Math 135B: HW #7

1. (From Feller, Vol. II) We wish to describe the growth of a large population in which the individuals are stochastically independent and the reproduction rate does not depend upon on the population size. For a very large population the process is approximately continuous; that is, governed by a diffusion equation. The independence of the individuals implies that the infinitesimal drift and variance must be proportional to the population size. Thus the system is governed by the backward (or forward) equation given by

$$\mu(x) = \beta x \text{ and } \sigma(x) = \alpha x$$

   (a) The *forward equation* for this process is

$$\frac{\partial v(s,y)}{\partial s} = \alpha \frac{\partial^2 y\, v(s,y)}{\partial y^2} - \beta \frac{\partial y\, v(s,y)}{\partial y}$$

   (b) Let $M(s)$ ($s =$time in the forward equation) denote the expected population size at time s. To calculate $M(s)$ multiply the forward equation by y and integrate with respect to y from 0 to ∞. Derive a differential equation for $M(s)$. What are the solutions to this DE?

2. Consider a Brownian motion impeded by two absorbing barriers at 0 and $a > 0$. This means that for $0 < y < a$ the transition probability $p_t(x, y)$ should satisfy the PDE

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions

$$p_t(0, y) = p_t(a, y) = 0$$

for all $0 < y < a$.

   (a) Explain why the fundamental solution is

$$p_t(x,y) = \sum_{k=-\infty}^{\infty} [q_t(x - 2ka, y) - q_t(-x - 2ka, y)]$$

   where

$$q_t(x,y) = \frac{1}{\sqrt{2\pi t}}\, e^{-(y-x)^2/(2t)}$$

   Interpret this solution in terms of the method of images. Would you expect this series to converge rapidly for small t or large t?

(b) Using the method of *separation of variables*[1] derive a different representation for $p_t(x, y)$ that satisfies the initial condition

$$p_0(x, y) = \delta(x - y).$$

Will this series converge rapidly for large or small t?

(c) Show that the probability that a particle starting at x, $0 < x < a$, will not be absorbed before time t is

$$\sum_{k=-\infty}^{\infty} \left[N\left(\frac{2ka + a - x}{\sqrt{t}}\right) - N\left(\frac{2ka - x}{\sqrt{t}}\right) - N\left(\frac{2ka + a + x}{\sqrt{t}}\right) + N\left(\frac{2ka + x}{\sqrt{t}}\right) \right]$$

where $N(x)$ is the normal distribution function of mean 0 and variance 1. Using the second representation for $p_t(x, y)$ derive a different representation for this probability. Can you graph (using MatLab or Mathematica) this probability as a function of time t for various values of x and a?

[1] This method can be found in any elementary textbook on partial differential equations.