Numerical Refinement of a Finite Mass Shock-Wave Cosmology

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References

- Exact solution incorporating a shock-wave into the standard FRW metric for cosmology...

- Smoller-Temple, *Shock-Wave Cosmology Inside a Black Hole*, PNAS Sept 2003.

- Smoller-Temple, *Cosmology, Black Holes, and Shock Waves Beyond the Hubble Length*, Meth. Appl. Anal., 2004.

References:

- Connecting the shock wave cosmology model with Guth's theory of inflation...

- *How inflationary spacetimes might evolve into spacetimes of finite total mass*, with J. Smoller, Meth. and Appl. of Anal., Vol. **12**, No. 4, pp. 451-464 (2005).

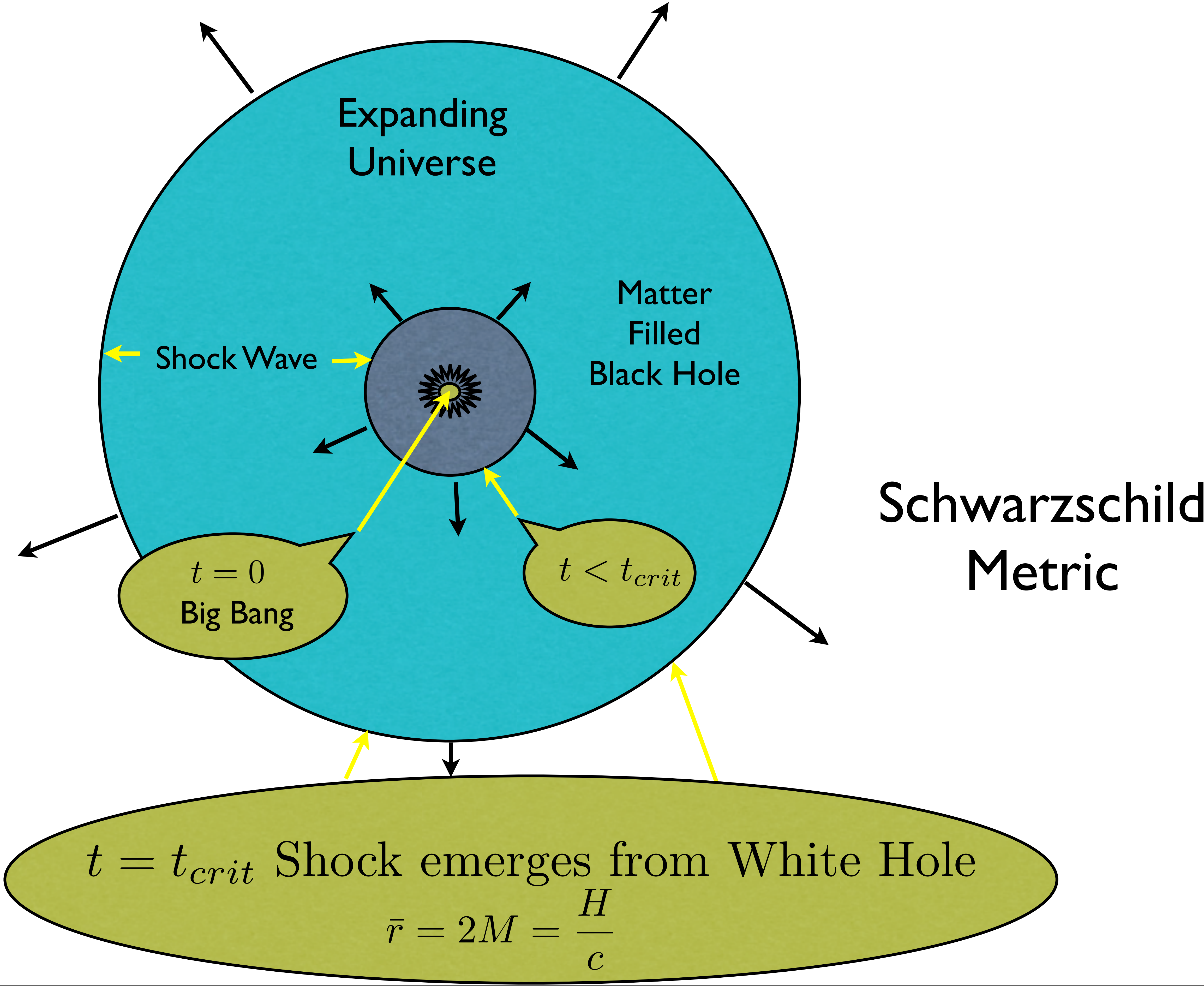
- *How inflation is used to solve the flatness problem*, with J. Smoller, Jour. of Hyp. Diff. Eqns. (JHDE) Vol. 3, no. 2, 375-386 (2006).

References:

- The locally inertial Glimm Scheme...

- *A shock-wave formulation of the Einstein equations*, with J. Groah, Meth. and Appl. of Anal., **7**, No. 4,(2000), pp. 793-812.
- *Shock-wave solutions of the Einstein equations: Existence and consistency by a locally inertial Glimm Scheme*, with J. Groah, Memoirs of the AMS, Vol. 172, No. 813, November 2004.
- *Shock Wave Interactions in General Relativity: A Locally Inertial Glimm Scheme for Spherically Symmetric Spacetimes*, with J. Groah and J. Smoller, Springer Monographs in Mathematics, 2007.

