again, $[f] = \bar{f} - f$ denotes the jump in the quantity f across Σ, and K is the second fundamental form on the shock surface.

4 FRW-TOV Shock Matching Outside the Black Hole—The Case $r_* = 0$

To construct the family of shock wave solutions for parameter values $0 < \sigma \leq 1$ and $r_* = 0$, we match the exact solution (2.20)-(2.22) of the FRW metric (1.1) to the TOV metric (1.3) *outside the Black Hole*, assuming $A > 0$. In this case, we can bypass the problem of deriving and solving the ODE's for the shock surface and constraints discussed above, by actually deriving the exact solution of the Einstein equations of TOV form that meets these equations. This exact solution

[9] I.e., we assume that the areas of the 2-spheres of symmetry change monotonically in the direction normal to the surface. E.g., if $c = r^2$, then $\frac{\partial}{\partial t} c = 0$, so the assumption $\mathbf{n}(c) \neq 0$ is valid except when $\mathbf{n} = \frac{\partial}{\partial t}$, in which case the rays of the shock surface would be spacelike. Thus the shock speed would be faster than the speed of light if our assumption $\mathbf{n}(c) \neq 0$ failed in the case $c = r^2$.

represents the general relativistic version of a static, singular isothermal sphere—singular because it has an inverse square density profile, and isothermal because the relationship between the density and pressure is $\bar{p} = \bar{\sigma}\bar{\rho}$, $\bar{\sigma} = const$.

Assuming the stress tensor for a perfect fluid, and assuming that the density and pressure depend only on $\bar{r}$, the Einstein equations for the TOV metric (1.3) *outside the Black Hole*, (that is, when $A = 1 - \frac{2M}{\bar{r}} > 0$), are equivalent to the Oppenheimer-Volkoff system

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\bar{\rho}, \tag{4.1}$$

$$-\bar{r}^2\frac{d}{d\bar{r}}\bar{p} = \mathcal{G}M\bar{\rho}\left\{1+\frac{\bar{p}}{\bar{\rho}}\right\}\left\{1+\frac{4\pi\bar{r}^3\bar{p}}{M}\right\}\left\{1-\frac{2\mathcal{G}M}{\bar{r}}\right\}^{-1}. \tag{4.2}$$

Integrating (4.1) we obtain the usual interpretation of M as the total mass inside radius $\bar{r}$,

$$M(\bar{r}) = \int_0^{\bar{r}} 4\pi\xi^2\bar{\rho}(\xi)d\xi. \tag{4.3}$$

The metric component $B \equiv B(\bar{r})$ is determined from $\bar{\rho}$ and M through the equation

$$\frac{B'(\bar{r})}{B} = -2\frac{\bar{p}'(\bar{r})}{\bar{p}+\bar{\rho}}. \tag{4.4}$$

Assuming

$$\bar{p} = \bar{\sigma}\bar{\rho}, \quad \bar{\rho}(\bar{r}) = \frac{\gamma}{\bar{r}^2}, \tag{4.5}$$

for some constants $\bar{\sigma}$ and γ, and substituting into (4.3), we obtain

$$M(\bar{r}) = 4\pi\gamma\bar{r}. \tag{4.6}$$

Putting (4.5)-(4.6) into (4.2) and simplifying yields the identity

$$\gamma = \frac{1}{2\pi\mathcal{G}}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right). \tag{4.7}$$

From (4.3) we obtain

$$A = 1 - 8\pi\mathcal{G}\gamma < 1. \tag{4.8}$$

Applying (4.4) leads to

$$B = B_0\left(\frac{\bar{\rho}}{\bar{\rho}_0}\right)^{-\frac{2\bar{\sigma}}{1+\bar{\sigma}}} = B_0\left(\frac{\bar{r}}{\bar{r}_0}\right)^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}}. \tag{4.9}$$

By rescaling the time coordinate, we can take $B_0 = 1$ at $\bar{r}_0 = 1$, in which case (4.9) reduces to

$$B = \bar{r}^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}}. \tag{4.10}$$

We conclude that when (4.7) holds, (4.5)-(4.8) and (4.9) provide an exact solution of the Einstein field equations of TOV type[10], for each $0 \le \bar{\sigma} \le 1$. By (4.8), these solutions are defined *outside the Black Hole*, since $\frac{2M}{\bar{r}} < 1$. When $\bar{\sigma} = 1/3$, (4.7) yields $\gamma = \frac{3}{56\pi\mathcal{G}}$, (c.f., [47], equation (11.4.13)).

