

FINAL EXAM

MATHEMATICS 22B, SPRING 2005

Instructions: Show all your work in your BLUE BOOK. Only the BLUE BOOK will be collected at the end of the exam. This is a closed book exam and no calculators or laptops are allowed.

1 General Information

Some of the questions involve the matrix differential equation

$$\frac{dx}{dt} = Ax \tag{1.1}$$

where x is a column vector of length n and A is a $n \times n$ matrix with constant entries. The value of n may vary from problem to problem.

2 Exam Questions

#1 (50 points). Consider the third-order, scalar, linear differential equation

$$\frac{d^3y}{dt^3} + y = 0 \tag{2.1}$$

What is the form of the general solution to (2.1)? Be as explicit as you can. Hint: You will need to know that the solutions of the equation $\lambda^3 + 1 = 0$ are $e^{-i\pi/3}$, $e^{i\pi/3}$ and -1 . Use Euler's formula to prove this hint—should you use it!

Useful Information You May Assume as Given for #2

Recall that the *weighted string problem* consists of N particles, each of mass m , equally spaced a distance d in the x -direction and constrained so that they only move in the y -direction. This string is held together by massless springs of tension T such that the first and last particles are tied (via a spring) to the points $x = 0$ and $x = (N + 1)d$. If u_j denotes the vertical displacement of the j^{th} particle, $j = 1, 2, \dots, N$, then the set of differential equations that describes the motion of these particles is

$$m \frac{d^2u_j}{dt^2} = \frac{T}{d} (u_{j+1} - 2u_j + u_{j-1}) \tag{2.2}$$

where we take $u_0 = u_{N+1} = 0$. Defining the $N \times N$ matrix

$$V_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \quad (2.3)$$

and the column vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \quad (2.4)$$

the equations (2.2) have the matrix form

$$\frac{d^2 \mathbf{u}}{dt^2} + \frac{T}{md} V_N \mathbf{u} = 0 \quad (2.5)$$

The eigenvalues λ_n and eigenvectors \mathbf{f}_n , $n = 1, 2, \dots, N$, of V_N are given by

$$\lambda_n = 2(1 - \cos \theta_n) = 4 \sin^2(\theta_n/2) \quad \text{with} \quad \theta_n = \frac{n\pi}{N+1}$$

and

$$\mathbf{f}_n = \begin{pmatrix} \sin(\theta_n) \\ \sin(2\theta_n) \\ \vdots \\ \sin(N\theta_n) \end{pmatrix}$$

In terms of this information, the general solution to (2.5) is

$$\mathbf{u}(t) = \sum_{n=1}^N (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \mathbf{f}_n$$

with

$$\omega_n^2 = \frac{T}{md} \lambda_n, \quad n = 1, 2, \dots, N.$$

#2 (50 pts): We now assume that the particles in the *weighted string problem* are subject to a force due to the presence of friction. (Imagine the particles are moving in a medium which offers resistance to the motion of the particles.) Assuming the frictional force is proportional to the velocity, the system of differential equations describing the motion is

$$m \frac{d^2 u_j}{dt^2} = \frac{T}{d} (u_{j+1} - 2u_j + u_{j-1}) - \gamma \frac{du_j}{dt}, \quad j = 1, 2, \dots, N \quad (2.6)$$

where γ is positive and, as before, $u_0 = u_{N+1} = 0$.

1. Rewrite the system (2.6) in matrix form such that when $\gamma = 0$ the equation becomes identical to the matrix equation (2.5). Hint: As before let V_N be the matrix (2.3) and \mathbf{u} the vector (2.4).

2. Assume a solution of the form

$$\mathbf{u}(t) = e^{i\omega t} \mathbf{f} \quad (2.7)$$

where \mathbf{f} is a column vector independent of t and ω is to be determined. For what values of ω is (2.7) a solution to the matrix equation derived in part (1)?

NOTE: This will not require a *complete* reworking of the eigenvalues since you may use the information given above to find the eigenvalues in this new problem. You should not have to solve anything more complicated than a quadratic equation.

