# A DECAY THEOREM FOR SOME SYMMETRIC HYPERBOLIC SYSTEMS

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Abstract. In this short note we consider smooth solutions to certain hyperbolic systems of equations. We present a condition which will ensure that no shocks develop and that solutions decay in $L^2$. The condition is restrictive in general; however when applied to the system of one dimensional gas dynamics it is shown that if the condition is satisfied initially then it will be satisfied for all time and therefore one obtains smooth solutions which decay.

## 1. Introduction

In this note we present a decay result for symmetric hyperbolic systems which satisfy certain conditions. The result is stated for $u(t, x) \in \mathbb{R}^n$, with $x \in \mathbb{R}^m$, for any $m, n$. The conditions under which the solutions will decay are fairly restrictive and thus the utility of this theorem is less clear for the most physically interesting cases of $m, n$. Nevertheless, one can show that this condition is satisfied for all time given that it is satisfied initially for the specific case of one dimensional gas dynamics (with $\gamma > 3$) and this example is presented in the sequel.

The result relies on the spatial and time translation invariance of our system, along with its' invariance under scaling. There is a long history of the use of such invariants in decay results. For example, in [9] the scaling invariance is used to get a decay result for Maxwell's equations. By considering additional symmetries one can do better. In [4] the Lorentz invariance of the wave equation leads to decay estimates which imply almost global existence for the wave equation. This method also works to obtain global existence results for the Born-Infeld equations which are a nonlinear version of Maxwell's equations, see [1], and also [2, 7]. In [10] the Galilean invariance of certain isotropic systems is used to obtain similar type decay estimates which can be used to obtain almost global existence for certain systems such as the equations of elasticity. The Galilean invariance of the wave equation is also used in [5, 8] to show decay of certain wave equations outside star-shaped obstacles. By considering only invariance under spatial and time translation and scaling, one might hope to develop tools which will apply to systems which are not isotropic, for example the equations of crystal optics. However at this stage we are far from a general result in that direction.

Quasi-linear hyperbolic equations frequently develop shocks. A tremendous body of work has been and continues to be developed surrounding conditions under which shocks develop, and how to deal with solutions after shocks have developed, see [6] and also [3]. In what follows we will work with the situation where no shocks develop and are concerned only with the behavior of smooth solutions to our equations.

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In the case of scalar equations in one space variable our condition is exactly the condition necessary to say that shocks will not develop.

## 2. Decay Result

**Theorem 1.** *Let $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$. Suppose $u$ satisfies a hyperbolic system*

$$\partial_t u_i + \sum_{k=1}^{m} \sum_{j=1}^{n} A_{ij}^k(u) \partial_k u_j = 0 \tag{2.1}$$

*and $u = u_0$ for $t = 0$, with $u = \{u_i\}$, $i = 1, ..., n$. If there are no shocks, $u \equiv 0$ for $r > r_0 + Mt$, $A_{ij}^k$ is symmetric for each $k$, and*

$$R_{ij} \equiv \frac{\partial A_{i\ell}^k}{\partial u_j} \partial_k u_\ell - \frac{1}{2} \partial_k (A_{ij}^k(u)) \tag{2.2}$$

*is nonnegative then we have the following*[1]*:*

- *$r\partial_r u + t\partial_t u, \nabla u, \partial_t u$ are all bounded in $L^2$ by their $L^2$ bounds at $t = 0$.*
- *$\int_{r<r_0} t^2 (\partial_t u)^2 dx$ is bounded and hence $\partial_t u \to 0$ in $L^2$ confined to a ball.*

We begin the proof with two lemmas.

**Lemma 1.** *Suppose that $u$ satisfies (2.1), then $\partial_t u, \nabla u$, and $r\partial_r u + t\partial_t u$ all satisfy the perturbation equation for $\delta u$ given by*

$$\partial_t(\delta u_i) + \frac{\partial A_{ij}^k(u)}{\partial u_\ell} \delta u_\ell \partial_k u_j + A_{ij}^k(u) \partial_k \delta u_j = 0 \tag{2.3}$$

**Lemma 2.** *If $u$ satisfies (2.1), then for any $t$ we have,*

$$\int |\delta u_i(t)|^2 dx \leq \int |\delta u_i(0)|^2 dx \tag{2.4}$$

*where $\delta u$ is any of $\partial_t u, \nabla u$, or $r\partial_r u + t\partial_t u$.*

