Some Background Material from Analysis[1]

1 Elementary point set topology

We denote by $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

- If $x, y \in \mathbb{R}$ the *distance between a and b* is defined to be

$$d(x, y) := |x - y| \,.$$

  If for $z \in \mathbb{C}$ we interpret $|z|$ as the absolute value of z,[2] then the same formula above holds for the distance between two complex numbers z and w.

- A *neighborhood* of a point p is a set $N_r(p)$ consisting of all points q such that $d(p, q) < r$. The number r is called the *radius* of $N_r(p)$.

- A point p is a *limit point* of the set E (a subset of either $\mathbb{R}$ or $\mathbb{C}$) if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$. (Note that p is not necessarily in the set E.)

- The set E is said to be *closed* if every limit point of E is a point of E. Thus, for example, the interval $[a, b] := \{x : a \leq x \leq b\}$ is a closed set in $\mathbb{R}$. We call $[a, b]$ a *closed interval*.

- A point $p \in E$ is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$. For example, the point $\frac{1}{2}$ is an interior point of $[0, 1]$ where as 0 and 1 are *not* interior points of $[0, 1]$.

- A set $E \subset X$ ($X = \mathbb{R}$ or $X = \mathbb{C}$) is *open* if every point of E is an interior point of E. For example, the interval $(a, b) := \{x : a < x < b\} \subset \mathbb{R}$, $a < b$, is an open set in $\mathbb{R}$.

- The *complement* of $E \subset X$ (here X is either $\mathbb{R}$ or $\mathbb{C}$) is the set of points $p \in X$ such that $p \notin E$.

- A set E is *bounded* if there is a real number M and a point q such that $d(p, q) < M$ for all $p \in E$.

- A set $E \subset X$ ($X = \mathbb{R}$ or $X = \mathbb{C}$) is *compact* if it is both closed and bounded.[3]

[1] For full details see *Principles of Analysis* by Walter Rudin.

[2] Recall that if $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, then $|z| = \sqrt{x^2 + y^2}$.

[3] In most analysis courses this is a theorem called the Heine-Borel theorem since a compact set is defined as follows: An *open cover* of a set $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$. A set K is said to be *compact* if every open cover of K contains a *finite* sub cover. We will sometimes use this property of compact sets.

2 Continuity

As earlier, by X and Y we mean either $\mathbb{R}$ or $\mathbb{C}$. Suppose f is a function whose domain E is a subset of X and whose range is a subset of Y.

- We say f is *continuous* at the point $p \in E$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$d_Y\left(f(x), f(p)\right) < \varepsilon$$
for all points $x \in E$ for which $d_X(x,p) < \delta$. Here d_X denotes the distance function in X and similarly for d_Y. Given the definition of a limit, this is equivalent to
$$\lim_{x \to p} f(x) = f(p).$$

- Suppose $f : X \to Y$. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$d_Y(f(p), f(q)) < \varepsilon$$
for all p and q in X for which $d_X(p,q) < \delta$.

  Uniform continuity is a property of a function on a set whereas continuity can be defined at a single point. The choice of δ in the above definition of uniform continuity can be taken to hold for all points in the set X.

  An example here is useful. Consider the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = 1/x$. Let $x_0 \in (0,1)$ be a fixed point. We claim that f is continuous at x_0. It's pretty clear intuitively that $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$, but let's give an explicit $\varepsilon - \delta$ proof: Given an $\varepsilon > 0$ we wish to find a $\delta > 0$ such that if $|x - x_0| < \delta$, then it follows that $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$. We do a little computation:
$$|\frac{1}{x} - \frac{1}{x_0}| = \frac{|x - x_0|}{x x_0} < \frac{\delta}{x x_0}$$
We want the last quantity to be less that ε. Since $x_0 - \delta < x$, we have $\frac{1}{x} < \frac{1}{x_0 - \delta}$. Thus we want
$$\frac{\delta}{x_0(x_0 - \delta)} < \varepsilon$$
Solving this inequality for δ gives that we can choose any δ satisfying
$$\delta < \frac{x_0^2}{1 + \varepsilon x_0}$$
it follows by reversing the above computation that
$$|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon.$$

Note that as x_0 becomes smaller, the choice of δ becomes smaller. It is not possible to find one $\delta > 0$ that works for all x_0. Thus $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

- A fundamental result in analysis says that if E is a compact subset of X and f is continuous for each point $p \in E$, then f is uniformly continuous on E. Note that in the above example the set E was not compact.

