Math 135B: HW #6

1. Let $\{X_n\}_{n\geq 1}$ be random variables such that the partial sums $S_n = X_1 + \cdots + X_n$ determine a martingale. Show that $\mathbb{E}(X_i X_j) = 0$ if $i \neq j$.

2. Let $\{X_n\}_{n\geq 0}$ denote a Markov chain with countable state space $\mathcal{S}$ and transition matrix P. Let $f : \mathcal{S} \to \mathbb{R}$ denote any bounded function on the state space $\mathcal{S}$. (Bounded means there exists a constant C so that $|f(i)| < C$ for all $i \in \mathcal{S}$.) The transition matrix P acts on bounded functions $f : \mathcal{S} \to \mathbb{R}$ as follows:

$$(Pf)(i) = \sum_{j\in\mathcal{S}} p_{ij} f(j), \;\; i \in \mathcal{S}.$$

If $f : \mathcal{S} \to \mathbb{R}$ is a bounded function, define the process

$$M_n^f = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P - I) f(X_m), \;\; n = 1, 2, \ldots.$$

(One can set $M_0^f = 0$.) Show that the process $\{M_n\}$ is a martingale with respect to $\{X_n\}$. Recall you must show (i) $\mathbb{E}(|M_n^f|) < \infty$ and the martingale property (ii) $\mathbb{E}(M_{n+1}^f | X_0, X_1, \ldots, X_n) = M_n^f$.

3. Let $S_n = a + \sum_{j=1}^n X_j$ ($S_0 = a$) be a simple symmetric random walk on $\mathbb{Z}$. The walk stops at the earliest time T when it reaches either of the two positions 0 or K where $0 < a < K$. Show that

$$M_n = \sum_{j=0}^{n} S_j - \frac{1}{3} S_n^3$$

is a martingale and deduce that

$$\mathbb{E}(\sum_{r=0}^{T} S_r) = \frac{1}{3}(K^2 - a^2)a + a.$$

4. An urn (called *Polya's urn model*) contains b black balls and r red balls. A ball is drawn at random. It is replaced and, moreover, c balls of the color drawn are added. Let

$$Y_0 = \frac{b}{b+r}$$

and let Y_n be the proportion of the black balls attained by the nth drawing.

(a) Show that $\{Y_n\}$ is a martingale. (With respect to what?)

(b) What is $\mathbb{E}(Y_n)$?

(c) Does $Y := \lim_{n\to\infty} Y_n$ exist? If so, why and in what sense (what sense of convergence). What can you say about $\mathbb{E}(Y)$?

(d) Let $X_j = 1$ if the jth drawing results in a black ball, and $X_j = 0$ if the result is a red ball. Let

$$S_n = X_1 + \cdots + X_n$$

Show that

$$Y_n = \frac{b + cS_n}{b + r + nc}$$

Conclude from this

$$\lim_{n\to\infty} \frac{S_n}{n} = Y$$