*Proof of Lemma 1.* If $u(x, t)$ is a solution of (2.1) then by the translation and scaling invariance of (2.1), $u(kx, kt)$, $u(x + ke_j, t)$, and $u(x, t + k)$ are also solutions of (2.1) for $j = 1, .., m$. Here $\{e_j\}_{j=1}^m$ are the standard basis vectors in $\mathbb{R}^m$. Differentiation with respect to $k$ and setting $k = 0$, or $k = 1$ as appropriate implies the result. □

*Proof of Lemma 2.* Following the energy method we multiply (2.3) by $\delta u_i$ and we have

$$\begin{aligned} \frac{1}{2}\partial_t(\delta u_i)^2 + \delta u_i \delta u_\ell \frac{\partial A_{ij}^k(u)}{\partial u_\ell} \partial_k u_j + \frac{1}{2} \partial_k (A_{ij}^k(u) \delta u_i \delta u_j) \\ + \frac{1}{2} \delta u_i A_{ij}^k(u) \partial_k \delta u_j - \frac{1}{2} \partial_k \delta u_i A_{ij}^k(u) \delta u_j - \frac{1}{2} \partial_k (A_{ij}^k(u)) \delta u_i \delta u_j = 0. \end{aligned} \tag{2.5}$$

Since $A_{ij}^k = A_{ji}^k$ for each $k$ we have

$$\frac{1}{2}\partial_t(\delta u_i)^2 + \frac{1}{2} \partial_k (A_{ij}^k(u) \delta u_i \delta u_j) + R_{ij} \delta u_i \delta u_j = 0. \tag{2.6}$$

Now if $R_{ij}$ is nonnegative, then we can integrate over the space variables to obtain

$$\int \partial_t (\delta u_i)^2 dx \leq 0.$$

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[1]Summation over repeated indices will always be understood.

Now integrating in time and using the fact that the solution is zero outside a ball we obtain the result. □

*Proof of Theorem 1.* Using Lemma 1 we can apply the result of Lemma 2 to any one of $\nabla u$, $\partial_t u$, and $r\partial_r u + t\partial_t u$. Now we have

$$(r\partial_r u + t\partial_t u)^2 = \frac{1}{2}t^2\partial_t u^2 + \frac{1}{2}(t\partial_t u + 2r\partial_r u)^2 - r^2\partial_r u^2$$
$$\geq \frac{1}{2}t^2\partial_t u^2 - r^2\partial_r u^2.$$

Which upon using Lemma 2 implies that on a ball $|x| < r_0$,

$$\int_{r\leq r_0} t^2\partial_t u^2 dx \leq C\int(|\nabla u|^2 + (r\partial_r u + t\partial_t u)^2)dx$$
$$\leq C\int(|\nabla u(0)|^2 + (r\partial_r u(0))^2)dx.$$

This implies that $\int_{r\leq r_0}\partial_t u^2 dx$ decays like $1/t^2$. □

The condition that $R_{ij}$ is nonnegative is very restrictive. One would like to know that if it is true initially than it remains true. It seems very unlikely that this would hold true in general, and we would like to see some examples where it does hold. In the case of scalar equations in one variable this is the standard condition for no shocks developing. We consider here the case of one dimensional gas dynamics (with $\gamma > 3$) where we are able to verify that if the condition holds initially then it will hold for all time.

## 3. One-dimensional Gas Dynamics

Consider the equations of one dimensional gas dynamics. We have

$$u_t + uu_x + p_x/\rho = 0, \tag{3.1}$$

$$\rho_t + u\rho_x + \rho u_x = 0. \tag{3.2}$$

with $p = A\rho^\gamma$. As is standard we can rewrite (3.1) using the speed $c^2 = \gamma A\rho^{\gamma-1}$, as

$$u_t + uu_x + c^2\rho_x/\rho = 0. \tag{3.3}$$

To make the system symmetric hyperbolic we will introduce a change of variable, $g(\rho)$, where $g'(\rho) = c/\rho$. Note that this implies that $g = \frac{\gamma^{1/2}A^{1/2}\rho^{(\gamma-1)/2}}{(\gamma-1)/2} = \frac{2c}{\gamma-1}$. This gives the equations in the form

$$\partial_t\begin{pmatrix} u \\ g \end{pmatrix} + \begin{pmatrix} u & c \\ c & u \end{pmatrix}\partial_x\begin{pmatrix} u \\ g \end{pmatrix} = 0. \tag{3.4}$$

