February 12, 2008

Math 135A Midterm Examination

Instructions: Do all your work in your BLUEBOOK. Only the Bluebook will be collected. Make sure you explain and justify your steps. Each question is worth 25 points. Calculus facts:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \text{ for all } x \in \mathbb{R}.$$

$$\sum_{n=k}^{\infty} x^n = \frac{x^k}{1-x} \text{ for all } k = 0, 1, 2, \ldots \text{ with } |x| < 1.$$

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. What is a **random variable** X defined on this probability space?

2. Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ denote two discrete random variables. We assume both first and second moments of X and Y are finite.

   (a) What does it mean to say X and Y are **independent random variables**?

   (b) Suppose X and Y are independent random variables. **Prove** that

   $$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y) \tag{1}$$

   where $\mathrm{var}(\cdot)$ denotes the **variance**.

3. In your pocket is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time. Show that the total number of heads after tossing all the coins in your pocket has the Poisson distribution with parameter λp.

   Recall: $N : \Omega \to \{0, 1, 2, \ldots\}$ is said to have Poisson distribution with parameter $\lambda > 0$ if

   $$\mathbb{P}(N = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

   Hint: Start with (and explain why it is true):

   $$\mathbb{P}(S_N = k) = \sum_{n \geq k} \mathbb{P}(S_N = k | N = n) \, \mathbb{P}(N = n)$$

4. A single (six sided fair) die and a fair coin are thrown repeatedly together. What is the expected waiting time for **both** a **3** and **heads** to simutaneously appear? (For example, on the first toss, the possible outcomes are 1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T.) Justify your answer as much as you can.