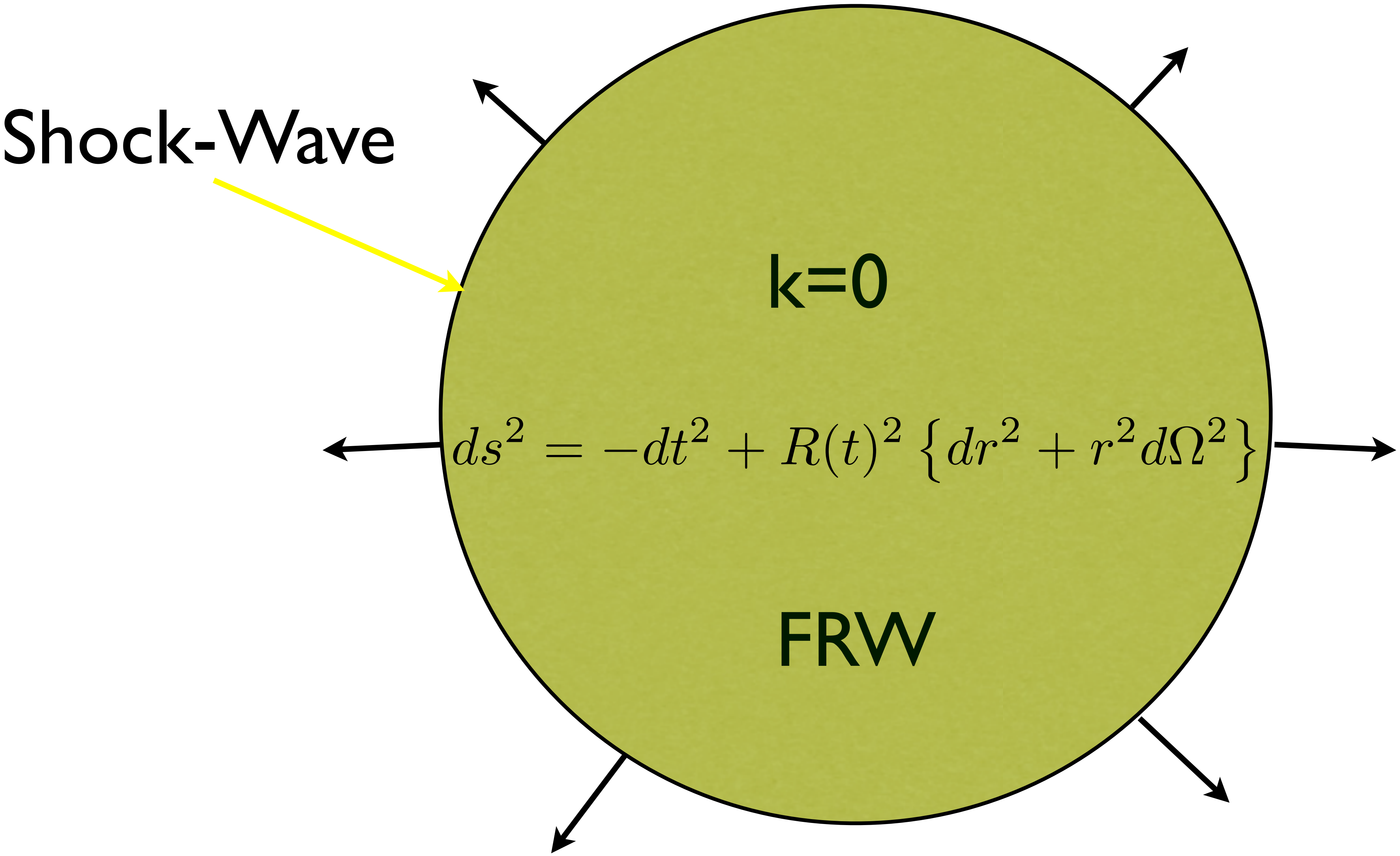
Our Shock Wave Cosmology Solution



- The solution can be viewed as a natural generalization of a $k=0$ Oppenheimer-Snyder solution to the case of non-zero pressure, inside the Black Hole----

$$2M/r>1$$

The Shock Wave Cosmology Solution



TOV: $$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$\frac{2M}{\bar{r}} > 1$ ⇒ $\bar{r}$ is timelike

- In [Smoller-Temple, PNAS] we constructed an exact shock wave solution of the Einstein equations by matching a (k=0)-FRW metric to a TOV metric *inside the Black Hole* across a subluminal, entropy-satisfying shock-wave, out beyond one Hubble length

- To obtain a large region of uniform expansion at the center consistent with observations we needed $\frac{2M}{\bar{r}} > 1$

- $$\frac{2M}{\bar{r}} = 1 \Longleftrightarrow \bar{r} = Hubble\ Length \equiv \frac{H}{c}$$

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- The TOV metric inside the Black Hole is the simplest metric that cuts off the FRW at a finite total mass.

- Approximately like a classical explosion of finite mass with a shock-wave at the leading edge of the expansion.

- The solution decays time asymptotically to Oppenheimer-Snyder----a finite ball of mass expanding into empty space outside the black hole, something like a gigantic supernova.

- Limitation of the model: TOV density and pressure are determined by the equations that describe the matching of the metrics

- $p = \frac{c^2}{3}\rho$ can only be imposed on the FRW side

- Imposing $p = \frac{c^2}{3}\rho$ on the TOV side introduces secondary waves which can not be modeled in an exact solution

- Question: How to model the secondary waves?

- OUR QUESTION: How to refine the model to incorporate the correct TOV equation of state, and thereby model the secondary waves in the problem?

- OUR PROPOSAL: Get the initial condition at the end of inflation

- Use the Locally Inertial Glimm Scheme to simulate the region of interaction between the FRW and TOV metrics

DETAILS
OF THE
EXACT SOLUTION

The Final Equations

$$p = \sigma\rho$$
$$\sigma = const.$$

Autonomous System

$$S' = 2S(1+3u)\left\{(\sigma - u) + (1+u)S\right\} \equiv F(S,u)$$

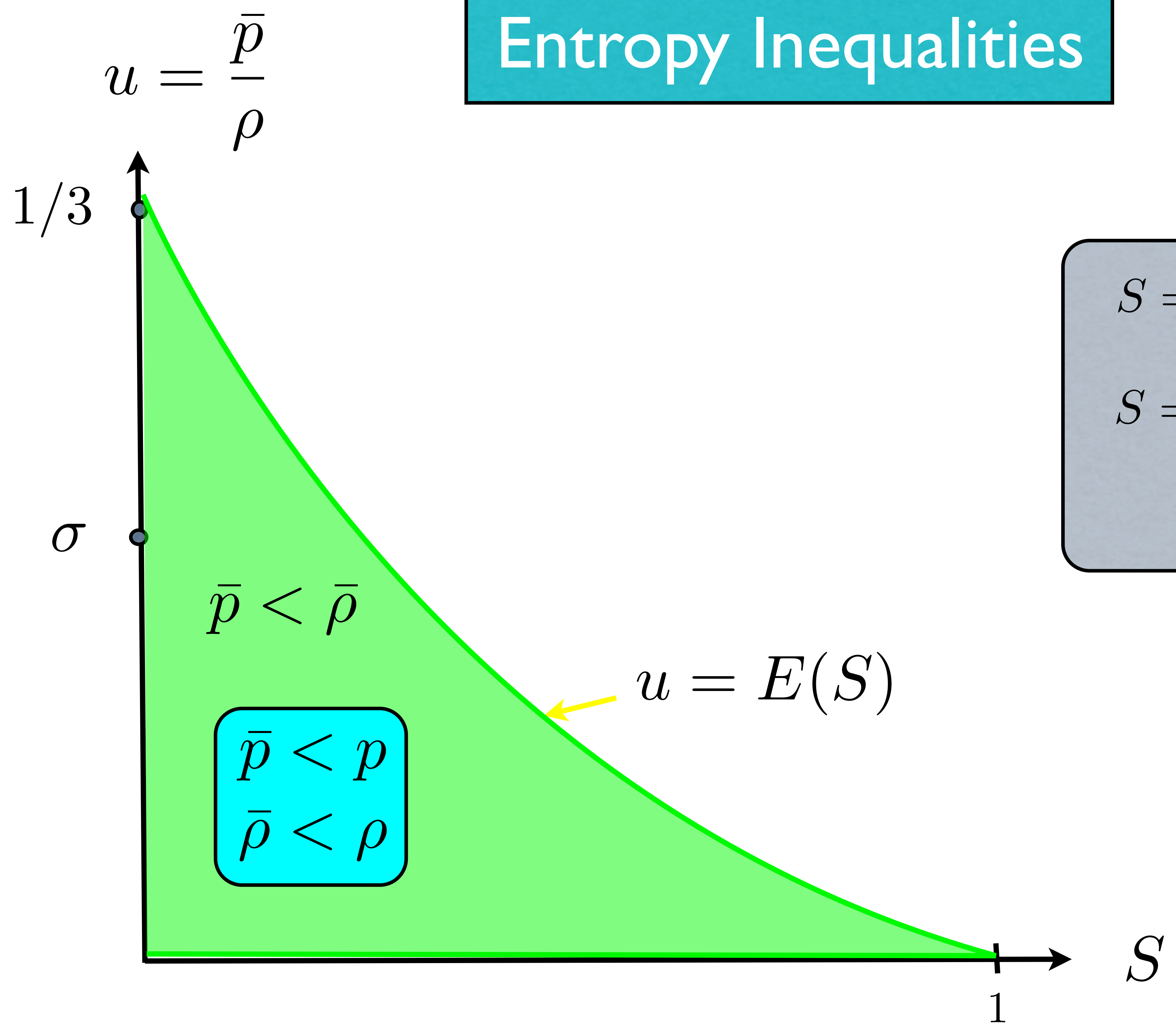
$$u' = (1+u)\left\{-(1-3u)(\sigma - u) + 6u(1+u)S\right\} \equiv G(S,u)$$

Big Bang $\equiv 0 \leq S \leq 1 \equiv$ Emerges From White Hole

Entropy Condition: $\bar{p} < p, \ \bar{\rho} < \rho, \ \bar{p} < \bar{\rho}$

$$S < \left(\frac{1-u}{1+u}\right)\left(\frac{\sigma - u}{\sigma + u}\right) \equiv E(u)$$

