Winter 2020

Math 135A: HW2
Due Monday, January 20

1. Two people toss a fair coin n times each.

   (a) Show that the probability, p_n, that they will score the same number of heads is

   $$p_n = \frac{1}{2^{2n}} \binom{2n}{n}$$

   where $\binom{n}{k}$ is the binomial coefficient.

   (b) Show that as $n \to \infty$,

   $$p_n \sim \frac{1}{\sqrt{\pi n}}.$$

   (c) Hints: Let $E_k^{(1)}$ denote the event person 1 tosses exactly k heads; and similarly, $E_k^{(2)}$ denotes the probability person 2 tosses exactly k heads. Then $E_k^{(1)} \cap E_k^{(2)}$ is the event that each toss exactly k heads. Then

   $$p_n = \sum_{k=0}^{n} \mathbb{P}(E_k^{(1)} \cap E_k^{(2)}).$$

   You will need to prove the binomial identity

   $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

   You will also need Stirling's formula for $n!$ which says

   $$\log(n!) = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + \mathrm{o}(1), \;\; n \to \infty,$$

   where $\mathrm{o}(1)$ means an error that goes to zero as $n \to \infty$.

2. A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses (by convention, zero is an even number). Show that $p_0 = 1$ and that

   $$p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1}, \;\; n \geq 1. \tag{1}$$

   Solve this difference equation for p_n. What is $\lim_{n \to \infty} p_n$? Plot p_n vs. n for $p = 1/4$ and for $p = 3/4$. Hint: Condition on the outcome of the first toss.

3. Let A_j, $j = 1, \ldots, n$ be events in a probability space $(\Omega, \mathbb{P}, \mathcal{F})$.

   (a) Prove

$$\mathbb{P}\left(\bigcup_{j=1}^{n} A_j\right) \leq \sum_{j=1}^{n} \mathbb{P}(A_j) \text{ and } \mathbb{P}\left(\bigcap_{j=1}^{n} A_j\right) \geq 1 - \sum_{j=1}^{n} \mathbb{P}(A_j^c)$$

   (b) Ten percent of the surface of a sphere S is colored blue, the rest is red. Show that irrespective of the manner in which the colors are distributed, it is possible to inscribe a cube in S with all its vertices red.

   (c) Hint for (b): Let A_r be the event that the rth vertex of a randomly selected cube is blue and note that $\mathbb{P}(A_r) = 0.1$. The event $\cup_{j=1}^{8} A_j$ is at least one vertex is blue. Use the inequality of part (a) to estimate this probability. What can you now conclude?

4. **Bayes's formula:** Suppose A_1, A_2,..., A_n is a partition of the sample space Ω and each A_j has positive probability. Prove that

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j)\mathbb{P}(A_j)}{\sum_{k=1}^{n} \mathbb{P}(B|A_k)\mathbb{P}(A_k)}$$

5. $2n$ points at equal distances are marked off on a circle. These points are randomly grouped into n pairs and the points of each pair are connected by a chord. What is the probability that each of the n chords constructed do not intersect?

   (a) Hint: Let M_n denote the number of favorable outcomes for $2n$ points. By direct enumeration we have

$$M_1 = 1, M_2 = 2, M_3 = 5, \text{and } M_4 = 14,$$

   Note that

$$M_4 = 1 \cdot M_3 + M_1 \cdot M_2 + M_2 \cdot M_1 + M_3 \cdot 1 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$$

   Defining $M_0 = 1$ show that

$$M_n = \sum_{r=0}^{n-1} M_r M_{n-r-1}. \tag{2}$$

   (b) Define

$$M(z) = \sum_{r=0}^{\infty} M_r z^r \tag{3}$$

   and show that (2) implies that

$$\frac{1}{z}\left(M(z) - 1\right) = M^2(z) \tag{4}$$

Solving this quadratic equation for $M(z)$ gives

$$M(z) = \frac{1}{2z}\left[1 - \sqrt{1-4z}\,\right] \qquad (5)$$

(c) Using Taylor's formula applied (5) to show that

$$M_n = \frac{(2n-1)!!}{(n+1)!}2^n$$

and hence

$$p_n = \frac{2^n}{(n+1)!}$$

6. **Background: Inclusion-Exclusion Principle:** Let $\mathbb{P}$ be a probability measure on the sample space Ω, and let $\{A_1, A_2, \ldots, A_n\}$ be a finite set of events. Then

$$\mathbb{P}\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) = \sum_{j=1}^{n} \mathbb{P}(A_j) - \sum_{1\le j<k\le n} \mathbb{P}(A_j \cap A_k) + \sum_{1\le j<k<\ell\le n} \mathbb{P}(A_j \cap A_k \cap A_\ell) - \cdots \qquad (6)$$

That is, to find the probability that at least one of n events A_j occurs, first add the probability of each event, then subtract the probabilities of all possible two-way intersections, then add the probability of all three-way intersection, and so forth down to adding $(-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$.

Random permutations: Recall that the number of permutations of n objects is $n!$. For example, there are 6 permutations of the letters a, b, and c; namely

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations of the letters a, b, c, and d. The number of permutations of n objects increases rapidly with n; for example, $10! = 3\,628\,800$. Another example: A deck of cards consists of 52 cards and the number of shuffled decks is $52! \approx 8.06 \times 10^{67}$.

Let Ω_n denote the set of all permutations of the numbers 1, 2,..., n. Thus Ω_n has $n!$ elements. By a *random permutation* we mean each permutation $\omega \in \Omega_n$ has probability $1/n!$ (this is the uniform measure). The number j, $1 \le j \le n$, is called a *fixed point* of the permutation ω if ω maps j to j. For example, of the six permutations of three elements, four have at least one fixed point:

$$123, 132, 213, 321$$

Exercise: What is the probability that a random permutation has no fixed points? For example the probability that a random permutation of three elements has no fixed point is $2/3! = 1/3$ since there are only two out of six permutations that have no fixed point.

Hints: This problem is solved by using the Inclusion-Exclusion Principle to calculate the probability that a random permutation has at least one fixed point. Let A_j be the event that the jth element is a fixed point. Look at the event $A_1 \cup A_2 \cup \cdots \cup A_n$ (which is the event there is at least one fixed point) and apply the Inclusion-Exclusion Principle.