February 17, 2017

Math 22B Midterm

Instructions: Work all problems in your Blue Book. This exam will not be collected. Part I is Short Answers and you need not justify your answer. You can simply write down the result. Part II is Long Answers and you should show all of your work.

Part I—Short Answers

In the mass-spring system without friction, if $x = x(t)$ denotes the displacement from equilibrium at time t, then $x(t)$ satisfies the DE

$$m\,\frac{d^2x}{dt^2} + k\,x = 0 \tag{1}$$

where $k > 0$ is the spring constant and $m > 0$ is the mass of the system. The following two questions are related to this mass-spring system.

#1 (10 pts): If the system starts with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = v_0,$$

what is the unique solution to (1) that satisfies these initial conditions?

#2 (10pts): If the mass m of the system is doubled (multiplied by 2), how does the *period of oscillation* of the system (1) change?

#3 (10pts): If the mass-spring system is subject to both frictional forces (proportional to the velocity) and an external driving force of the form $F(t) = F_0\cos(\omega t)$ the differential equation for $x(t)$ becomes

$$m\,\frac{d^2x}{dt^2} + \beta\,\frac{dx}{dt} + k\,x = F_0\cos(\omega t), \quad \beta > 0. \tag{2}$$

If you seek a *particular solution* to (2) what is the *form* of the solution you guess? *You do not have to determine the constants appearing in your guess.*

Part II—Long Answers

#1 (30 pts): An unusual spring has a force law $F(x) = -\gamma\,x^3$ where x is the displacement from equilibrium and $\gamma > 0$ is a constant. The displacement $x = x(t)$ of this mass-spring system obeys the DE (we assume there are no frictional forces)

$$m\,\frac{d^2x}{dt^2} + \gamma\,x^3 = 0 \tag{3}$$

where $m > 0$ is the mass. Define the energy function

$$E = \frac{1}{2}\,m\left(\frac{dx}{dt}\right)^2 + \frac{\gamma}{4}\,x^4.$$

We proved, and you may assume as given, that this energy is conserved. Assume the initial conditions

$$x(0) = x_0 > 0, \quad \frac{dx}{dt}(0) = 0. \tag{4}$$

Show that the *period of oscillation*, T, of the solution $x = x(t)$ to (3) with initial conditions (4) is

$$T = \frac{4}{x_0}\sqrt{\frac{2m}{\gamma}} \int_0^1 \frac{du}{\sqrt{1-u^4}}$$

Recall that T is the smallest time for which $x(t+T) = x(t)$ for all $t > 0$.

#2 (40 pts:) There are three tanks that hold water. Tank I flows into tank II at rate $\lambda_1 > 0$. Tank II flows into tank III at rate $\lambda_2 > 0$ and tank III flows into tank I at rate $\lambda_3 > 0$. If $x_1(t)$, denotes the amount of water in tank I at time t, $x_2(t)$ the amount of water in tank II at time t and similarly for $x_3(t)$, then the system is modeled (in dimensionless units) by the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -\lambda_1 x_1(t) + \lambda_3 x_3(t), \\ \frac{dx_2}{dt} &= \lambda_1 x_1(t) - \lambda_2 x_2(t), \\ \frac{dx_3}{dt} &= \lambda_2 x_2(t) - \lambda_3 x_3(t). \end{aligned}$$

1. Define the column vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Find a 3×3 matrix A so that the matrix equation

$$\frac{dx}{dt} = Ax$$

is equivalent to the system of three equations above.

2. For the special case $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 1$, the eigenvalues of the matrix A found in the previous part are 0, $-2+\mathrm{i}$ and $-2-\mathrm{i}$. We now use our linear algebra algorithms to compute $\exp(tA)$. To make the results easier to read, define

$$f_c(t) = \mathrm{e}^{-2t}\cos t \quad \text{and} \quad f_s(t) = \mathrm{e}^{-2t}\sin t.$$

Then we have

$$\exp(tA) = \begin{pmatrix} \frac{2}{5} + \frac{3}{5}f_c(t) + \frac{1}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) - \frac{4}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) + \frac{1}{5}f_s(t) \\ \frac{1}{5} - \frac{1}{5}f_c(t) + \frac{3}{5}f_s(t) & \frac{1}{5} + \frac{4}{5}f_c(t) - \frac{2}{5}f_s(t) & \frac{1}{5} - \frac{1}{5}f_c(t) - \frac{2}{5}f_s(t) \\ \frac{2}{5} - \frac{2}{5}f_c(t) - \frac{4}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) + \frac{6}{5}f_s(t) & \frac{2}{5} + \frac{3}{5}f_c(t) + \frac{1}{5}f_s(t) \end{pmatrix}. \tag{5}$$

(a) If the initial condition is

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{(all the water is initially in tank I)} \tag{6}$$

find $x(t)$. (Hint: You should be able to write down the answer with little or no computations.)

(b) For this same initial condition (6), find

$$\lim_{t \to +\infty} x(t)$$

(c) If all the water is initially in tank II and $x(t)$ is the solution for this initial condition, what is $\lim_{t \to +\infty} x(t)$? Same question if all the water is initially in tank III.

3. Returning to $\exp(tA)$ as given in (5), what is the *sum of the matrix elements in each column* of $\exp(tA)$? Should you have been able to predict this result without first computing $\exp(tA)$? That is, give a physical reason for this result.