- Suppose f is a continuous function on the compact set $E \subset X$ (and hence by the previous item, uniformly continuous on E). Let

$$m = \inf_{x \in E} f(p), \;\; M = \sup_{p \in E} f(p)$$

then there exist points $p_1, p_2 \in E$ such that $f(p_1) = m$ and $f(p_2) = M$. Recall that inf is the infinum which is the greatest lower bound and sup is the supremum which is the least upper bound.

3 Uniform Convergence

Many times in analysis we have a sequence of functions $\{f_n\}$ and we define a new function by $f := \lim_{n \to \infty} f_n$; and we then ask questions about the properties of f. For example, suppose each f_n is a continuous function, then is f necessarily continuous? To make this question precise we must first answer in what sense does f_n converge to f.

The first notion of convergence is *pointwise convergence*; that is suppose f_n are defined on some set E, then we say f_n converges pointwise on E to f if for each $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$. Since we are evaluating $f_n(x)$ at a point x, this is convergence of a sequence of *numbers* (we assume f_n are either real- or complex-valued). The following examples show that pointwise convergence of a sequence of functions does not imply the limiting function necessarily inherits the properties of the sequence functions:

1. $$f_n(x) := x^n, \;\; 0 \leq x \leq 1.$$

   For $|x| < 1$, $\lim_{n\to\infty} f_n(x) = 0$. Since $f_n(1) = 1$, $\lim_{n\to\infty} f_n(1) = 1$. Thus the limiting function is
   $$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$$
   which is clearly a discontinuous function of x.

2. Let
   $$f_m(x) := \lim_{n\to\infty} \left(\cos m!\pi x\right)^{2n}.$$

When $m!x$ is an integer, $\cos(m!\pi x) = \pm 1$ so that $f_m(x) = \lim_{n\to\infty}(\pm 1)^{2n} = 1$. For other values of x, the cosine of absolute value less than 1; and hence, $f_m(x) = 0$ for those values of x. Now let

$$f(x) = \lim_{m\to\infty} f_m(x)$$

If x is a rational number; say $x = p/q$ where p and q are integers, then for sufficiently large m (namely, $m \geq q$) $m!x$ is an integer and hence $f(x) = 1$. If x is irrational, then $m!x$ is never an integer and we obtain for every m $f_m(x) = 0$. Thus we've shown that

$$f(x) = \begin{cases} 0, x \text{ irrational} \\ 1, x \text{ rational} \end{cases}$$

Thus the limiting function f is neither continuous nor Riemann-integrable.

3. Let

$$f_n(x) := \frac{\sin nx}{\sqrt{n}}, \quad x \in \mathbb{R}, n = 1, 2, 3, \ldots.$$

Since the absolute value of the sine function is bounded by 1,

$$f(x) = \lim_{n\to\infty} f_n(x) = 0, x \in \mathbb{R}.$$

Thus $f'(x) = 0$. However

$$f_n'(x) = \sqrt{n}\cos(nx)$$

so that $\{f_n'\}$ does *not* converge to f'.