Entropy Inequalities



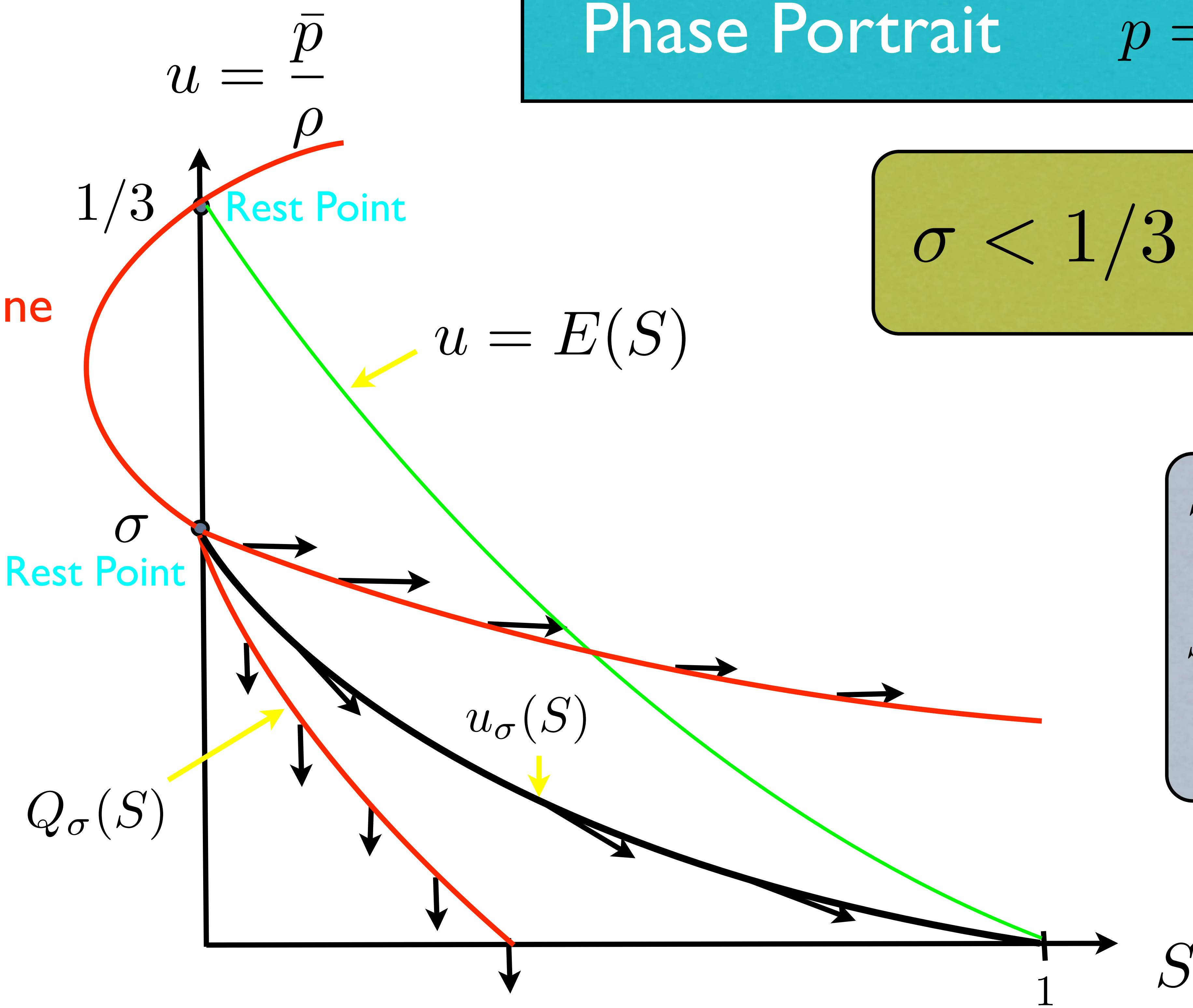
$S = 0$ *Big Bang*

$S = 1$ *Solution emerges from White Hole*

The entropy conditions for an outgoing shock hold when $u_\sigma(S) < E(S)$

Phase Portrait $\quad p = \sigma\rho$

$$\sigma < 1/3$$



$S = 0$ *Big Bang*

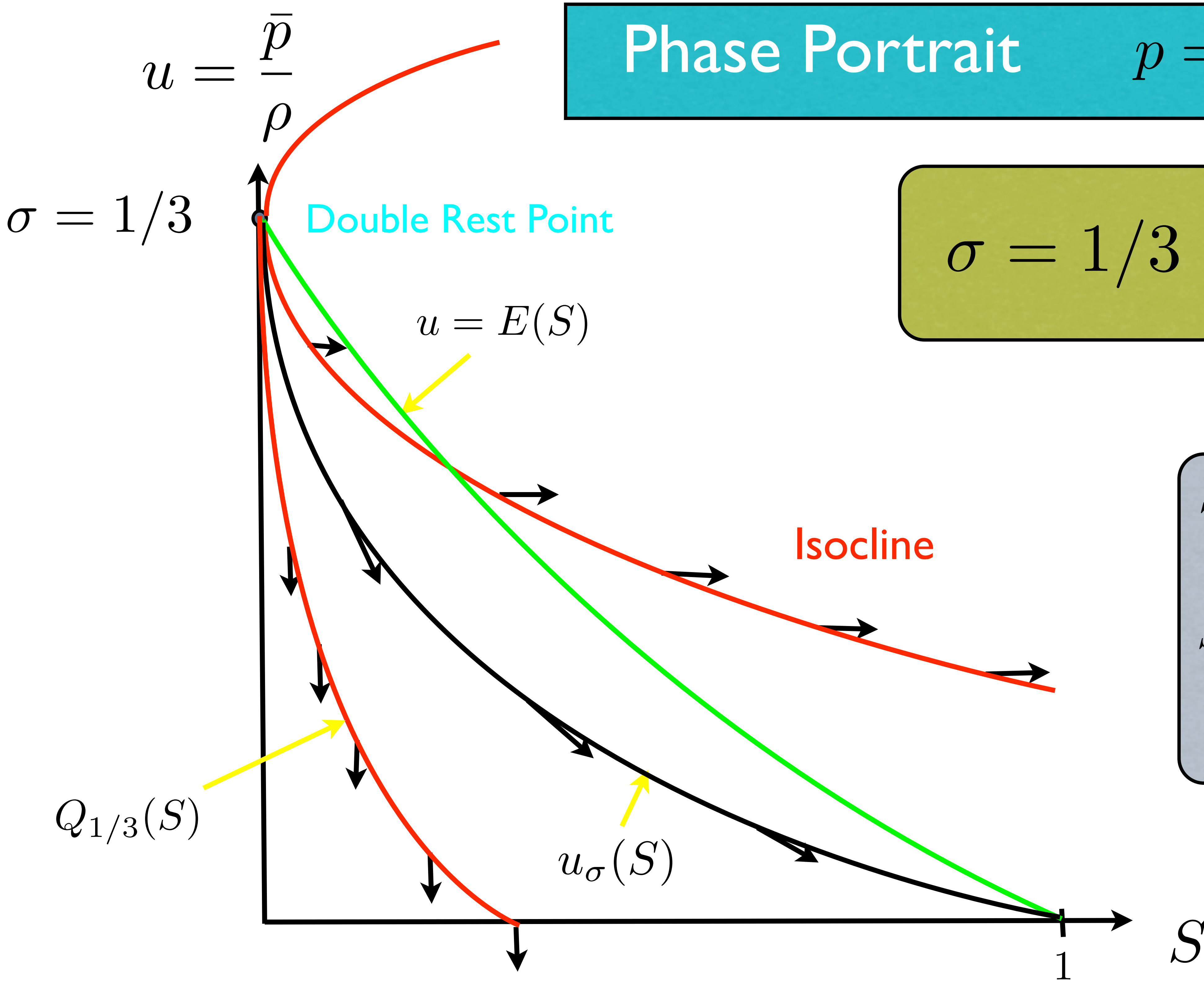
$S = 1$ *Solution emerges from White Hole*