3. Explain the significance of the fact that the ω 's you obtain are *complex* numbers.
4. For a large system $N \gg 1$ explain why you expect some of the allowed ω 's to be *purely imaginary*. Explain the significance of this result, i.e. what is the implication for the motion?

#3 (50 pts): Consider the DE (1.1) for $n = 3$ and

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

where α is a real number. A linear algebra computation shows that for this matrix A we have the following results:

$$Af_1 = f_1 \quad \text{where } f_1 = \begin{pmatrix} 2\alpha \\ -\alpha \\ 1 \end{pmatrix}$$

$$Af_2 = -\sqrt{\alpha} f_2 \quad \text{where } f_2 = \begin{pmatrix} 1 - \sqrt{\alpha} \\ -1 \\ 1 \end{pmatrix}$$

$$Af_3 = \sqrt{\alpha} f_3 \quad \text{where } f_3 = \begin{pmatrix} 1 + \sqrt{\alpha} \\ -1 \\ 1 \end{pmatrix}$$

- For what values of α will we have *oscillatory* solutions?
- For these choices of α , find the subspace V of \mathbf{R}^3 such that for initial conditions x_0 lying in this subspace V , the solutions are purely oscillatory. (That is, the solutions with these initial conditions oscillate but neither decay nor grow exponentially.)
- Write down the form of the most general solution.

#4 (50 pts): Consider the quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2} \hat{p}^2 + \frac{1}{2} x^2 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \quad (2.8)$$

where we have chosen units so that $\hbar = 1$, $m = 1$ and $\omega = 1$ (to make the problem notationally simpler). Here $\hat{p} = -i \frac{d}{dx}$ is the momentum operator. We showed in class, and you may assume as given, that

$$H\psi_n = \left(n + \frac{1}{2}\right) \psi_n, \quad n = 0, 1, 2, \dots \quad (2.9)$$

where

$$\psi_n(x) = \left(\frac{1}{\sqrt{\pi} n! 2^n}\right)^{1/2} H_n(x) e^{-x^2/2} \quad (2.10)$$

and $H_n(x)$ are the Hermite polynomials. Furthermore, we showed that (and you may also assume as given)

$$\begin{aligned} (\psi_m, \psi_n) &= \int_{-\infty}^{\infty} \psi_m(x) \bar{\psi}_n(x) dx = \delta_{m,n}, \\ x \psi_n(x) &= \sqrt{\frac{n}{2}} \psi_{n-1}(x) + \sqrt{\frac{n+1}{2}} \psi_{n+1}(x), \\ \hat{p} \psi_n(x) &= -i\sqrt{\frac{n}{2}} \psi_{n-1}(x) + i\sqrt{\frac{n+1}{2}} \psi_{n+1}(x). \end{aligned}$$

Here $\delta_{m,n}$ is the Kronecker delta function, i.e. equal to 1 if $m = n$ and zero otherwise. The “bar” denotes complex conjugation, i.e. $\bar{\psi}_n$ is the complex conjugate of ψ_n .

If \mathcal{O} is a hermitian operator (self-adjoint), define the average in the state ψ_n by

$$\langle \mathcal{O} \rangle_n = (\mathcal{O}\psi_n, \psi_n)$$

where (\cdot, \cdot) is the inner product defined above. Define

$$\Delta \mathcal{O} = \sqrt{\langle \mathcal{O}^2 \rangle_n - (\langle \mathcal{O} \rangle_n)^2}.$$

We put the subscript n on the averages to denote that the average is taken in state ψ_n .

1. Explain why it follows (almost immediately) that $\langle H \rangle_n = n + \frac{1}{2}$. Prove that $\Delta H = 0$ for any eigenstate ψ_n . What is the meaning of this last result?
2. Prove that for any eigenstate ψ_n ,

$$\Delta x \Delta \hat{p} \geq \frac{1}{2}.$$

3. Compute $\langle x^4 \rangle$ for the state ψ_n .

Hint: For question 2, first compute $x^2\psi_n$ and $\hat{p}^2\psi_n$. For question 3 write $(x^4\psi_n, \psi_n) = (x^2\psi_n, x^2\psi_n)$ and then start computing. Be sure to explain why this identity is true.