4. Let

$$f_n(x) = n^2 x(1-x^2)^n, \quad 0 \leq x \leq 1, n = 1, 2, \ldots.$$

For $0 < x < 1$, $\log f_n(x) = n\left[\log(1-x^2) + 2\log x\right] + \log x$. Since the quantity in square brackets is negative for $0 < x < 1$, $\lim_{n\to\infty} \log f_n(x) = -\infty$; and hence, $\lim_{n\to 1} f_n(x) = 0$. Since $f_n(0) = f_n(1) = 0$, we get $\lim_{n\to\infty} f_n(x) = 0$ for all $0 \leq x \leq 1$. Now

$$\int_0^1 x(1-x^2)^n \, dx = \frac{1}{2n+2}$$

so that

$$\int_0^1 f_n(x) \, dx = \frac{n^2}{2n+2}$$

Thus

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = \infty$$

but

$$\int_0^1 f(x) \, dx = 0.$$

Thus we have for this example

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

We now introduce a stronger notion of convergence of a sequence of functions, *uniform convergence*, that will guarantee nice limiting properties.

Definition. We say that a sequence of functions $\{f_n\}$, $n = 1, 2, \ldots$ converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

The difference between pointwise convergence and uniform convergence is that in pointwise convergence the N can depend both upon ε and x whereas for uniform convergence the N can depend only upon ε; that is, we can find one integer N that holds for *all* $x \in E$.

Definition. We say that the series $\sum_n f_n(x)$ converges uniformly on E if the sequence of partial sums $\{S_n\}$ defined by

$$S_n(x) := \sum_{j=1}^{n} f_j(x)$$

converges uniformly on E.

The Cauchy criterion for uniform convergence is as follows:

Theorem. The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies

$$|f_m(x) - f_n(x)| \leq \varepsilon.$$

Proof. Suppose $\{f_n\}$ converges uniformly to f on E. Then there is an integer N such that for all $n \geq N$ and all $x \in E$,

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}.$$

Then for all $m, n \geq N$ and all $x \in E$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(We used the triangle inequality $|x - y| \leq |x - z| + |z - y|$.) This proves one-half of the theorem.

Now suppose $\{f_n\}$ satisfies the above Cauchy criterion. By the Cauchy criterion for convergence of a sequence of numbers, we have the existence (pointwise) of a limiting function

f. We must show the convergence to f is uniform. Let $\varepsilon > 0$ be given and choose N such that for all $x \in E$ and all $m, n \geq N$ we have

$$|f_m(x) - f_n(x)| \leq \varepsilon.$$

Since $f_m(x) \longrightarrow f(x)$ pointwise, we take the limit $m \to \infty$ in the above inequality to obtain

$$|f(x) - f_n(x)| \leq \varepsilon$$

which holds for all $x \in E$ and all $n \geq N$. $\square$

It is sometimes useful to introduce the sup-norm: Suppose f is defined on a set E, the

$$\|f\| = \sup_{x \in E} |f(x)|$$

Theorem. The sequence $\{f_n\}$ converges uniformly on E if and only if $\|f_n - f\| \to 0$ as $n \to \infty$.

Proof. Suppose $\|f_n - f\| \to 0$ as $n \to 0$, then for every $\varepsilon > 0$ there exists an integer N such that for all $n \geq N$ we have

$$\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon.$$

Since for all $x \in E$

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)|$$

by definition of sup (the least upper bound), the sequence $\{f_n\}$ converges uniformly to f on E.

Now suppose $\{f_n\}$ converges uniformly to f on E. Let $\varepsilon > 0$ be given. Then there exists an integer N such that for all $x \in E$ and all $n \geq N$

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

The least upper bound of this inequality satisfies

$$\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$$

for all $n \geq N$. $\square$

Theorem.(Weierstrass M-test). Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose there exist nonnegative real numbers M_n such that

$$|f_n(x)| \leq M_n, \;\; x \in E, n = 1, 2, \ldots.$$

Then $\sum_n f_n$ converges uniformly on E if $\sum_n M_n$ converges.

Proof. If $\sum_n M_n$ converges then for arbitrary $\varepsilon > 0$

$$\left|\sum_{j=n}^{m} f_j\right| \leq \sum_{j=n}^{n} M_j \leq \varepsilon$$

provided m, n are large enough. By the Cauchy criterion we obtain uniform convergence of the series. □

Examples of the use of the Weierstrass M-test:

1. Let

   $$f(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}, \quad x \in \mathbb{R}.$$

   For any closed interval $[a, b] \subset \mathbb{R}$

   $$\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$$

   and since the series $\sum_n 1/n^2$ is convergent, the above series converges uniformly on every closed interval of $\mathbb{R}$.