$s_\sigma(S) \equiv$ *shock speed* $< c$ for $0 < S < 1$

$s_\sigma(S) \to 0$ as $S \to 0$

Phase Portrait $p = \sigma\rho$

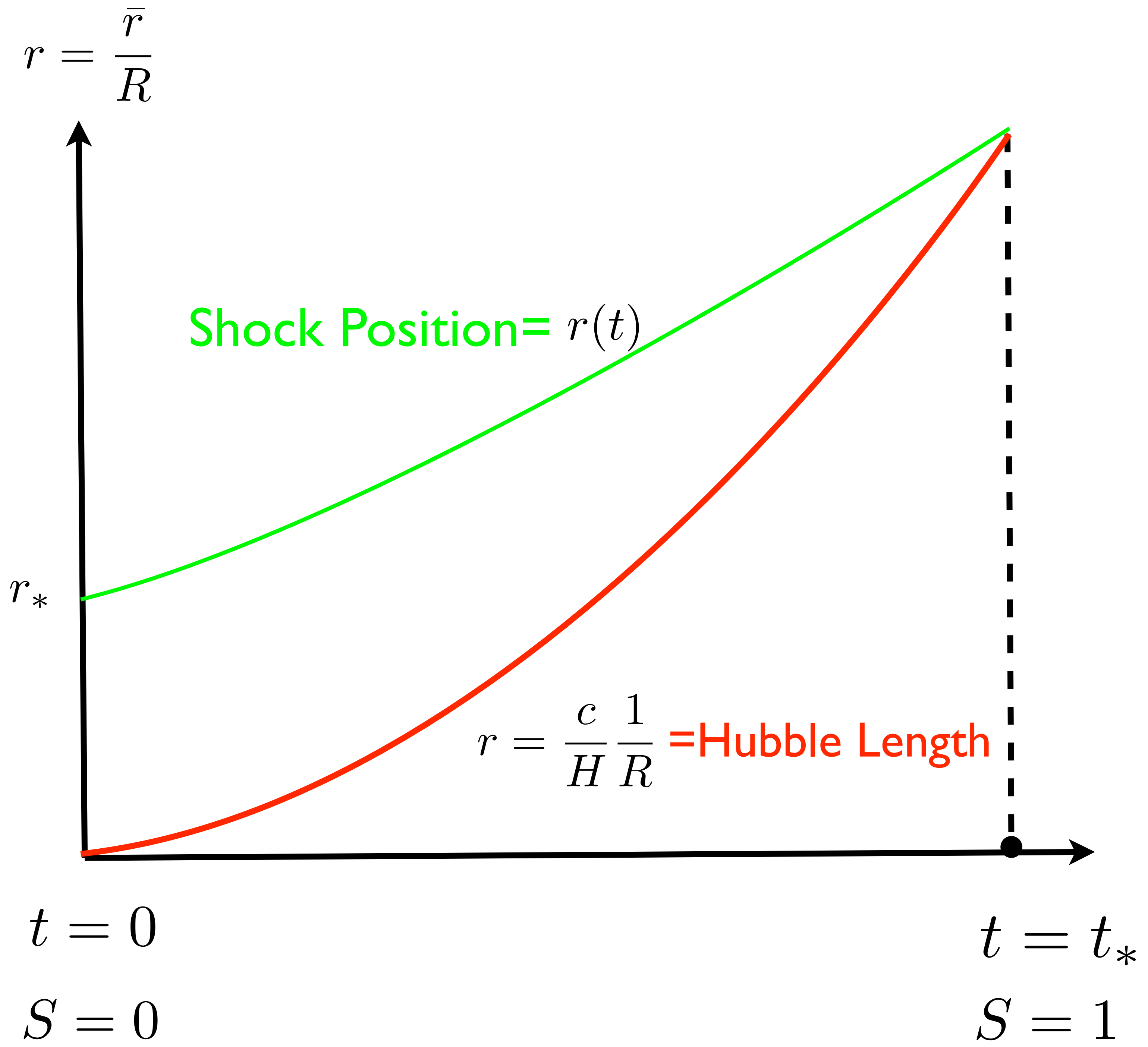
$\sigma = 1/3$



$S = 0$ *Big Bang*

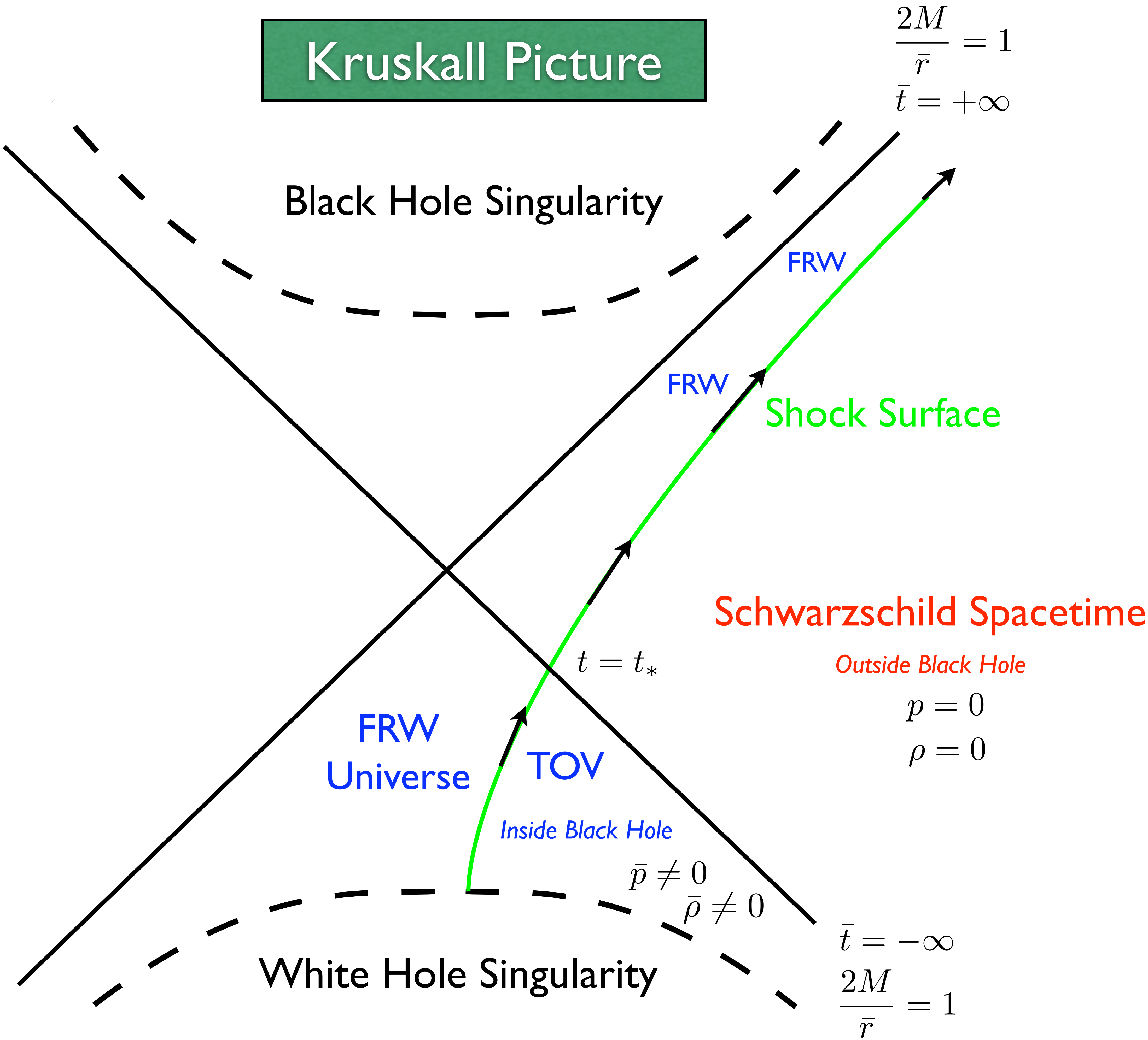
$S = 1$ *Solution emerges from White Hole*

$$s_\sigma(S) \to c \quad \text{as} \quad S \to 0$$



- The Hubble length catches up to the shock-wave at S=1, the time when the entire solution emerges from the White Hole