To match the FRW exact solution (2.20)-(2.22) with equation of state $p = \sigma\rho$ to the TOV exact solution (4.5)-(4.10) with equation of state $\bar{p} = \bar{\sigma}\bar{\rho}$ across a shock interface, we first set $\bar{r} = Rr$ to match the spheres of symmetry, and then match the timelike and spacelike components of the corresponding metrics in standard Schwarzschild coordinates. The matching of the $d\bar{r}^2$ coefficient A^{-1} yields the conservation of mass condition that implicitly gives the shock surface $\bar{r} = \bar{r}(t)$,

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3. \tag{4.11}$$

Using this together with (4.6) and (4.6) gives the following two relations that hold at the shock surface:

$$\bar{r} = \sqrt{\frac{3\gamma}{\rho(t)}}, \tag{4.12}$$

$$\rho = \frac{3}{4\pi}\frac{M}{\bar{r}(t)^3} = \frac{3\gamma}{\bar{r}(t)^2} = 3\bar{\rho}. \tag{4.13}$$

Matching the $d\bar{t}^2$ coefficient B on the shock surface determines the integrating factor ψ in a neighborhood of the shock surface by assigning initial conditions for (2.9). Finally, the conservation constraint $[T_{ij}]n_i n_j = 0$ leads to the single condition

$$0 = (1-A)(\rho+\bar{p})(p+\bar{\rho})^2 + \left(1 - \frac{1}{A}\right)(\bar{\rho}+\bar{p})(\rho+p)^2 + (p-\bar{p})(\rho-\bar{\rho})^2, \tag{4.14}$$

which upon using $p = \sigma\rho$ and $\bar{p} = \bar{\sigma}\bar{\rho}$ is satisfied assuming the condition

$$\bar{\sigma} = \frac{1}{2}\sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2}\sigma - \frac{7}{2} \equiv H(\sigma). \tag{4.15}$$

[10] In this case, an exact solution of TOV type was first found by Tolman [45], and rediscovered in the case $\bar{\sigma} = 1/3$ by Misner and Zapolsky, c.f. [47], page 320.

Alternatively, we can solve for σ in (4.15) and write this relation as

$$\sigma = \frac{\bar{\sigma}(\bar{\sigma}+7)}{3(1-\bar{\sigma})}. \tag{4.16}$$

This guarantees that conservation holds across the shock surface, and so it follows from Theorem 4 that all of the equivalencies in Theorem 3 hold across the shock surface. Note that $H(0) = 0$, and to leading order $\bar{\sigma} = \frac{3}{7}\sigma + O(\sigma^2)$ as $\sigma \to 0$. Within the physical region $0 \leq \sigma, \bar{\sigma} \leq 1$, $H'(\sigma) > 0$, $\bar{\sigma} < \sigma$, and $H(1/3) = \sqrt{17} - 4 \approx .1231$, $H(1) = \frac{\sqrt{112}}{2} - 5 \approx .2915$.

Using the exact formulas for the FRW metric in (2.20)-(2.22), and setting $R_0 = 1$ at $\rho = \rho_0$, $t = t_0$, we obtain the following exact formulas for the shock position:

$$\bar{r}(t) = \alpha t, \tag{4.17}$$

$$r(t) = \bar{r}(t)R(t)^{-1} = \beta\, t^{\frac{1+3\sigma}{3+3\sigma}}, \tag{4.18}$$

where

$$\alpha = 3(1+\sigma)\sqrt{\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}},$$

$$\beta = \alpha^{\frac{1+3\sigma}{3+3\sigma}} \left(\frac{3\gamma}{\rho_0}\right)^{\frac{1}{3+3\sigma}}. \tag{4.19}$$

It follows from (4.6) that $A > 0$, and from (4.18) that $r_* = \lim_{t\to 0}\, r(t) = 0$. The entropy condition that the shock wave be compressive follows from the fact that $\bar{\sigma} = H(\sigma) < \sigma$. Thus we conclude that for each $0 < \sigma \leq 1$, $r_* = 0$, the solutions constructed in (4.5)-(4.19) define a one parameter family of shock wave solutions that evolve everywhere *outside the Black Hole*, which implies that the distance from the shock wave to the FRW center is less than one Hubble length for all $t > 0$.

