Central Limit Theorem

These notes give a *heuristic* derivation of the central limit theorem. They are heuristic since we need to be more careful[1] with the error estimates as well as some other points discussed below. However, these "gaps" can be filled in (and they are in an advanced course); the ideas presented here are the basic ideas that go into the proof of the central limit theorem. This derivation shows why only information relating to the mean and variance of the underlying distribution function are relevant in the central limit theorem. (That is, one sees why, for instance, the third moment does not appear in the statement of the central limit theorem.)

Let X_j, $j = 1, 2, \ldots$ be independent random variables with common continuous density function f_X. We assume that all the moments of X_j are finite; that is, we assume

$$\mu_n := \int_{\mathbb{R}} x^n f_X(x)\, dx < \infty$$

for $n = 0, 1, 2, \ldots$. Of course, $\mu_0 = 1$. We introduce the *moment generating function*

$$\begin{aligned} M_X(\xi) &= \sum_{n\geq 0} \frac{\mu_n}{n!}\, \xi^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{\mathbb{R}} x^n f_X(x) \right) \xi^n \, dx \\ &= \int_{\mathbb{R}} \left(\sum_{n\geq 0} \frac{(x\xi)^n}{n!} \right) f_X(x)\, dx \\ &= \int_{\mathbb{R}} \mathrm{e}^{x\xi} f_X(x)\, dx. \end{aligned}$$

To get some feeling for the moment generating function let's compute it in the special case of a gaussian distributed random variable of mean μ and variance σ^2. The density is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}.$$

[1] The estimates we give are correct pointwise, but we need some *uniform* estimates since we integrate the estimates. A careful treatment of the central limit theorem can be found in Feller, Vol. 2.

Denote the moment generating function by $M_{\mu,\sigma}$; thus for the special case of $\mu = 0$ and $\sigma = 1$

$$\begin{aligned}
M_{0,1}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-x^2/2+x\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left((x-\xi)^2 - \xi^2\right)\right) dx \\
&= \mathrm{e}^{\xi^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(x-\xi)^2/2}\, dx \\
&= \mathrm{e}^{\xi^2/2}
\end{aligned}$$

More generally,

$$M_{\mu,\sigma}(\xi) = \mathrm{e}^{\sigma^2\xi^2/2+\xi\mu}$$

as can easily be proved from the above result.

Let

$$S_n = X_1 + X_2 + \cdots + X_n$$

then it's moment generating function is

$$\begin{aligned}
M_{S_n}(\xi) &= E\left(\mathrm{e}^{\xi S_n}\right) \\
&= E\left(\mathrm{e}^{X_1+\cdots+X_n}\right) \\
&= \left[E\left(\mathrm{e}^{\xi X_1}\right)\right]^n
\end{aligned}$$

where the last equality follows from the independence of the X_j. Define

$$S_n^* = \frac{S_n - n\mu_1}{\sigma\sqrt{n}}$$

where σ^2 is the variance of X_1. Then

$$\begin{aligned}
M_{S_n^*}(\xi) &= E\left(\mathrm{e}^{\xi S_n^*}\right) \\
&= E\left(\exp\left(\frac{\xi}{\sqrt{n}\sigma} S_n - \frac{\mu_1}{\sigma}\sqrt{n}\xi\right)\right) \\
&= \mathrm{e}^{-(\mu_1/\sigma)\sqrt{n}\,\xi}\, E\left(\exp\left(\frac{\xi S_n}{\sigma\sqrt{n}}\right)\right) \\
&= \mathrm{e}^{-(\mu_1/\sigma)\sqrt{n}\,\xi}\, M_{S_n}\left(\frac{\xi}{\sqrt{n}\,\sigma}\right) \\
&= \mathrm{e}^{-(\mu_1/\sigma)\sqrt{n}\,\xi} \left[M_X\left(\frac{\xi}{\sqrt{n}\,\sigma}\right)\right]^n \qquad (1)
\end{aligned}$$

We now take $n \to \infty$ in the expression for the moment generating function for S_n^*. We will show that

$$M_{S_n^*}(\xi) \to \mathrm{e}^{\xi^2/2} \tag{2}$$

as $n \to \infty$. The generating function on the right is the moment generating function of the gaussian of mean zero and variance one. We then appeal to a result (which we don't prove) that says all the moments of S_n^* converge to the moments of the gaussian; and hence the distribution function of S_n^* converges to the distribution function of the gaussian. That is,

$$\mathrm{P}\left(\frac{S_n - n\mu_1}{\sigma\sqrt{n}} \le x\right) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-t^2/2}\,dt \text{ as } n \to \infty.$$

This last result is the central limit theorem.

We now show (2). Returning to (1) and taking the logarithm of both sides gives

$$\log\left(M_{S_n^*}(\xi)\right) = -\frac{\mu_1}{\sigma}\sqrt{n}\,\xi + n\log\left(M_X\left(\frac{\xi}{\sigma\sqrt{n}}\right)\right) \tag{3}$$

Now

$$M_X(\xi) = 1 + \mu_1\xi + \frac{\mu_2}{2}\,\xi^2 + \mathrm{O}(\xi^3)$$

so that

$$M_X\left(\frac{\xi}{\sigma\sqrt{n}}\right) = 1 + \mu_1\frac{\xi}{\sigma\sqrt{n}} + \frac{\mu_2}{2\sigma^2}\,\frac{\xi^2}{n} + \mathrm{O}(\frac{\xi^3}{n^{3/2}})$$

We now use

$$\log(1+x) = x - x^2/2 + \mathrm{O}(x^3)$$

to find that

$$\begin{aligned} n\log\left(M_X\left(\frac{\xi}{\sigma\sqrt{n}}\right)\right) &= n\left\{\mu_1\frac{\xi}{\sigma\sqrt{n}} + \frac{\mu_2}{2\sigma^2}\,\frac{\xi^2}{n} - \frac{\mu_1^2}{2\sigma^2 n}\,\xi^2 + \mathrm{O}(\frac{1}{n^{3/2}})\right\} \\ &= \frac{\mu_1}{\sigma}\,\xi\sqrt{n} + \frac{1}{2}\,\xi^2 + \mathrm{O}(\frac{1}{\sqrt{n}}) \end{aligned}$$

(In the last step we used that $\mu_2 - \mu_1^2 = \sigma^2$.) Substituting this last expression into (3) and noting that the term involving $\sqrt{n}$ cancels shows that

$$\lim_{n\to\infty} \log\left(M_{S_n^*}(\xi)\right) = \frac{1}{2}\,\xi^2$$

or

$$\lim_{n\to\infty} M_{S_n^*}(\xi) = \mathrm{e}^{\xi^2/2}$$

which was to be proved.