Kruskall Picture

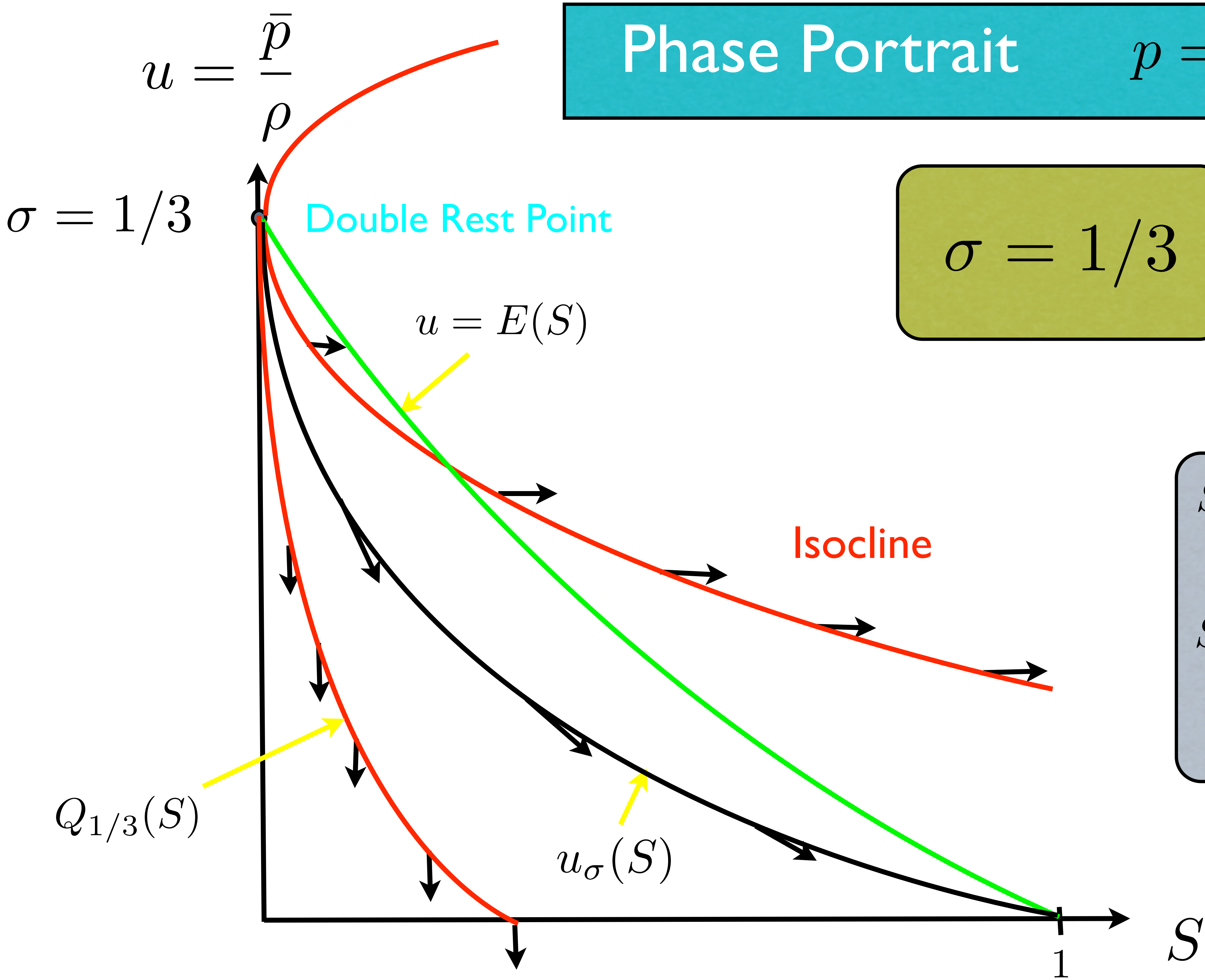


- We are interested in the case $\sigma = 1/3$
  $\approx$ correct for $t = Big\ Bang$ to $t = 10^5 yr$

- ρ and p on the FRW and TOV side
  tend to the same values as $t \to 0$

- It is as though the solution is emerging from a spacetime of constant density and pressure at the Big Bang $\approx$ Inflation

Phase Portrait $\quad p = \sigma\rho$



$$\sigma = 1/3$$

$S = 0$ *Big Bang*

$S = 1$ *Solution emerges from White Hole*

$$s_\sigma(S) \to c \quad \text{as} \quad S \to 0$$

Conclude: A solution like this would emerge at the end of inflation if the fluid at the end of inflation became co-moving wrt a ($k = 0$) FRW metric for $\bar{r} < \bar{r}_0$, and co-moving wrt the simplest spacetime of finite total mass for $\bar{r} > \bar{r}_0$.

The inflationary diSitter spacetime has all of the symmetries of a vaccuum, and so there is no preferred frame at the end of inflation