If we next let $U_1 = u + g$, and $U_2 = u - g$, we can diagonalize the system to:

$$\partial_t U_1 + \frac{1}{4}\left[(\gamma+1)U_1 - (\gamma-3)U_2\right]\partial_x U_1 = 0, \tag{3.5}$$

$$\partial_t U_2 + \frac{1}{4}\left[-(\gamma-3)U_1 + (\gamma+1)U_2\right]\partial_x U_2 = 0. \tag{3.6}$$

where $u = \frac{1}{2}(U_1 + U_2)$, $g = \frac{1}{2}(U_1 - U_2)$, and hence $c = \frac{\gamma-1}{4}(U_1 - U_2)$. It is in this form that we can apply Theorem 1.

The perturbation equation in this case is then

(3.7)
$$\partial_t \delta U_1 + \frac{1}{4}\left\{\left[(\gamma+1)U_1 - (\gamma-3)U_2\right]\partial_x \delta U_1 + \left[(\gamma+1)\delta U_1 - (\gamma-3)\delta U_2\right]\partial_x U_1\right\} = 0,$$

(3.8)
$$\partial_t \delta U_2 + \frac{1}{4}\left\{\left[-(\gamma-3)U_1 + (\gamma+1)U_2\right]\partial_x \delta U_2 + \left[-(\gamma-3)\delta U_1 + (\gamma+1)\delta U_2\right]\partial_x U_2\right\} = 0.$$

If we multiply the equations by $\delta U_1$, and $\delta U_2$, respectively and then add them together we will have

(3.9)
$$\begin{aligned}&\partial_t(\delta U_1^2 + \delta U_2^2) + \\ &\frac{1}{4}\{\partial_x\left[((\gamma+1)U_1 - (\gamma-3)U_2)\,\delta U_1^2\right] + \partial_x\left[(-(\gamma-3)U_1 + (\gamma+1)U_2)\,\delta U_2^2\right] \\ &- ((\gamma+1)\partial_x U_1 - (\gamma-3)\partial_x U_2)\,\delta U_1^2 - (-(\gamma-3)\partial_x U_1 + (\gamma+1)\partial_x U_2)\,\delta U_2^2\} \\ &+\frac{1}{2}\{\left((\gamma+1)\delta U_1^2 - (\gamma-3)\delta U_1\delta U_2\right)\partial_x U_1 + \left(-(\gamma-3)\delta U_1\delta U_2 + (\gamma+1)\delta U_2^2\right)\partial_x U_2\} = 0\end{aligned}$$

Thus

(3.10)
$$\begin{aligned}8R = \left[(\gamma+1)\partial_x U_1 + (\gamma-3)\partial_x U_2\right]\delta U_1^2 - 2(\gamma-3)(\partial_x U_1 + \partial_x U_2)\delta U_1\delta U_2 \\ + \left[(\gamma-3)\partial_x U_1 + (\gamma+1)\partial_x U_2\right]\delta U_2^2\end{aligned}$$

We are now ready to state the following result:

**Proposition 1.** *Let* $\gamma > 3$. *Let* $U = (U_1, U_2) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, *be a solution to* (3.5)-(3.6). *Suppose in addition that* $\partial_x U_1$, $\partial_x U_2$ *are positive initially, then* $R$ *defined in* (3.10) *is also nonnegative and hence Theorem 1 applies. Furthermore,* $|\partial_x U_1|$ *and* $|\partial_x U_2|$ *remain bounded for all time.*

