March 21, 2008

# Math 135A Final Exam, Winter 2008

**Instructions**: Work all problems in your Blue Book. This exam will not be collected.

**Useful Information** which you may assume as given:

1.
$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n.$$

2. If $X$ has Gaussian distribution $N(0,1)$ (mean zero, variance one), then the characteristic function of $X$ equals
$$\phi_X(t) = \mathbb{E}(e^{itX}) = e^{-t^2/2}$$

3. If $X$ is a Poisson random variable with parameter $\lambda$, i.e.
$$\mathbb{P}(X=n) = \frac{\lambda^n}{n!}e^{-\lambda}, \;\; n = 0, 1, 2, \dots$$
then the characteristic function is
$$\phi_X(t) = \mathbb{E}(e^{itX}) = \exp\left(-\lambda + \lambda e^{it}\right)$$
From this or by direct computation it follows that
$$\mathbb{E}(X) = \lambda \text{ and } \operatorname{var}(X) = \lambda.$$

4. The random variable $X$ is said to have *Cauchy distribution* if the density of $X$ is given by
$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \;\; x \in \mathbb{R}.$$
The characteristic function of a Cauchy random variable $X$ is
$$\phi_X(t) = \mathbb{E}(e^{itX}) = e^{-|t|}$$

5. Let
$$\Psi(x) := \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}\,dy.$$
Here is a small table of values of $\Psi(x)$

| $x$ | $\Psi(x)$ |
|---|---|
| 0 | 0.5 |
| 1 | 0.1587 |
| 2 | 0.0228 |
| 3 | 0.00135 |
| 4 | $3.17 \times 10^{-5}$ |
| 5 | $2.87 \times 10^{-7}$ |
| 6 | $9.86 \times 10^{-10}$ |
| 7 | $1.28 \times 10^{-12}$ |
| 8 | $6.22 \times 10^{-16}$ |
| 9 | $1.13 \times 10^{-19}$ |
| 10 | $7.62 \times 10^{-24}$ |

.............................. **End of Useful Information** ..............................

**Beginning of Exam Questions**

**#1 (30 pts)** Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables. Let $\mu = \mathbb{E}(X_j) < \infty$ denote the mean. State both the *weak law of large numbers* and the *strong law of large numbers*. Explain how they are different.

**#2 (50 pts)**

1. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables. Let $\mu = \mathbb{E}(X_j) < \infty$ and $\sigma^2 = \text{var}(X_j) < \infty$. State the central limit theorem.
2. In 10,000 tossings, a coin fell heads 5400 times. Is it reasonable to assume the coin is skew (not fair)? Give a reason for your answer. Hint: Assume a fair coin and use the central limit theorem and the table above to estimate the probability of the event $\{S_{10,000} \geq 5400\}$.

**#3, (30 pts)** Show that the average

$$Z_n = \frac{1}{n} \sum_{j=1}^{n} X_j$$

of $n$ independent *Cauchy* random variables has Cauchy distribution too. Why does this not violate the laws of large numbers (both weak and strong)?

**#4 (40 pts)** Let $X$ denote a Poisson random variable with parameter $\lambda$.

Define the random variable

$$Y := \frac{X - \mathbb{E}(X)}{\sqrt{\text{var}(X)}}$$

Prove that $Y$ converges in distribution to the normal law $N(0,1)$ as $\lambda \to \infty$. Hint: Use characteristic functions and recall that $\mathbb{E}(X) = \lambda$ and $\text{var}(X) = \lambda$ can be assumed as given.

**#5 (50 pts)** Consider a Bernoulli process $\{X_j\}_{j \geq 1}$ where

$$\mathbb{P}(X_j = 1) = p \text{ and } \mathbb{P}(X_j = 0) = q = 1 - p, \; j = 1, 2, \ldots$$

(We assume $0 < p < 1$.) Let $\mathcal{T}$ denote the *first time* that the combination 10 (HT in coin tossing language) occurs. Thus, for example, if $\omega = 00100110\ldots$, then $\mathcal{T}(\omega) = 4$. Another example: if $\omega = 101110\cdots$ then $\mathcal{T}(\omega) = 2$.

1. Find the generating function of $\mathcal{T}$, i.e.

   $$G_{\mathcal{T}}(s) = \mathbb{E}(s^{\mathcal{T}}) = \sum_{n \geq 2} \mathbb{P}(\mathcal{T} = n) s^n$$

   Hint: Here's *one* approach. First condition on the outcome of the first trial, e.g.

   $$\mathbb{E}(s^{\mathcal{T}}) = \mathbb{E}(s^{\mathcal{T}} | X_1 = 1)\mathbb{P}(X_1 = 1) + \mathbb{E}(s^{\mathcal{T}} | X_1 = 0)\mathbb{P}(X_1 = 0)$$

   You will then have to condition part of the resulting expression on the outcome of the second trial.
2. Will the event "head" followed by "tail" sooner or later occur with probability one? How does this follow from your expression for $G_{\mathcal{T}}(s)$?
3. Using your result for $G_{\mathcal{T}}(s)$, find $\mathbb{E}(\mathcal{T})$. As a check on your calculation you should get $\mathbb{E}(\mathcal{T}) = \frac{1}{pq}$.