Finite-Mass time-slice at the end of Inflation

$\frac{2M}{\bar{r}} > 1$ *and* $\bar{r}$ *timelike*

diSitter spacetime
in (k=0)-FRW coordinates

diSitter spacetime
in TOV-coordinates

No preferred coordinates

$t = const.$

$M, \bar{r} = const.$

Inflation $T_{ij} = -\rho_* g_{ij}$

End of Inflation $T_{ij} = (\rho + p)u^i u^j + p g_{ij}$

u co-moving wrt
$t = const.$

u co-moving wrt
$M = const.$

Shock

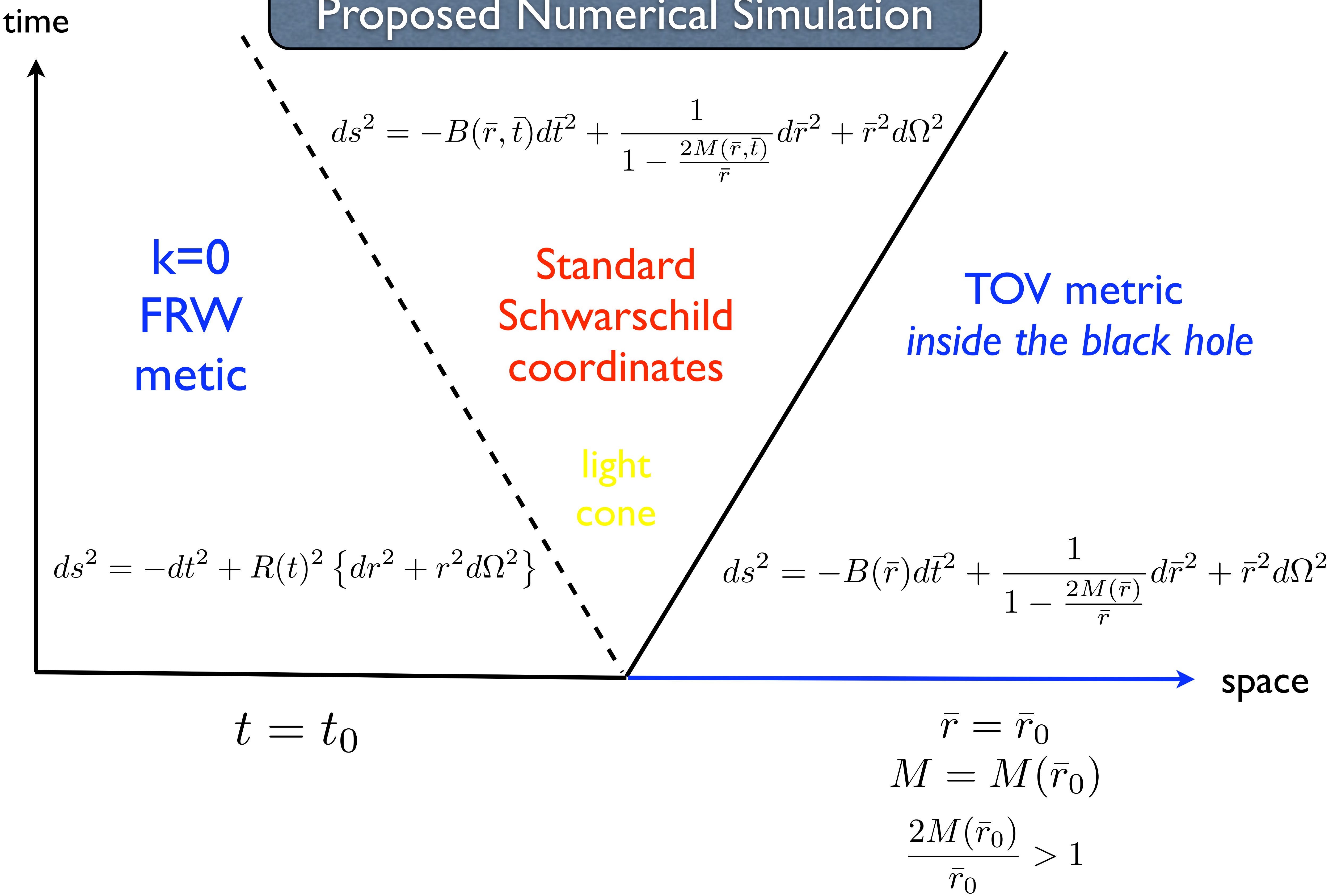
$(k = 0)$-FRW

$$ds^2 = -dt^2 + R(t)^2 \left\{ dr^2 + r^2 d\Omega^2 \right\}$$

TOV

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

Proposed Numerical Simulation



$$t = \text{the end of inflation} \approx 10^{-30} s = t_0$$

A Locally Inertial Method for Computing Shocks

Einstein equations-Spherical Symmetry

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)$$

$$G = 8\pi T$$

$$\begin{aligned}
\frac{A}{r^2 B}\left\{r\frac{B'}{B} + B - 1\right\} &= \kappa A^2 T^{00} && (1)\\
-\frac{B_t}{rB} &= \kappa A B T^{01} && (2)\\
\frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} &= \kappa B^2 T^{11} && (3)\\
-\frac{1}{rAB^2}\left\{B_{tt} - A'' + \Phi\right\} &= \frac{2\kappa r}{B} T^{22}, && (4)
\end{aligned}$$

$$B = \frac{1}{1 - \frac{2M}{r}}$$

$$\begin{aligned}
\Phi &= -\frac{BA_tB_t}{2AB} - \frac{B}{2}\left(\frac{B_t}{B}\right)^2 - \frac{A'}{r} + \frac{AB'}{rB}\\
&\quad + \frac{A}{2}\left(\frac{A'}{A}\right)^2 + \frac{A}{2}\frac{A'}{A}\frac{B'}{B}.
\end{aligned}$$

(1)+(2)+(3)+(4) ⟺ (1)+(3)+div T=0

(weakly)

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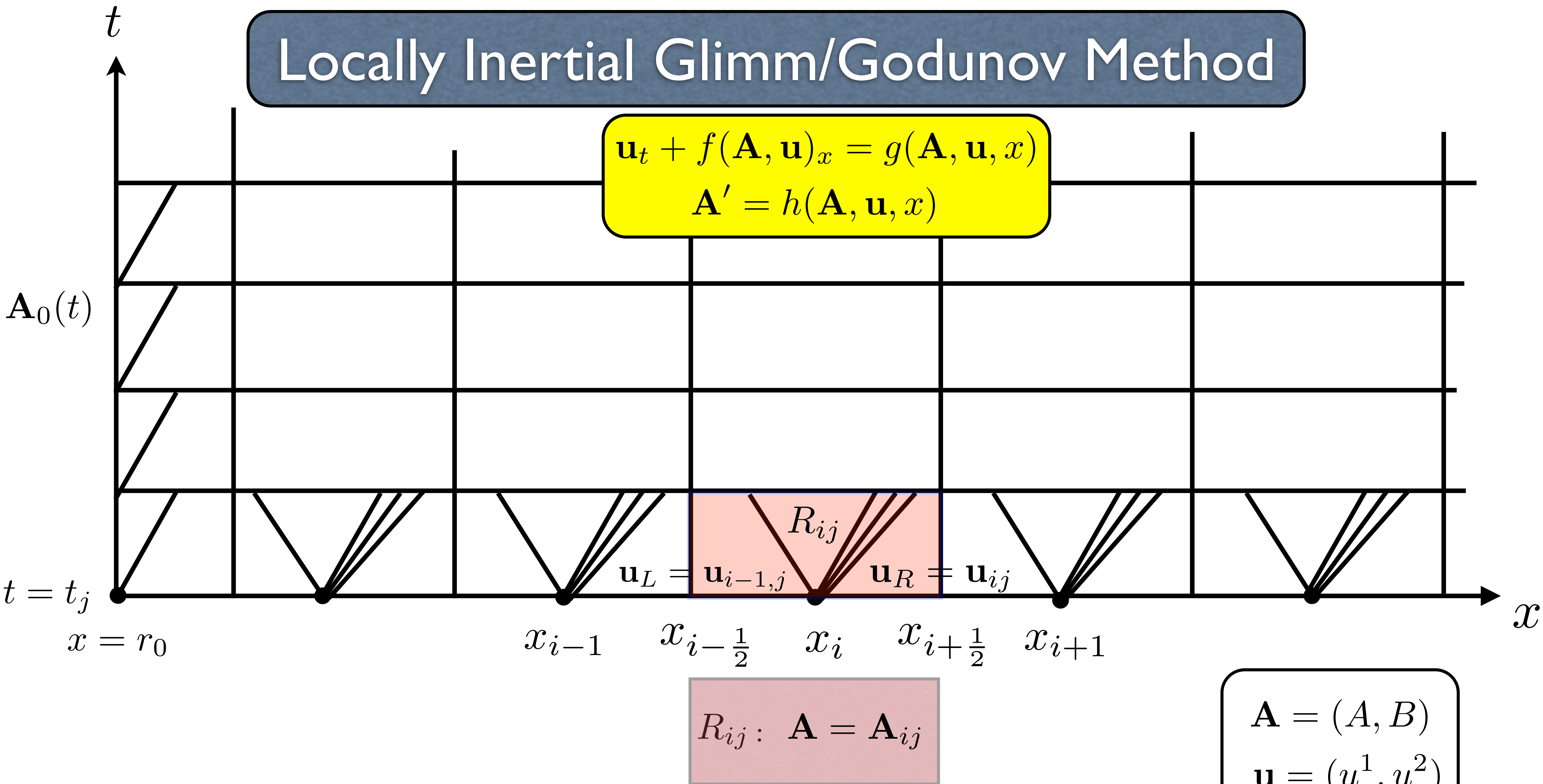
- *Shock Wave Interactions in General Relativity: A Locally Inertial Glimm Scheme for Spherically Symmetric Spacetimes*, with J. Groah and J. Smoller, Springer Monographs in Mathematics, 2007.