*Proof.* To show that $R$ is positive it is sufficient to show that
$$(\gamma-3)\partial_x U_1^2 + 2(\gamma-1)\partial_x U_1\partial_x U_2 + (\gamma-3)\partial_x U_2^2 \geq 0,$$
and that
$$\begin{aligned}(\gamma+1)\partial_x U_1 + (\gamma-3)\partial_x U_2 > 0 \\ (\gamma-3)\partial_x U_1 + (\gamma+1)\partial_x U_2 > 0,\end{aligned}$$
However, since we are considering $\gamma > 3$, it is sufficient to show that if $\partial_x U_1, \partial_x U_2$ are positive initially, then they remains positive. From (3.5),(3.6) we can derive the equations for $\partial_x U_1, \partial_x U_2$ by differentiation. This gives

(3.11)
$$\begin{aligned}\partial_t(\partial_x U_1) + \frac{1}{4}\{((\gamma+1)U_1 - (\gamma-3)U_2)\partial_x(\partial_x U_1) \\ + ((\gamma+1)\partial_x U_1 - (\gamma-3)\partial_x U_2)\partial_x U_1\} = 0,\end{aligned}$$

(3.12)
$$\begin{aligned}\partial_t(\partial_x U_2) + \frac{1}{4}\{(-(\gamma-3)U_1 + (\gamma+1)U_2)\partial_x(\partial_x U_2) \\ + (-(\gamma-3)\partial_x U_1 + (\gamma+1)\partial_x U_2)\partial_x U_2\} = 0.\end{aligned}$$

Setting
$$\frac{d}{ds_1} = \partial_t + \frac{1}{4}[(\gamma+1)U_1 - (\gamma-3)U_2]\partial_x,$$

and

$$\frac{d}{ds_2} = \partial_t + \frac{1}{4}[-(\gamma-3)U_1 + (\gamma+1)U_2]\partial_x,$$

we obtain

$$\frac{d}{ds_1}(\partial_x U_1) + f_1(\partial_x U)\partial_x U_1 = 0, \tag{3.13}$$

$$\frac{d}{ds_2}(\partial_x U_2) + f_2(\partial_x U)\partial_x U_2 = 0, \tag{3.14}$$

with

$$f_1(\partial_x U) = \frac{1}{4}[(\gamma+1)\partial_x U_1 - (\gamma-3)\partial_x U_2]$$

and

$$f_2(\partial_x U) = \frac{1}{4}[-(\gamma-3)\partial_x U_1 + (\gamma+1)\partial_x U_2].$$

Equations (3.13),(3.14) are homogeneous ordinary differential equations along the characteristics for $\partial_x U_1$, $\partial_x U_2$. Hence the sign of $\partial_x U_1$, respectively $\partial_x U_2$, is determined by its sign initially.

To show that $|\partial_x U_1|$ and $|\partial_x U_2|$ remain bounded for all time, we again look at the equations in the form (3.13),(3.14), and hence we can write

$$\partial_x U_1(t,x) = \partial_x U_1(0,x)e^{\int -f_1(\partial_x U)ds_1}, \tag{3.15}$$

$$\partial_x U_2(t,x) = \partial_x U_2(0,x)e^{\int -f_2(\partial_x U)ds_2}. \tag{3.16}$$

Now, suppose $\partial_x U_1(t,x)$ remains bounded for all $t < t^*$ and $t^*$ is the first time where $\lim_{t\to t^*} \partial_x U_1(t,x) \to \infty$ and $\partial_x U_2$ remains bounded up to time $t = t^*$. Then $\lim_{t\to t^*} f_1(\partial_x U) \to -\infty$. But this is a contradiction given the definition of $f_1$, and the fact that $\partial_x U_2$ is bounded, and $\partial_x U_1 > 0$. Similarly one can show that if $\partial_x U_1$ is bounded than so is $\partial_x U_2$.

Suppose that both $\partial_x U_1$, $\partial_x U_2 \to \infty$ as $t \to t^*$. That implies that $f_1, f_2 \to -\infty$. Since we know that $\partial_x U_1, \partial_x U_2 > 0$, by the definition of $f_i$ this implies that

$$\frac{\partial_x U_1}{\partial_x U_2} \le \frac{\gamma-3}{\gamma+1} \text{ and } \frac{\partial_x U_1}{\partial_x U_2} \ge \frac{\gamma+1}{\gamma-3}.$$

This implies that $(\gamma-3)^2 \ge (\gamma+1)^2$, which is a contradiction for $\gamma > 3$, and hence $\partial_x U_1$, $\partial_x U_2$ remain bounded for all time. □

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