Using (4.17) and (4.18), one can determine the shock speed, and check when the Lax Characteristic condition holds at the shock. The result is the following[11], (see [33] for details),

[11]Note that even when the shock speed is larger than c, only the wave, and not the sound speeds or any other physical motion, exceeds the speed of light. See [29] for the case when the shock speed is equal to the speed of light

Theorem 5 *There exist values* $0 < \sigma_1 < \sigma_2 < 1$, *(*$\sigma_1 \approx .458$, $\sigma_2 = \sqrt{5}/3 \approx .745$*), such that, for* $0 < \sigma \leq 1$, *the Lax characteristic condition holds at the shock if and only if* $0 < \sigma < \sigma_1$; *and the shock speed is less than the speed of light if and only if* $0 < \sigma < \sigma_2$.

The explicit solution in the case $r_* = 0$ can be interpreted as a *general relativistic version* of a shock wave explosion into a static, singular, isothermal sphere, known in the Newtonian case as a simple model for star formation, (see [36, 4]). As the scenario goes, a star begins as a diffuse cloud of gas. The cloud slowly contracts under its own gravitational force by radiating energy out through the gas cloud as gravitational potential energy is converted into kinetic energy. This contraction continues until the gas cloud reaches the point where the mean free path for transmission of light is small enough that light is scattered, instead of transmitted, through the cloud. The scattering of light within the gas cloud has the effect of equalizing the temperature within the cloud, and at this point the gas begins to drift toward the most compact configuration of the density that balances the pressure when the equation of state is isothermal. This configuration is a static, singular, isothermal sphere, the general relativistic version of which is the exact TOV solution beyond the shock wave when $r_* = 0$. This solution in the Newtonian case is also inverse square in the density and pressure, and so the density tends to infinity at the center of the sphere. Eventually, the high densities at the center ingnite thermonuclear reactions. The result is a shock-wave explosion emanating from the center of the sphere, and this signifies the birth of the star. The exact solutions when $r_* = 0$ represent a *general relativistic version* of such a shock-wave explosion.

5 Shock Wave Solutions Inside the Black Hole—The case $r_* > 0$.

When the shock wave is beyond one Hubble length from the FRW center, we obtain a family of shock wave solutions for each $0 < \sigma \leq 1$ and $r_* > 0$ by shock matching the FRW metric (1.1) to a TOV metric of form (1.3) under the assumption that

$$A(\bar{r}) = 1 - \frac{2M(\bar{r})}{\bar{r}} \equiv 1 - N(\bar{r}) < 0. \tag{5.1}$$

In this case, $\bar{r}$ is the timelike variable. Assuming the stress tensor T is taken to be that of a perfect fluid co-moving with the TOV metric, the Einstein equations

$G = \kappa T$, inside the Black Hole, take the form, (see [41] for details),

$$\bar{p}' = \frac{\bar{p}+\bar{\rho}}{2}\frac{N'}{N-1}, \tag{5.2}$$

$$N' = -\left\{\frac{N}{\bar{r}} + \kappa\bar{p}\bar{r}\right\}, \tag{5.3}$$

$$\frac{B'}{B} = -\frac{1}{N-1}\left\{\frac{N}{\bar{r}} + \kappa\bar{\rho}\right\}. \tag{5.4}$$

The system (5.2)-(5.4) defines the simplest class of gravitational metrics that contain matter, evolve *inside the Black Hole*, and such that the mass function $M(\bar{r}) < \infty$ at each fixed time $\bar{r}$. System (5.2)-(5.4) for $A < 0$ differs substantially from the TOV equations for $A > 0$ because, for example, the energy density T^{00} is equated with the timelike component G^{rr} when $A < 0$, but with G^{tt} when $A > 0$. In particular, this implies that, *inside the Black Hole*, the mass function $M(\bar{r})$ does not have the interpretation as a total mass inside radius $\bar{r}$ as it does *outside the Black Hole*.