Locally Inertial Glimm/Godunov Method



- $\mathbf{A}_{ij} = const. \implies$ Locally Flat in each Grid Cell
- Fractional Step Method: — Solve RP for 1/2 timestep, Solve ODE for 1/2 time step → Sample or Average → $\mathbf{A}' = h(\mathbf{A},\mathbf{u},x)$, $\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$
- $p = \dfrac{c^2}{3}\rho \implies$ Nishida System $\implies$ Global Exact Soln of RP, [Smol,Te]

Remarkable Change of Variables

- Equations close under change to Local Minkowski variables: T $\longrightarrow$ $\mathbf{u} = T_M$

- I.e., Div T=0 reads:

$$0 = T^{00}_{,0} + T^{01}_{,1} + \frac{1}{2}\left(\frac{2A_t}{A} + \frac{B_t}{B}\right)T^{00} + \frac{1}{2}\left(\frac{3A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) + \frac{B_t}{2A}T^{11}$$

$$0 = T^{01}_{,0} + T^{11}_{,1} + \frac{1}{2}\left(\frac{A_t}{A} + \frac{3B_t}{B}\right)T^{01} + \frac{1}{2}\left(\frac{A'}{A} + \frac{2B'}{B} + \frac{4}{r}\right)T^{11} + \frac{A'}{2B}T^{00} - 2\frac{r}{B}T^{22}$$

- Time derivatives A_t and B_t cancel out under change $T \rightarrow \mathbf{u}$

- Good choice because o.w. there is no A_t equation to close $Div\ T = 0$!

$$\frac{A}{r^2B}\left\{r\frac{B'}{B} + B - 1\right\} = \kappa A^2 T^{00} \quad (1)$$

$$-\frac{B_t}{rB} = \kappa ABT^{01} \quad (2)$$

$$\frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} = \kappa B^2 T^{11} \quad (3)$$

$$-\frac{1}{rAB^2}\left\{B_{tt} - A'' + \Phi\right\} = \frac{2\kappa r}{B}T^{22}, \quad (4)$$

Locally Inertial Formulation

$$\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{\frac{A}{B}}T_M^{01}\right\}_{,1} = -\frac{2}{x}\sqrt{\frac{A}{B}}T_M^{01}, \qquad (1)$$

$$\left\{T_M^{01}\right\}_{,0} + \left\{\sqrt{\frac{A}{B}}T_M^{11}\right\}_{,1} = -\frac{1}{2}\sqrt{\frac{A}{B}}\left\{\frac{4}{x}T_M^{11} + \frac{(B-1)}{x}(T_M^{00} - T_M^{11}) + 2\kappa x B(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4xT^{22}\right\}, \qquad (2)$$

$$\frac{B'}{B} = -\frac{(B-1)}{x} + \kappa x B T_M^{00}, \qquad (3)$$

$$\frac{A'}{A} = \frac{(B-1)}{x} + \kappa x B T_M^{11}. \qquad (4)$$

$$\mathbf{u} = (T_M^{00}, T_M^{01})$$

$$\mathbf{A} = (A, B)$$

$$T_M^{00} = \frac{c^4 + \sigma^2 v^2}{c^2 - v^2}\rho$$

$$T_M^{01} = \frac{c^2 + \sigma^2}{c^2 - v^2}cv\rho$$

$$T_M^{11} = \frac{v^2 + \sigma^2}{c^2 - v^2}\rho c^2$$

Flat Space Relativistic Euler

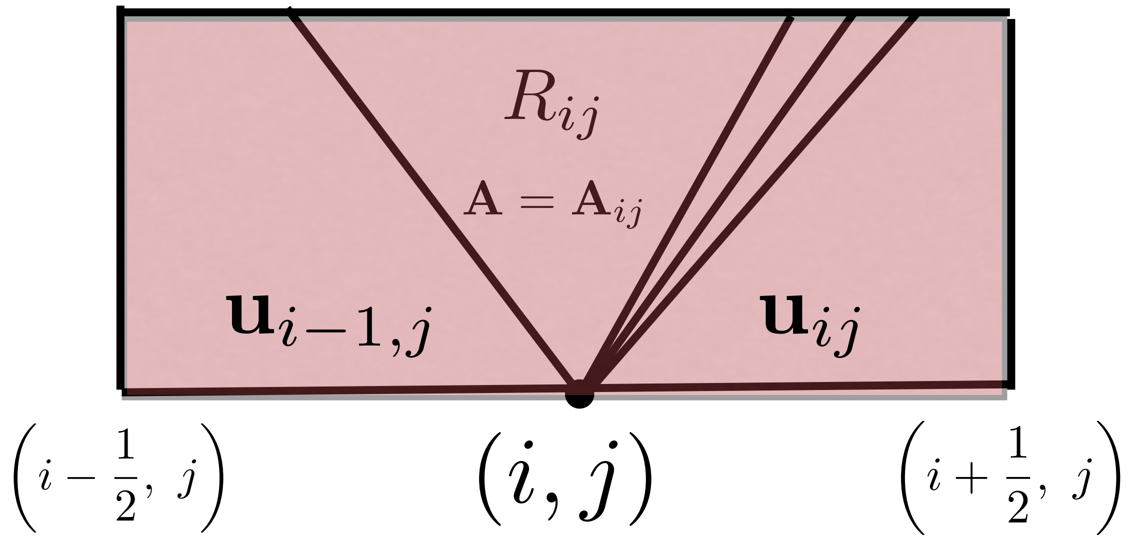
$$\left\{T_M^{00}\right\}_{,0} + \left\{T_M^{10}\right\}_{,1} = 0$$

$$\left\{T_M^{01}\right\}_{,0} + \left\{T_M^{11}\right\}_{,1} = 0$$

$$\mathbf{u}_t + f(\mathbf{A}, \mathbf{u})_x = g(\mathbf{A}, \mathbf{u}, x)$$

$$\mathbf{A}' = h(\mathbf{A}, \mathbf{u}, x)$$

Grid Rectangle



- Solve RP for $\frac{1}{2}$-timestep

$$\mathbf{u}_t + f(\mathbf{A}_{ij}, \mathbf{u})_x = 0$$

$$\mathbf{u} = \begin{cases} \mathbf{u}_{i-1,j} & x \le x_i \\ \mathbf{u}_{ij} & x > x_i \end{cases} \rightarrow u_{ij}^{RP}$$

- Solve ODE for $\frac{1}{2}$-timestep

$$\mathbf{u}_t = g(\mathbf{A}_{ij}, \mathbf{u}, x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$$

