Math 135A: HW Assignment #10

1. Show that the average
$$Z_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$
of n independent *Cauchy* variables has Cauchy distribution too. Why does this not violate the law of large numbers? Recall a random variable has Cauchy distribution if the density is
$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

2. Let X_1, X_2, ..., X_n be independent random variables with characteristic functions ϕ_1, ϕ_2,..., ϕ_n. Describe random variables which have the following characteristic functions
   (a) $\phi_1(t)\phi_2(t)\cdots\phi_n(t)$.
   (b) $|\phi_1(t)|^2$
   (c) $\sum_{j=1}^{n} p_j\phi_j(t)$ where $p_j \geq 0$ and $\sum_j p_j = 1$.
   (d) $(2-\phi_1(t))^{-1}$.
   (e) $\int_0^\infty \phi_1(ut)e^{-u}\,du$.

3. Use generating functions to show that is is not possible to load two dice in such a way that the sum of the values they show is equally likely to take any value between 2 and 12.

4. A biased coin shows heads with probability p. It is flipped repeatedly until the first time W_n by which it has shown n consecutive heads. (Set $q = 1-p$.) Let
$$G_n(s) = \mathbb{E}\left(s^{W_n}\right).$$
Show that
$$G_n = \frac{psG_{n-1}}{1-qsG_{n-1}},$$
and deduce that
$$G_n(s) = \frac{(1-ps)p^n s^n}{1-s+q\,p^n s^{n+1}}$$
From this find the expected waiting time in the case $p = q = 1/2$.