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Use of Mathematica in Complex Analysis

Example 1: Let $\mathbb{D} := \{|z| < 1 : z \in \mathbb{C}\}$ denote the open unit disk. Consider the mapping

$$F : z \to \frac{z - w}{1 - z\overline{w}}, \;\; z \in \mathbb{D},$$

where $w \in \mathbb{D}$ is fixed. Suppose we wish to get some geometrical understanding of the function F. For example, suppose z traces out a circle centered at the origin whose radius is less than 1. Under the mapping F, what does the image of this circle look like? Later we will show that the image is also a circle. But let's see how we could come to guess this using the software Mathematica.

Below is a short Mathematica program I wrote to solve this problem. The first line defines the function F. The second line uses the command `ParametricPlot` to draw both the original circle in the z-plane and the resulting image under F. The input values are r, the radius of the circle in the z-plane and the parameter $w \in \mathbb{D}$. The third line runs the program for $r = 1/2$ and $w = i/2$. Clearly the image is also a circle, but no longer centered at the origin. Can you find the center and radius of the new circle?

Answer to last question: If $z = re^{i\theta}$, then the image circle has center

$$z_c = -\frac{1 - r^2}{1 - r^2|w|^2}\, w$$

with radius R equal to

$$R = \frac{r(1 - |w|^2)}{1 - r^2|w|^2}$$

Observe that z_c lies in $\mathbb{D}$ and $R < 1$.

```
F[z_, w_] := (z - w) / (1 - z * Conjugate[w]);

imageF[r_, w_] := ParametricPlot[
   {{Re[r * Exp[I * theta]], Im[r * Exp[I * theta]]},
    {Re[F[r * Exp[I * theta], w]], Im[F[r * Exp[I * theta], w]]}},
   {theta, 0, 2 * Pi}, AxesOrigin → {0, 0}];
imageF[1 / 2, I / 2]
```