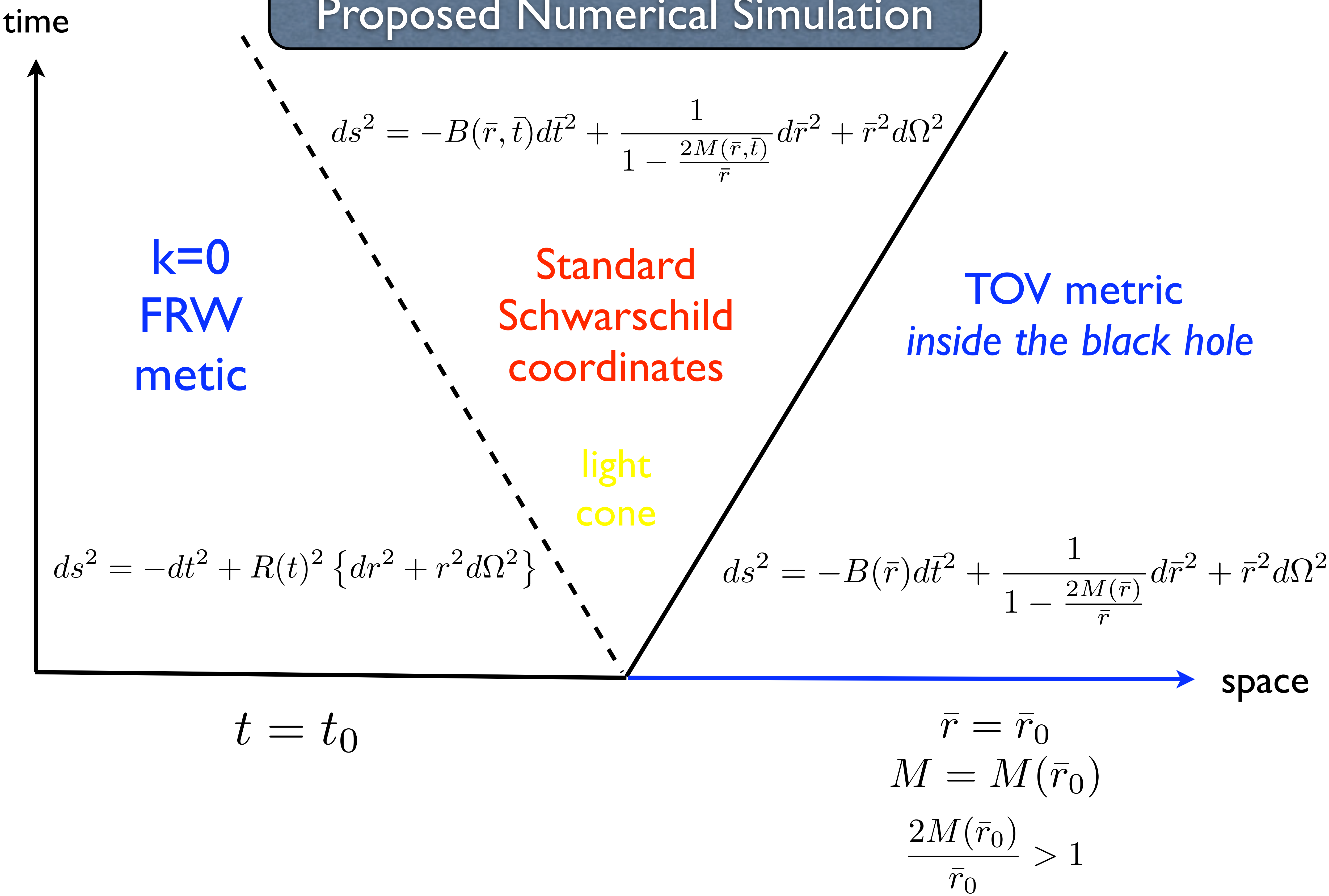
$$\mathbf{u}(0) = \mathbf{u}_{ij}^{RP}$$

- Sample/Average then update $\mathbf{A}$ to time t_{j+1}

$$\mathbf{A}' = h(\mathbf{A}, \mathbf{u}, x)$$

$$\mathbf{A}(r_0, t_{j+1}) = \mathbf{A}_0(t_{j+1})$$

Proposed Numerical Simulation



$$t = \text{the end of inflation} \approx 10^{-30}s = t_0$$

Speculative Question: Could the anomolous acceleration of the Galaxies and Dark Energy be explained within Classical GR as the effect of looking out into a wave?

This model represents the simplest simulation of such a wave

Standard Model for Dark Energy

- Assume Einstein equations with a cosmological constant:

$$G_{ij} = 8\pi T_{ij} + \Lambda g_{ij}$$

- Assume $k = 0$ FRW: $$ds^2 = -dt^2 + R(t)^2 \left\{dr^2 + r^2 d\Omega^2\right\}$$

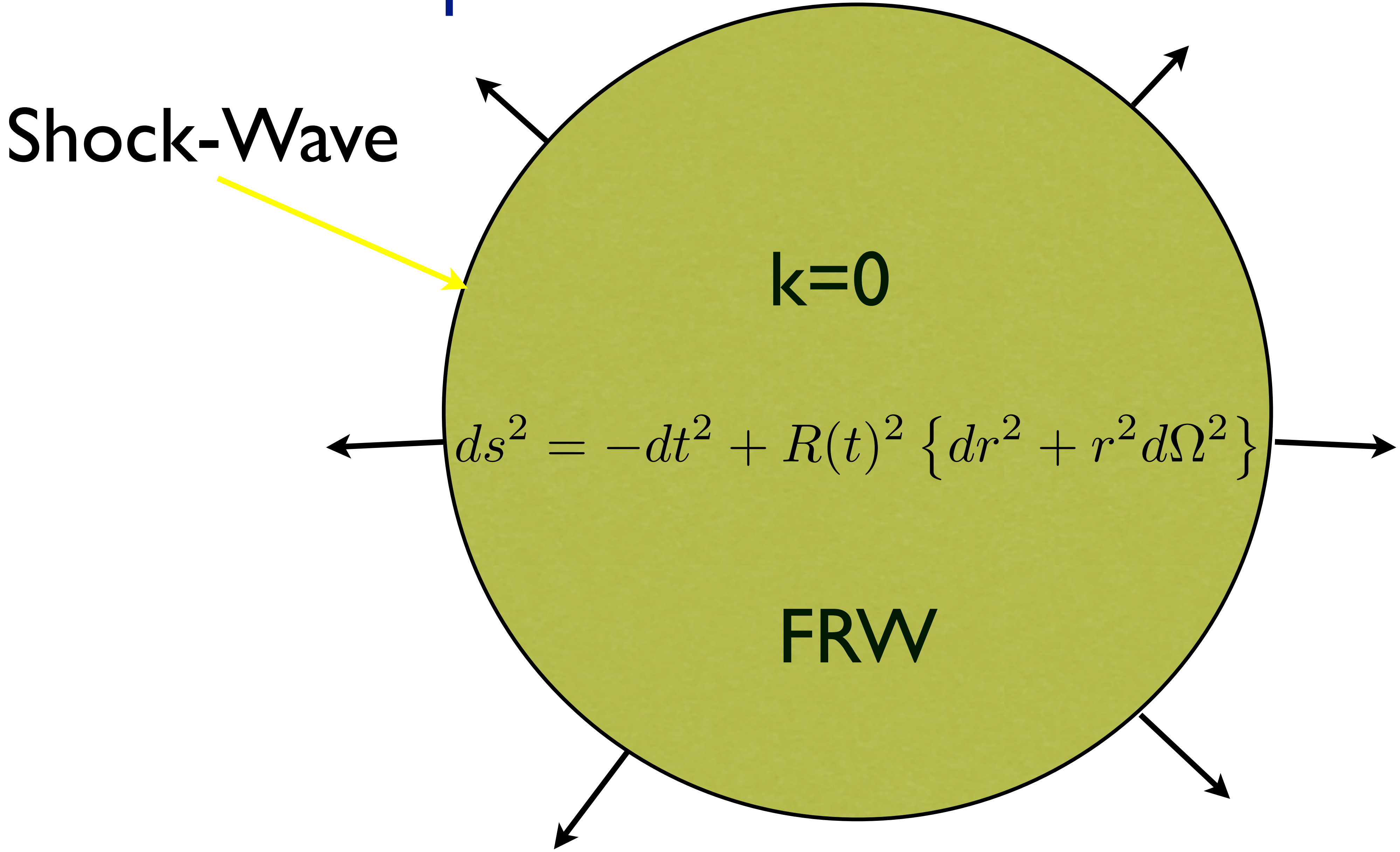
- Leads to: $$H^2 = \frac{\kappa}{3}\rho + \frac{\kappa}{3}\Lambda$$

- Divide by $H^2 = \rho_{crit}$: $$1 = \Omega_M + \Omega_\Lambda$$

- Best data fit leads to $\Omega_\Lambda \approx .73$ and $\Omega_M \approx .27$

- Implies: The universe is 73 percent dark energy

Could the Anomalous acceleration be accounted for by an expansion behind the Shock Wave?



TOV: $$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{1 - \frac{2M(\bar{r})}{\bar{r}}} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$$\frac{2M}{\bar{r}} > 1 \quad \Longrightarrow \quad \bar{r} \text{ is timelike}$$

Conclusion

- We think this numerical proposal represents a natural mathematical starting point for numerically resolving the secondary waves neglected in the exact solution.
- Also a possible starting point for investigating whether the anomalous acceleration/"Dark Energy" could be accounted for within classical GR with classical sources?