2. Suppose f is defined by the power series

   $$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R,$$

   where R is the radius of convergence (which may be infinite). Let $0 < r < R$. Then the series converges uniformly in the disk $|z| \leq r$. To see this note that

   $$\left|\sum_n a_n z^n\right| \leq \sum_n |a_n| r^n$$

   The series on the right-hand side (a series of real numbers) converges since r is less than the radius of convergence. (It is a fact that if the power series converges it converges absolutely.)

3. If $s = \sigma + \mathrm{i}t$, $\sigma > 1$, then the Riemann zeta-function $\zeta(s)$ is defined by setting

   $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \qquad (\star)$$

   If $\sigma \geq \sigma_0 > 1$, the the series on the right in $(\star)$ converges uniformly and absolutely in the half-plane $\sigma \geq 1 + \sigma_0$ for all $\sigma_0 > 0$. This follows from the following estimates and the Weierstrass M-test:

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+\mathrm{i}t}}\right| &\leq \sum_{n=1}^{\infty}\frac{1}{|n^{\sigma+\mathrm{i}t}|}=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}}\\ &\leq \sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}}<1+\int_1^{\infty}\frac{dx}{x^{\sigma_0}}\\ &= 1+\frac{1}{\sigma_0-1}<\infty.\end{aligned}$$

Theorem. Suppose $\{f_n\}$ converges uniformly on E. Let x be a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t)=A_n,\ \ n=1,2,\ldots.$$

Then $\{A_n\}$ converges, and

$$\lim_{t\to x} f(t)=\lim_{n\to\infty} A_n=\lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Proof. Let $\varepsilon>0$ be given. By the uniform convergence of $\{f_n\}$, there exists N such that $n\geq N$, $m\geq N$, $t\in E$ imply

$$|f_n(t)-f_m(t)|\leq\varepsilon.$$

Let $t\to x$ in the above to obtain

$$|A_n-|A_m|\leq\varepsilon$$

for $m,n\geq N$. Thus $\{A_n\}$ is a Cauchy sequence and therefore converges to a number, say A. Now

$$|f(t)-A|\leq|f(t)-f_n(t)|+|f_n(t)-A_n|+|A_n-A| \qquad (\star)$$

by use of the triangle inequality. We first choose n such that

$$|f(t)-f_n(t)|\leq\frac{\varepsilon}{3}$$

for all $t\in E$ (this is possible by the uniform convergence), and such that

$$|A_n-A|\leq\frac{\varepsilon}{3}.$$

Then, for this n, we choose a neighborhood V of x such that $t\in V$ implies

$$|f_n(t)-A_n|\leq\frac{\varepsilon}{3}.$$

Substituting these three inequalities into the inequality $(\star)$ implies

$$|f(t)-A|\leq\frac{\varepsilon}{3}.$$

$\square$

An immediate corollary of this theorem is

Corollary. If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \longrightarrow f$ uniformly on E, then f is continuous on E.

The following theorem shows we can interchange the limit with integration under the hypothesis of uniform convergence.

Theorem. Suppose f_n are Riemann integrable on $[a, b]$ for $n = 1, 2, \ldots$, and suppose $f_n \longrightarrow f$ uniformly on $[a, b]$, then f is Riemann-integrable on $[a, b]$ and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx. \tag{1}$$

Proof. We first show that the limiting function f is Riemann integrable.

