Math 205A

Homework #6—Due March 4, 2013

#1. Problem #9, page 202 of Stein & Shakarchi. Hint: The distributed notes might prove useful.

#2. Problem #1a, page 203 of Stein & Shakarchi.

Remarks: For $F(s) = \zeta(s)$, the formula proved becomes

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 \, dt = \zeta(2\sigma), \ \ \sigma > 1.$$

Since $\zeta(s)$ has a pole at $s = 1$ it is not clear what happens at $\sigma = 1$. One can prove[1] that

$$\int_1^T |\zeta(1 + it)|^2 \, dt = \zeta(2)T + \mathrm{O}(\log^2 T)$$

It is a much deeper result that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^2 \, dt = T \log T + \mathrm{O}(T \log^{1/2} T).$$

In 1926 A. E. Ingham proved

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^4 \, dt = (2\pi^2)^{-1} T \log^4 T + \mathrm{O}(T \log^3 T).$$

It is *conjectured* that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^{2k} \, dt \sim a_k T (\log T)^{k^2}$$

where there are explicit conjectures for the coefficients a_k.

#3. Let $\omega_1, \omega_2 \in \mathbb{C}$, $\Im(\omega_2/\omega_1) > 0$ and set

$$\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

[1] A. Ivić, The Riemann Zeta-Function: Theory and Applications, Dover 2003. See page 29.

Define

$$\sigma(z) = \sigma(z, \Omega) := z \prod_{0 \neq \omega \in \Omega} E_2(\frac{z}{\omega}) \tag{1}$$

where $E_2(z)$ is the Weierstrass canonical factor

$$E_2(z) = (1 - z)e^{z + z^2/2}.$$

For (1) to be well-defined we must show the product converges.

1. Let $U = \{(u, v) \in \mathbb{C}^2 : \Im(u/v) > 0\}$. Let $K \subset U$ be compact and let $\alpha > 2$. Prove there exists a bound $M > 0$ such that

   $$\sum_{0 \neq \omega \in \Omega} \left|\omega\right|^{-\alpha} \leq M \tag{2}$$

   for all $(\omega_1, \omega_2) \in K$.

   Useful linear algebra lemma: Let u, v be real variables and consider the quadratic form

   $$Q(u, v) = a\, u^2 + 2\, b\, u\, v + c\, v^2$$

   Suppose that $Q(\cdot, \cdot)$ is *positive definite*; that is,

   $$Q(u, v) > 0 \text{ for all } u, v,\ (u, v) \neq (0, 0).$$

   Then there exist positive constants λ_1 and λ_2 (depending upon a, b and c but not u and v) such that

   $$\lambda_1(u^2 + v^2) \leq Q(u, v) \leq \lambda_2(u^2 + v^2).$$

   If you wish to use this lemma, first prove it. Using this lemma prove there exist positive constants c_1 and c_2 such that

   $$c_1 \sqrt{m^2 + n^2} \leq \left| m\omega_1 + n\omega_2 \right| \leq c_2 \sqrt{m^2 + n^2}. \tag{3}$$

   Use (3) to prove (2).

2. Show the convergence of (1) follows now from (2) and conclude $\sigma(z)$ is an entire function of z whose zero set is Ω. The entire function $\sigma(z, \Omega)$ is called the *Weierstrass sigma-function*.

3. Set
$$\wp(z) = \wp(z; \omega_1, \omega_2) := -\frac{d^2}{dz^2} \log \sigma(z) \tag{4}$$
and show that
$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Omega} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{5}$$

4. Show that $\wp(z)$ is doubly periodic, i.e.
$$\wp(z+\omega) = \wp(z), \;\; \omega \in \Omega.$$
The elliptic function $\wp$ is called the *Weierstrass $\wp$-function*.

5. Remarks: The infinite product for $\sigma(z)$ is similar to that of $\sin z$ given by
$$\sin z = z \prod_{0 \neq m \in \mathbb{Z}} E_1(\frac{z}{m\pi})$$
and $\wp$ is similar to
$$\csc^2 z = -\frac{d^2}{dz^2} \log \sin z = \frac{1}{z^2} + \sum_{0 \neq m \in \mathbb{Z}} \frac{1}{(z-m\pi)^2}.$$