Math 135A Final Exam: Winter 2007

Instructions: Show all your work in your BLUEBOOK.

1. Let X_j, $j = 1, 2, 3, \ldots$ be a sequence of independent and identically distributed random variables. Assume the moments $\mathbb{E}(X^k)$ exist for as many k as you need.

   (a) (10pts) State the *weak law of large numbers.*

   (b) (10pts) State the *strong law of large numbers.*

   (c) (10pts) State the *central limit theorem.*

2. Let X and Y denote discrete random variables (on the same probability space). Define the joint density function

$$f_{Y,X}(y, x) = \mathbb{P}(Y = y, X = x)$$

   (a) (15 pts.) Give the definition of the *conditional density* $f_{Y|X}(y|x)$ where we assume $\mathbb{P}(X = x) > 0$.

   (b) (15 pts.) Give the definition of $\mathbb{E}(Y|X = x)$.

3. Consider a simple one-dimensional random walk on the integer lattice $\mathbb{Z}$. Recall that the particle jumps to the right with probability p and to the left with probability q, $p + q = 1$. Each jump is independent of all the other jumps.

   (a) (30pts) Assuming the walker starts at lattice site 0, let T_1 denote the *first passage time* to the lattice site $+1$. Find the generating function for T_1.

   (b) (10pts) Find the expected value of T_1.

   (c) (10pts) What is the generating function for the first passage time, T_r, to the site r, $r > 0$, assuming the walker starts at 0. What is the expected value of T_r?

4. (30 pts) Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having density

$$\mathbb{P}\left(X_j = x\right) = (1 - p_j)p_j^{x-1} \text{ for } x = 1, 2, 3, \ldots \text{ and } j = 1, 2, 3.$$

   (Here $0 < p_j < 1$ for $j = 1, 2, 3$.) Show that

$$\mathbb{P}\left(X_1 < X_2 < X_3\right) = \frac{(1 - p_1)(1 - p_2)p_2p_3^2}{(1 - p_2p_3)(1 - p_1p_2p_3)}.$$

Remark: A useful series, which you may assume as given, is

$$\sum_{j=n}^{\infty} x^j = \frac{x^n}{1-x}, \quad |x| < 1.$$

5. Recall that T is said to have *exponential distribution* with parameter $\lambda > 0$ if it has distribution

$$F_T(x) = 1 - e^{-\lambda x}, \;\; x \geq 0.$$

   (a) (10pts) If T has exponential distribution with parameter λ, compute $\mathbb{E}(T)$.

   (b) Let T_1 and T_2 be independent random variables with common exponential distribution with parameter λ. Define the new random variable

$$\mathcal{T} = \max(T_1, T_2)$$

   where "max" denotes the maximum of its arguments.

      i. (30pts) Compute the *distribution function* of $\mathcal{T}$.

      ii. (10pts) Compute the *density function* of $\mathcal{T}$.

      iii. (10pts) Compute $\mathbb{E}(\mathcal{T})$.

   Remark: Some useful integrals

$$\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}, \;\; \int_0^{\infty} xe^{-ax}\,dx = \frac{1}{a^2}, \;\; a > 0.$$