Math 135B: HW #5

1. Birth processes have been used since the time of Rutherford to model radioactive decay. (Radioactive decay occurs when an unstable isotope transforms to a more stable isotope, generally by emitting a subatomic particle.) In many cases a radioactive nuclide A decays into a nuclide B which is also radioactive; and hence, B decays into a nuclide C, etc. The nuclides B, C, etc. are called the progeny (formerly called daughters). This continues until the decay chain reaches a stable nuclide. For example, uranium-238 decays through α-emission to thorium-234 which in turn decays to protactinium-234 through β-emission. This chain continues until the stable nuclide lead-206 is reached.[1]

   (a) Let the decay states be $E_1 \to E_2 \to \cdots \to E_N$ where E_N is the final stable state. We can relabel these states to be simply $1, 2, \ldots, N$. The Markov process $\{N(t)\}_{t\geq 0}$ satisfies (definition of a birth process)

   $$\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda_n h + \mathrm{o}(h) & \text{if } m = 1 \\ \mathrm{o}(h) & \text{if } m > 1 \\ 1 - \lambda_n h + \mathrm{o}(h) & \text{if } m = 0 \end{cases}$$

   Explain why $\lambda_0 = 0$ and $\lambda_N = 0$. We assume $\lambda_i \neq \lambda_j$ for $i, j = 1, \ldots, N-1$. Let $p_j(t) = \mathbb{P}(N(t) = j | N(0) = 1)$ Give an argument (along the lines of the Lecture Notes) why the Kolmogorov forward equations are

   $$\frac{dp_j}{dt} = \lambda_{j-1} p_{j-1}(t) - \lambda_j p_j(t), \; j = 1, 2, \ldots, N. \tag{1}$$

   for $j = 1, 2, \ldots, N$. In applications to radioactive decay, if N_1 is the number of initial nuclides (the the number of nuclides in state E_1), then $N_1 p_j(t)$ is the number of nuclides in state E_j at time t.

   (b) Introduce the Laplace transform

   $$\hat{p}_j(s) = \int_0^\infty e^{-ts} p_j(t)\, dt$$

[1] The complete decay chain can be found at `http://www.ead.anl.gov/pub/doc/natural-decay-series.pdf`

and show that the Laplace transform of (1) is

$$s\hat{p}_j(s) - \delta_{j,1} = \lambda_{j-1}\hat{p}_{j-1}(s) - \lambda_j\hat{p}_j(s),\ j = 1, \ldots, N. \tag{2}$$

Solve these equations for $\hat{p}_j(s)$ and show that

$$\hat{p}_j(s) = \frac{\lambda_1}{s+\lambda_1}\,\frac{\lambda_2}{s+\lambda_2}\cdots\frac{\lambda_{j-1}}{s+\lambda_{j-1}}\,\frac{1}{s+\lambda_j}$$

(c) Using the above expression for $\hat{p}_j(s)$ partial fraction the result:

$$\hat{p}_j(s) = \sum_{k=1}^{j} \frac{c_{j,k}}{s+\lambda_k}$$

See if you can find expressions for $c_{j,k}$. You might want to take some special cases to see if you can make a guess for the $c_{j,k}$. (The Mathematica command `Apart` will prove useful.)

(d) From the partial fraction decomposition of $\hat{p}_j(s)$ explain why you can almost immediately conclude

$$p_j(t) = \sum_{k=1}^{j} c_{j,k}\, e^{-\lambda_j t},\ j = 1, 2, \ldots, N. \tag{3}$$

(e) For the special case of $N = 4$: $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4$ find explicitly the probabilities $p_j(t)$. (You can use Mathematica if you wish. Note there is a command `InverseLaplaceTransform`.)

i. Show that $p_2(t)$ has a maximum at $t = t_m$

$$t_m = \frac{\log(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2} > 0.$$

In terms of the radioactive decay interpretation, this is the time when the first progeny has a maximum population.

ii. Using Mathematica (recall the command `Series`) show that as $t \longrightarrow 0$

$$\begin{aligned}
p_1(t) &= 1 - \lambda_1 t + \mathrm{O}(t^2) \\
p_2(t) &= \lambda_1 t + \mathrm{O}(t^2) \\
p_3(t) &= \frac{1}{2}\lambda_1\lambda_2 t^2 + \mathrm{O}(t^3) \\
p_4(t) &= \frac{1}{3!}\lambda_1\lambda_2\lambda_3 t^3 + \mathrm{O}(t^4)
\end{aligned}$$

(f) Radon 222 gas is a chemically inert radioactive gas that is part of the Uranium 238 decay chain. Radon and its radioactive progeny are known carcinogens. Here is part of the decay chain[2]

$$\cdots \longrightarrow \text{Rn } 222 \longrightarrow \text{Po } 218 \longrightarrow \text{Pb } 214 \longrightarrow \text{Bi } 214 \longrightarrow \cdots$$

The half-life of each nuclide is known:[3]

$$\begin{aligned} \text{Rn } 222\text{:} &\quad T_{1/2} = 3.8235 \text{ days} \\ \text{Po } 218\text{:} &\quad T_{1/2} = 3.10 \text{ minutes} \\ \text{Pb } 214\text{:} &\quad T_{1/2} = 26.8 \text{ minutes} \\ \text{Bi } 214\text{:} &\quad T_{1/2} = 19.9 \text{ minutes} \end{aligned}$$

Let N_{Rn} denote the initial amount of Rn 220 and assume the other nuclides are not present at time $t = 0$. Solve the Kolmogorov forward equations for this particular birth process. (Note that here the probabilites do not sum to one since the Bi 214 also decays.) This is not so messy if you use Mathematica. Find the times when each of the progeny have maximum population. (Highest probability) You might want to use Mathematica's `FindRoot`.

2. Let $B(t) = (p_{ij}(t))$ be a simple birth process as defined in your textbook on page 250, example (b) with $p_{ij}(0) = \delta_{ij}$ and birth rates $\lambda_n = n\lambda$. Write down the forward equations for this process. Solve these differential equations subject to the given initial conditions. You should find

$$p_{ij}(t) = e^{-i\lambda t} \binom{j-1}{i-1} (1 - e^{-\lambda t})^{j-i}$$

Hint: Introduce the generating functions

$$G_i(s,t) = \sum_{j=0}^{\infty} p_{ij}(t) s^j$$

and derive a partial differential equation for $G_i(s,t)$.

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[2] Po=polonium, Pb=lead, Bi=bismuth.

[3] Recall the relation between the half-life $T_{1/2}$ and the decay rate λ

$$T_{1/2} = \frac{\log 2}{\lambda}, \ \log 2 = 0.693147\ldots$$

Why is this?

3. Let $\{B(t)\}_{t\geq 0}$ be a simple birth process with immigration as defined in your textbook on page 250, example (c). The birthrates are given by $\lambda_n = n\lambda + \nu$ where ν models the constant immigration rate. Assume $B(0) = 0$. Write down the forward equations for $p_n(t) = \mathbb{P}(B(t) = n)$. Let

$$m(t) = \mathbb{E}(B(t))$$

Show that $m(t)$ satisfies the differential equation

$$\frac{dm(t)}{dt} = \lambda m(t) + \nu$$

Solve this DE for $m(t)$.