May 2, 2007

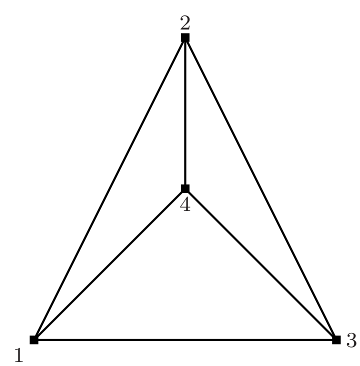
Math 135B Midterm

Instructions: Work all problems in your Blue Book. This exam will not be collected.

#1 (40 pts) Let $X = \{X_n\}_{n \geq 0}$ denote a sequence of random variables defined on a common probability space; that is, a discrete-time process.

1. Give the definition that X is a *Markov chain*.
2. If X is a Markov chain, what is the *transition matrix* P?
3. Again assuming X is a Markov chain, what is the *n-step transition matrix*?
4. Let j denote a state of the Markov chain X. What does it mean to say the state j is *recurrent* (or *persistent*)?
5. Suppose j is a state of the Markov chain X. State a condition on the n-step transition probabilities that implies that the state j is recurrent.
6. What is the *mean recurrence time* μ_i for a state i of the Markov chain X?
7. What is the difference between a *null recurrent* state and a *non-null recurrent* state of a Markov chain?
8. What does it mean to say a Markov chain is *irreducible*?

#2. (30pts) Consider a random walk on the 4-site lattice shown below:



At each time step the walker is equally likely to go to one of its neighbors. (A neighbor is any lattice site connected by a bond to the given lattice site.)

1. How can this be formulated as a Markov chain X? (Give definition of $X = \{X_n\}_{n \geq 0}$, the state space $\mathcal{S}$ and the transition matrix P.)
2. Find a stationary distribution π so that

$$\pi P = \pi.$$

   (Hint: If you think before computing, you should be able to guess the form of π.)
3. On the basis of your solution for π, what can you conclude about the Markov chain X?

#3. (30 pts) Consider a 2-state Markov chain with transition matrix P

$$P = \begin{pmatrix} 1-p & p \\ \alpha & 1-\alpha \end{pmatrix} \tag{1}$$

where $0 < p < 1$ and $0 < \alpha < 1$. A straightforward calculation shows that the *eigenvalues* of P are

$$1 \text{ and } 1-\alpha-p.$$

Using these eigenvalues and their associated eigenvectors, P can be diagonalized:

$$P = \frac{1}{\alpha+p} \begin{pmatrix} 1 & -p/\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-p \end{pmatrix} \begin{pmatrix} \alpha & p \\ -\alpha & \alpha \end{pmatrix} \tag{2}$$

1. Explain why (1) is the most general transition matrix for a 2-state *irreducible* Markov chain.
2. Show, using the above (given) diagonalization (2), that

$$P^n = \frac{1}{\alpha+p} \begin{pmatrix} \alpha & p \\ \alpha & p \end{pmatrix} + \frac{(1-\alpha-p)^n}{\alpha+p} \begin{pmatrix} p & -p \\ -\alpha & \alpha \end{pmatrix}$$

3. Compute the limiting value of P^n as $n \to \infty$ based upon your above calculation of P^n.
4. What is the connection of this limiting matrix to the stationary distribution π of the chain?
5. How fast is the convergence of P^n to its limiting value? (The answer is exponentially fast. Why? Where does this information reside in the eigenvalues?)