June 11, 2007

# Math 135B Final Exam, Spring 2007

Instructions: Work all problems in your Blue Book. This exam will not be collected.

**#1 (30 pts)** Let $X = \{X_n\}_{n\geq 0}$ denote a sequence of random variables defined on a common probability space; that is, a discrete-time process.

1. Give the definition that $X$ is a *Markov chain*.
2. If $X$ is a Markov chain, what is the *transition matrix* $P$?
3. Again assuming $X$ is a Markov chain, what is the *n-step transition matrix*?
4. Let $j$ denote a state of the Markov chain $X$. What does it mean to say the state $j$ is *recurrent* (or *persistent*)?
5. Give the definition that the sequence of random variables $\{S_n\}_{n\geq 1}$ is a *martingale* with respect to the sequence $\{X_n\}_{n\geq 1}$.
6. What is the *definition* of Brownian motion, $W_t$?

**#2. (35pts)** Consider a simple random walk on the the (one-dimensional) lattice points $\{0, 1, 2, \dots, N\}$. We assume the walker hops to the right with probability $p$ and to the left with probability $q$. The points 0 and $N$ are assumed to be *absorbing* sites. (Once there always remain there.) We take $p \neq q$; and, of course, $p + q = 1$. Let $S_n$ denote the position of the walker at time $n$. We assume $S_0 = k$, $0 < k < N$. Show that

$$Y_n = \left(\frac{q}{p}\right)^{S_n}$$

is a martingale with respect to the underlying Bernoulli process $\{X_n\}$. ($S_n = X_1 + \cdots + X_n$, $X_j$ i.i.d. and $\mathbb{P}(X_j = 1) = p, \mathbb{P}(X_j = -1) = q$.)

**#3. (50pts)** An urn (called *Polya's urn model*) contains $b$ black balls and $r$ red balls. A ball is drawn at random. It is replaced and, moreover, $c$ balls of the color drawn are added. Let

$$Y_0 = \frac{b}{b+r}$$

and let $Y_n$ be the proportion of the black balls attained by the $n$th drawing.

1. Show that $\{Y_n\}$ is a martingale. (With respect to what?)
2. What is $\mathbb{E}(Y_n)$?
3. Does $Y := \lim_{n\to\infty} Y_n$ exist? If so, why and in what sense (what sense of convergence). What can you say about $\mathbb{E}(Y)$?
4. Let $X_j = 1$ if the $j$th drawing results in a black ball, and $X_j = 0$ if the result is a red ball. Let

   $$S_n = X_1 + \cdots + X_n$$

   Show that

   $$Y_n = \frac{b + cS_n}{b + r + nc}$$

   Conclude from this

   $$\lim_{n\to\infty} \frac{S_n}{n} = Y$$

**#4. (50pts)** This problem modifies the usual simple random walk on the integer lattice $\mathbb{Z}$ as follows: The walker carries an alarm clock that goes off at random times. When the clock rings the walker takes one step to the right with probability $p$ or one step to the left with probability $q$. After completing a step the clock is reset to zero and the waiting time for the next ring begins anew. Formally, we are given a sequence of independent and identically distributed waiting times

$$T_1, T_2, T_3, \ldots, T_n, \ldots$$

where

$$\mathbb{P}(T_j \leq t) = 1 - e^{-t}.$$

Thus these waiting times define a Poisson process, $N(t)$. Recall (you may assume as given)

$$\mathbb{P}\left(N(t) = n\right) = \frac{t^n}{n!} e^{-t}, \;\; n = 0, 1, 2, \ldots.$$

Let $S_t$ denote the position of the walker at time $t$ assuming $S_0 = 0$. Find as explicit expression as you can for

$$\mathbb{P}(S_t = x), \;\; x \in \mathbb{Z}.$$

**#5. (35pts)** A deck of $n$ cards can be ordered in $n!$ ways. The deck comes in some order, we (perhaps repeatedly) shuffle it, and then it comes out in some order. The outcome order of the deck is dependent on the type of shuffle, the random data fed into it, and the number of shuffles. An ordered deck of cards can be identified with a permutation and a shuffle moves from one permutation (one ordering of the deck) to another permutation (another ordering of the deck). To make a Markov chain we let the state space be the set of permutations of $\{1, 2, \ldots, n\}$ and the shuffle have some random element in it.

Consider the *Top-in shuffle*: Take a card from the top and insert it at a random position in the deck. For the simple case of $n = 3$ cards, write out explicitly the transition matrix $P$. (Note: The transition matrix is $6 \times 6$.) Be sure to label your states. Find a row vector $\pi$ so that

$$\pi P = \pi.$$

Given that you have found this row vector $\pi$, what does this tell you about the chain; and hence, the shuffle?