The equations (5.3), (5.4) do not have the same character as (4.1), (4.2) and the relation $\bar{p} = \bar{\sigma}\bar{\rho}$ with $\bar{\sigma} = const.$ is inconsistent with (5.3), (5.4) together with the conservation constraint and the FRW assumption $p = \sigma\rho$ for shock matching. Thus, instead of looking for an exact solution of (5.3), (5.4) ahead of time, as in the case $r_* = 0$, we assume the FRW solution (2.20)-(2.22), and derive the ODE's that describe the TOV metrics that match this FRW metric Lipschitz continuously across a shock surface, and then impose the conservation, entropy and equation of state constraints at the end. Matching a given $k = 0$ FRW metric to a TOV metric *inside the Black Hole* across a shock interface, leads to the system of ODE's, (see [41]) for details),

$$\frac{du}{dN} = -\left\{\frac{(1+u)}{2(1+3u)N}\right\}\left\{\frac{(3u-1)(\sigma-u)N + 6u(1+u)}{(\sigma-u)N + (1+u)}\right\}, \tag{5.5}$$

$$\frac{d\bar{r}}{dN} = -\frac{1}{1+3u}\frac{\bar{r}}{N}, \tag{5.6}$$

with conservation constraint

$$v = \frac{-\sigma\,(1+u) + (\sigma-u)N}{(1+u) + (\sigma-u)N}, \tag{5.7}$$

where

$$u = \frac{\bar{p}}{\rho},\ v = \frac{\bar{\rho}}{\rho},\ \sigma = \frac{p}{\rho}. \tag{5.8}$$

Here ρ and p denote the (known) FRW density and pressure, and all variables are evaluated at the shock. Solutions of (5.5)-(5.7) determine the (unknown) TOV metrics that match the given FRW metric Lipschitz continuously across a shock interface, such that conservation of energy and momemtum hold across the shock, and such that there are no delta function sources at the shock, [14, 34]. Note that the dependence of (5.5)-(5.7) on the FRW metric is only through the variable σ, and so the advantage of taking $\sigma = const.$ is that the whole solution is determined by the inhomogeneous scalar equation (5.5) when $\sigma = const.$ We take as the entropy constraint the condition that

$$0 < \bar{p} < p, \qquad 0 < \bar{\rho} < \rho, \tag{5.9}$$

and to insure a physically reasonable solution, we impose the equation of state constriant on the TOV side of the shock

$$0 < \bar{p} < \bar{\rho}. \tag{5.10}$$

Condition (5.9) implies that outgoing shock waves are compressive. Inequalities (5.9) and (5.10) are both implied by the single condition, (see [41])),

$$\frac{1}{N} < \left(\frac{1-u}{1+u}\right)\left(\frac{\sigma-u}{\sigma+u}\right). \tag{5.11}$$

Since σ is constant, equation (5.5) uncouples from (5.6), and thus solutions of system (5.5)-(5.7) are determined by the scalar non-autonomous equation (5.5). Making the change of variable $S = 1/N$, which transforms the "Big Bang" $N \to \infty$ over to a rest point at $S \to 0$, we obtain,

$$\frac{du}{dS} = \left\{\frac{(1+u)}{2(1+3u)S}\right\}\left\{\frac{(3u-1)(\sigma-u)+6u(1+u)S}{(\sigma-u)+(1+u)S}\right\}. \tag{5.12}$$

Note that the conditions $N > 1$ and $0 < \bar{p} < p$ restrict the domain of (5.12) to the region $0 < u < \sigma < 1$, $0 < S < 1$. The next theorem gives the existence of solutions for $0 < \sigma \leq 1$, $r_* > 0$, *inside the Black Hole*, c.f. [40]:

Theorem 6 *For every σ, $0 < \sigma < 1$, there exists a unique solution $u_\sigma(S)$ of (5.12), such that (5.11) holds on the solution for all S, $0 < S < 1$, and on this solution,* $0 < u_\sigma(S) < \bar{u}$, $\lim_{S\to 0} u_\sigma(S) = \bar{u}$, *where*

$$\bar{u} = Min\left\{1/3, \sigma\right\}, \tag{5.13}$$

and

$$\lim_{S\to 1}\bar{p} = 0 = \lim_{S\to 1}\bar{\rho}. \tag{5.14}$$

For each of these solutions $u_\sigma(S)$, the shock position is determined by the solution of (5.6), which in turn is determined uniquely by an initial condition which can be taken to be the FRW radial position of the shock wave at the instant of the Big Bang,

$$r_* = \lim_{S\to 0} r(S) > 0. \tag{5.15}$$

Concerning the the shock speed, we have:

Theorem 7 *Let $0 < \sigma < 1$. Then the shock wave is everywhere subluminous, that is, the shock speed $s_\sigma(S) \equiv s(u_\sigma(S)) < 1$ for all $0 < S \le 1$, if and only if $\sigma \le 1/3$.*

Concerning the shock speed near the Big Bang $S = 0$, the following is true:

Theorem 8 *The shock speed at the Big Bang $S = 0$ is given by:*

$$\lim_{S\to 0} s_\sigma(S) = 0, \quad \sigma < 1/3, \tag{5.16}$$

$$\lim_{S\to 0} s_\sigma(S) = \infty, \quad \sigma > 1/3, \tag{5.17}$$

$$\lim_{S\to 0} s_\sigma(S) = 1, \quad \sigma = 1/3. \tag{5.18}$$

Theorem 8 shows that the equation of state $p = \frac{1}{3}\rho$ plays a special role in the analysis when $r_* > 0$, and only for this equation of state does the shock wave emerge at the Big Bang at a finite non-zero speed, the speed of light. Moreover, (5.13) implies that in this case, the correct relation $\frac{\bar{p}}{\bar{\rho}} = \bar{\sigma}$ is also achieved in the limit $S \to 0$. The result (5.14) implies that, (neglecting the pressure p at this time onward), the solution continues to a $k = 0$ Oppenheimer-Snyder solution outside the Black Hole for $S > 1$.

It follows that the shock wave will first become visible at the FRW center $\bar{r} = 0$ at the moment $t = t_0$, $(R(t_0) = 1)$, when the Hubble length $H_0^{-1} = H^{-1}(t_0)$ satisfies

$$\frac{1}{H_0} = \frac{1+3\sigma}{2} r_*, \tag{5.19}$$

where r_* is the FRW position of the shock at the instant of the Big Bang. At this time, the number of Hubble lengths $\sqrt{N_0}$ from the FRW center to the shock wave at time $t = t_0$ can be estimated by

$$1 \leq \frac{2}{1+3\sigma} \leq \sqrt{N_0} \leq \frac{2}{1+3\sigma} e^{\sqrt{3\sigma}\left(\frac{1+3\sigma}{1+\sigma}\right)}.$$

Thus, in particular, the shock wave will still lie beyond the Hubble length $1/H_0$ at the FRW time t_0 when it first becomes visible. Furthermore, the time $t_{crit} > t_0$ at which the shock wave will emerge from the White Hole given that t_0 is the first instant at which the shock becomes visible at the FRW center, can be estimated by

$$\frac{2}{1+3\sigma} e^{\frac{1}{4}\sigma} \leq \frac{t_{crit}}{t_0} \leq \frac{2}{1+3\sigma} e^{\frac{2\sqrt{3\sigma}}{1+\sigma}}, \tag{5.20}$$

for $0 < \sigma \leq 1/3$, and by the better estimate

$$e^{\frac{\sqrt{6}}{4}} \leq \frac{t_{crit}}{t_0} \leq e^{\frac{3}{2}}, \tag{5.21}$$

in the case $\sigma = 1/3$. Inequalities (5.20), (5.21) imply, for example, that at the Oppenheimer-Snyder limit $\sigma = 0$,

$$\sqrt{N_0} = 2, \quad \frac{t_{crit}}{t_0} = 2,$$

and in the limit $\sigma = 1/3$,

$$1.8 \leq \frac{t_{crit}}{t_0} \leq 4.5, \quad 1 < \sqrt{N_0} \leq 4.5.$$

We can conclude that the moment t_0 when the shock wave first becomes visible at the FRW center, the shock wave must lie within 4.5 Hubble lengths of the FRW center. Throughout the expansion up until this time, the expanding universe must lie entirely within a *White Hole*—the universe will eventually emerge from this *White Hole*, but not until some later time t_{crit}, where t_{crit} does not exceed $4.5t_0$.

6 Conclusion

We believe that the existence of a wave at the leading edge of the expansion of the galaxies is the most likely possibility. The alternatives are that either the universe of expanding galaxies goes on out to infinity, or else the universe is not simply connected. Although the first possibility has been believed for most of the history of cosmology based on the Friedmann universe, we find this implausible and arbitrary in light of the shock wave refinements of the FRW metric discussed here. The second possibility, that the universe is not simply connected, has received considerable attention recently[12]. However, since we have not seen, and cannot create, any non-simply connected 3-spaces on any other length scale, and since there is no observational evidence to support this, we view this as less likely than the existence of a wave at the leading edge of the expansion of the galaxies, left over from the Big Bang. Recent analysis of the microwave background radiation data shows a cut-off in the angular frequencies consistent with a length scale of around one Hubble length, [1]. This certainly makes one wonder whether this cutoff is evidence of a wave at this length scale, especially given the consistency of this possibility with the case $r_* > 0$ of the family exact solutions discussed here.

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