Let $\varepsilon > 0$ be given. Choose $\eta > 0$ such that

$$\eta(b - a) \le \frac{\varepsilon}{3}.$$

By the uniform convergence, there exists an integer n such that

$$|f_n(x) - f(x)| \le \eta, \; a \le x \le b.$$

For this fixed n, we choose a partition P of $[a, b]$ such that

$$U(P, f_n) - L(P, f_n) \le \frac{\varepsilon}{3}$$

where U and L are the upper and lower Riemann sums, respectively. Now $f(x) \le f_n(x) + \eta$ for all $a \le x \le b$. Thus

$$U(P, f) \le U(P, f_n) + (b - a)\eta \le U(P, f_n) + \frac{\varepsilon}{3}.$$

Similarly, the inequality $f(x) \ge f_n(x) - \eta$ implies

$$L(P, f) \ge L(P, f_n) - \frac{\varepsilon}{3}.$$

Thus

$$U(P, f) - L(P, f) \le \left\{U(P, f_n) + \frac{\varepsilon}{3}\right\} - \left\{L(P, f_n) - \frac{\varepsilon}{3}\right\} = \{U(P, f_n) - L(P, f_n)\} + \frac{2\varepsilon}{3} \le \varepsilon$$

This proves that f is Riemann integrable.

We now prove (1). Choose N such that $n \ge N$ implies

$$|f_n(x) - f(x)| \le \varepsilon, \; a \le x \le b.$$

Then for $n \ge N$

$$\left|\int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx\right| = \left|\int_a^b (f(x) - f_n(x))\, dx\right| \le \int_a^b |f(x) - f_n(x)|\, dx \le \varepsilon(b - a)$$

Since ε is arbitrary, (1) follows. $\square$

As a corollary of this theorem we have

Corollary. If f_n are Riemann-integrable on $[a,b]$ and if the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \;\; a \le x \le b,$$

converges uniformly on $[a,b]$, then

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Example. Suppose we define

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}, \;\; 0 \le x \le \pi.$$

We showed earlier that the above series converges uniformly on $[0,\pi]$. By use of the above corollary we can compute

$$\begin{aligned}
\int_0^{\pi} f(x)\,dx &= \int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\,dx \\
&= \sum_{n=1}^{\infty} \int_0^{\pi} \frac{1}{n^2} \sin(nx)\,dx = \sum_{n=1}^{\pi} \frac{1}{n^2} \int_0^{\pi} \sin(nx)\,dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1-\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1+(-1)^{n+1}}{n} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} + 2\sum_{n=1}^{\infty} \frac{1}{(2n)^3} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^3} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3} \\
&= \frac{7}{4}\zeta(3)
\end{aligned}$$

where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $\Re(s) > 1$, is the Riemann zeta-function.

For differentiation some additional hypotheses beyond uniform convergence are required. The following theorem can be found in Rudin.

Theorem. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x), \;\; a \le x \le b.$$

Remarks: Note that the above analysis on interchange of limits does *not* apply to

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \;\; 0 \le x \le \pi,$$

since the obvious bound

$$\left|\frac{\sin nx}{n}\right| \le \frac{1}{n}$$

does not lead to a useful bound to which we can apply the Weierstrass M-test. This series will be studied in the lectures and we will show that the convergence is not uniform on $[0,\pi]$. To get some preview of the difference between the two series, let's define the partial sums

$$S_1(n,x) := \sum_{j=1}^{n} \frac{\sin(j\pi x)}{j} \;\; \text{and} \;\; S_2(n,x) := \sum_{j=1}^{n} \frac{\sin(j\pi x)}{j^2}, \;\; 0 \le x \le 1.$$

In Figures 1 and 2 we plot the partial sums for $n = 10, 20, 50, 100$. As these pictures indicate, the nature of convergence is quite different for these two series.

To see better the behavior near zero, we plot the partial sums for $n = 500$ in the interval $0 \le x \le 1/10$ in Figures 3 and 4.

