Math 135B: HW # 1

1. Statistical Hypothesis Testing: In class we showed that if

$$S_n = X_1 + \cdots + X_n$$

where X_j are independent and identically distributed Bernoulli random variables and $1 > a > p > 0$ where $p = \mathbb{E}(X_j)$, then[1]

$$\mathbb{P}(S_n \geq na) \approx e^{-n\Lambda^*(a,p)}$$

where

$$\Lambda^*(a,p) = a\log(\frac{a}{p}) + b\log(\frac{b}{q})$$

where $q = 1 - p$ and $b = 1 - a$.

Suppose now that p is unknown and that there are two competing hypotheses concerning its value, the hypothesis H_1 that $p = p_1$ and the hypothesis H_2 that $p = p_2$ where $p_1 < p_2$. Given the observed results $X_1, \ldots, X_n$ of n Bernoulli trials, one decides in favor of H_2 if $S_n \geq na$ and in favor of H_1 if $S_n < na$, where a is some number satisfying $p_1 < a < p_2$. The question is, how do we choose a? Her is one strategy.

Given the hypothesis H_1: $p = p_1$, then

$$\mathbb{P}(S_n \geq na | H_1) \approx e^{-n\Lambda^*(a,p_1)}$$

whereas given the hypothesis H_2: $p = p_2$,

$$\mathbb{P}(S_n < na | H_2) \approx e^{-n\Lambda^*(a,p_2)}$$

These give us the probabilities of *erroneously* deciding in favor of H_2 whe H_1 is, in fact, true and of *erroneously* deciding in favor of H_1 when H_2 is in fact true.

Since we don't want to favor one event over another we choose a so that the two error probabilities are approximately equal, e.g.

$$\Lambda^*(a,p_1) = \Lambda^*(a,p_2).$$

This gives an equation for the unknown a.

Show that the a that satifies this equation is

$$a = \frac{\log(q_1/q_2)}{\log(p_2/p_1) + \log(q_1/q_2)}$$

[1] If $a < p$ then a similar argument shows that

$$\mathbb{P}(S_n < na) \approx e^{-n\Lambda^*(a,p)}$$

Let $p_2 = p_1 + t$. $t > 0$. Show that for small t

$$a = p + \frac{1}{2}t + \frac{p-q}{12pq}t^2 + \frac{1-2p+2p^2}{24q^2p^2}\,t^3 + \mathrm{O}(t^4)$$

and that

$$\Lambda^*(a,p) = \frac{t^2}{8pq} + \frac{p-q}{16p^2q^2}t^3 + \frac{23-65p+65p^2}{576q^3p^3}t^4 + \mathrm{O}(t^5).$$

You might want to use Mathematica to perform these last two computations.

2. Grimmett & Stirzaker: page 219, #1.

3. Grimmett & Stirzaker: page 219, #2.