Math 135A: Homework Assignment #2

1. In a ten-question true-false exam, find the probability that a student gets a grade of 70 percent or better by guessing. Answer the same question if the test has 30 questions.

2. Suppose that in a bushel of 550 apples there are 2% rotten ones. What is the probability that a random sample of 25 apples contains two rotten apples? Hint: Hypergeometric distribution.

3. (a) Assume that every time you buy a box of Wheaties, you receive one of the pictures of the k players of the SF Giants. Over a period of time you buy $n \geq k$ boxes of Wheaties. What is the probability that you get all k pictures? Hint: Let E_j, $j = 1, 2, \ldots, k$, denote the event you do *not* get the j^{th} player's picture. Then $E_1 \bigcup \cdots \bigcup E_k$ is the event you do not get at least one picture. Calculate $\mathrm{P}(E_1 \bigcup \cdots \bigcup E_k)$ by inclusion-exclusion.

   (b) Consider the set $\mathcal{F}_O$ of all functions

   $$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}, n \geq k,$$

   with the additional restriction that f is *onto*.[1] Show that this is just an abstract reformulation of the Wheaties problem and the number of such onto functions is

   $$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \tag{1}$$

   Hint: Explain why in the Wheaties problem an onto function corresponds to an event where all cards are collected. Another way of saying this is that (1) is the number of ways of placing n balls into k cells so that each cell has at least one ball. Show that the probability of this occuring is

   $$p_0(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1 - \frac{j}{k})^n \tag{2}$$

   (c) Consider now the situation when *exactly* m cells are empty. Distribute the n balls among the remaining $k - m$ cells so that no cell is empty. Show that the probability that exactly m cells are empty is

   $$p_m(n,k) = \frac{1}{k^n} \binom{k}{m} (k-m)^n \, p_0(n, k-m)$$

[1] Recall a function $f : \mathcal{D} \to \mathcal{R}$ is *onto* if $f(\mathcal{D}) = \mathcal{R}$; that is, every value in the range is taken on by the function f.

4. A census in the United States is an attempt to count everyone in the country. It is inevitable that many people are not counted. The U.S. Census Bureau proposed a way to estimate the number of people who were not counted by the latest census. Their proposal was as follows: In a given locality, let N denote the actual number of people who live there. Assume that the census counted n_1 people living in this area. Now, another census was taken in the locality, and n_2 people were counted. In addition, n_{12} people were counted both times.

   (a) Given N, n_1, and n_2, let X denote the number of people counted both times. Find the probability that $X = k$, where k is a fixed positive integer between 0 and n_2.

   (b) Now assume that $X = n_{12}$. Find the value of N which maximizes the expression in part (a). Hint: Consider the ratio of the expressions for successive values of N.

5. A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses (by convention, zero is an even number). Show that $p_0 = 1$ and that

$$p_n = p\,(1 - p_{n-1}) + (1 - p)\,p_{n-1},\; n \geq 1. \tag{3}$$

   Solve this difference equation.