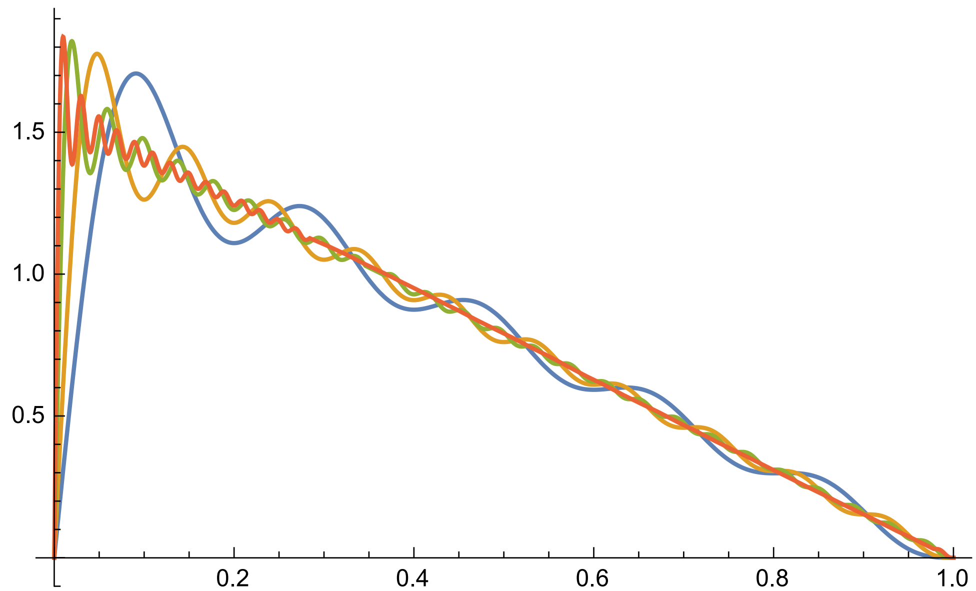

Figure 1: The partial sums $\sum_{j=1}^{n} \sin(j\pi x)/j$ for $0 \leq x \leq 1$ and $n = 10, 20, 50, 100$.

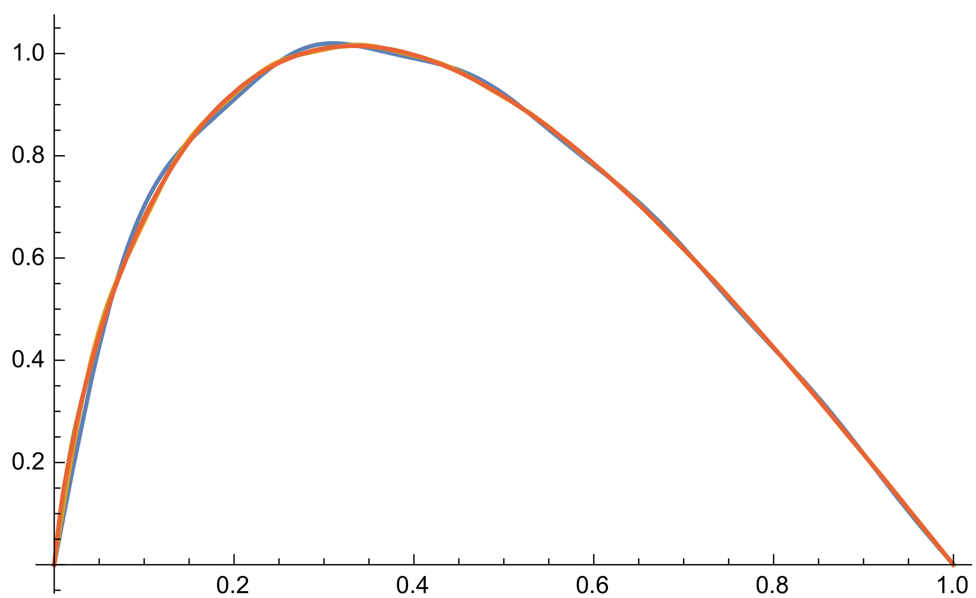

Figure 2: The partial sums $\sum_{j=1}^{n} \sin(j\pi x)/j^2$ for $0 \leq x \leq 1$ and $n = 10, 20, 50, 100$.

