Math 135B: HW # 2

1. Grimmett & Stirzaker, page 226, #3.
2. In a sequence of Bernoulli trials we say that at time n the state E_1 is observed if the trials number $n-1$ and n result in SS. (Success–Success). Similarly, E_2, E_3 and E_4 stand for SF, FS, FF. Find the transition matrix P and all its powers. Referring to Theorem (17) on page 232 of GS, find the mean recurrence time for each of these states.
3. Consider the three-state Markov chain defined by the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

   (a) Diagonalize the matrix P. That is, write

$$P = UDU^{-1}$$

   where D is a diagonal matrix. (You can use Mathematica to do these computations.)

   (b) Explain why

$$P^n = UD^nU^{-1}, \;\; n = 1, 2, 3, \ldots$$

   (c) Show that

$$p_{11}^{(n)} = (P^n)_{11} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left\{\frac{4}{5}\cos(n\pi/2) - \frac{2}{5}\sin(n\pi/2)\right\}$$

   (d) Find the invariant distribution π; that is, find the row vector satisfying

$$\pi P = \pi$$

   Show that from the previous result that $\lim_{n\to\infty} p_{11}^{(n)} = \pi_1$ where $\pi = (\pi_1, \pi_2, \pi_3)$.

4. A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability 1/5. If the first score is 6, what is the probability p that the nth score is 6? What is the probability that the nth score is 1?
5. Explain why the most general two-state Markov chain has a transition matrix P of the form

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

   where $0 \leq \alpha, \beta \leq 1$. Find P^n (assume $0 < \alpha, \beta < 1$).