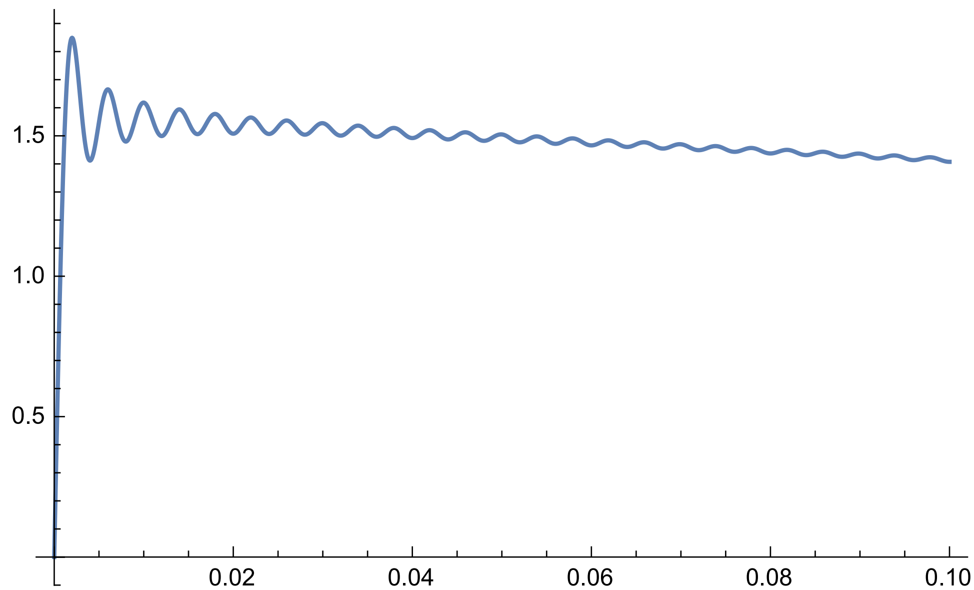

Figure 3: The partial sum $\sum_{j=1}^{500} \sin(j\pi x)/j$ for $0 \leq x \leq 1/10$.

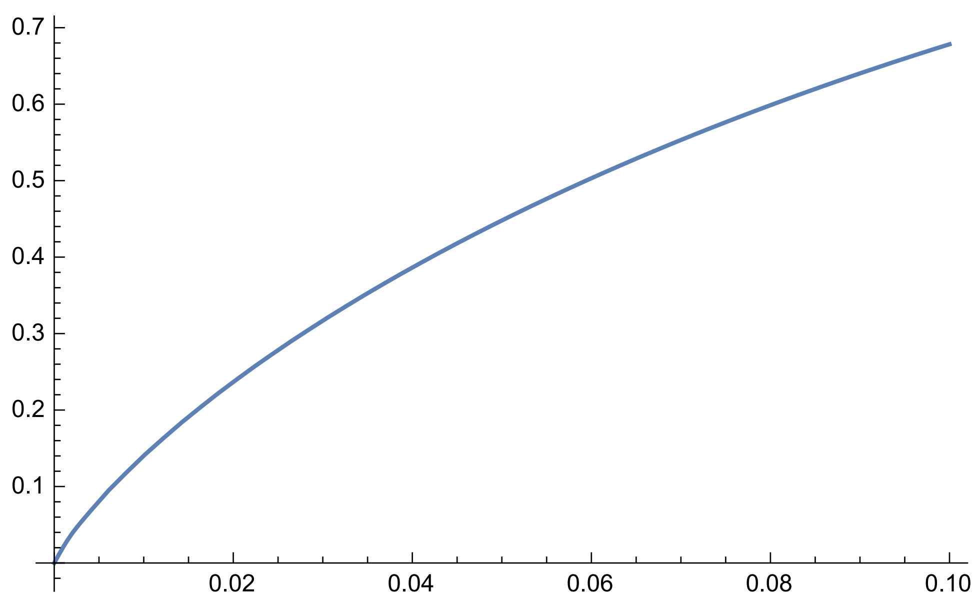

Figure 4: The partial sum $\sum_{j=1}^{500} \sin(j\pi x)/j^2$ for $0 \leq x \leq 1/10$.