Final Examination, Winter 2009

Mathematics 22B

Instructions: Put all answers in your Blue Book. The exam consists of two parts: Part 1 has five multiple choice/short answer questions. You need not justify your answer. Simply write your answer in your Blue Book. Each question is worth 15 points. Part 2 consists of three problems. Show all your work in your Blue Book.

Notation: For real or complex z, e^z is the exponential function. If A is a square matrix $\exp(A)$ denotes the exponential of the matrix which is sometimes written as e^A.

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1 Multiple Choice/Short Answers, 75 points

Some of the questions involve the matrix differential equation

$$\frac{dx}{dt} = A\,x + f(t) \tag{1.1}$$

where x is a column vector of length n, A is a $n \times n$ matrix with *constant* entries and $f = f(t)$ is a given column vector of length n whose entries are functions of the independent variable t. The value of n may vary from problem to problem.

1. Every solution of (1.1) can be written in the form: (choose one)

   (a) $x(t) = c_1\, e^{\lambda_1 t} + c_2\, e^{\lambda_2 t}$ where $\lambda_{1,2}$ are roots to the equation $a\lambda^2 + b\lambda + c = 0$.

   (b) $x(t) = \exp(tA)x_0$ for some fixed vector x_0.

   (c) $x(t) = \exp(tA)x_0 + \exp(tA)\int_0^t \exp(-sA)f(s)\,ds$ for some fixed vector x_0.

   (d) There is no general formula that one can write down when the matrix is bigger than 2×2.

2. The general solution to the scalar equation

$$\frac{d^2y}{dx^2} - 2y = 0$$

   is (c_1 and c_2 are constants):

   (a) $c_1 \sin(2x) + c_2 \cos(2x)$

   (b) $c_1 \sin(\sqrt{2}x) + c_2 \cos(\sqrt{2}x)$

   (c) $c_1\, e^{2x} + c_2\, e^{-2x}$

   (d) $c_1\, e^{\sqrt{2}x} + c_2\, e^{-\sqrt{2}x}$

   (e) None of the above

3. Consider the mass-spring system (with no external forcing). Let $x = x(t)$ denote the displacement from equilibrium at time t. Assuming that the spring follows Hooke's Law and that the frictional force is proportional to the velocity, we showed that x satisfies the differential equation

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + k\,x = 0 \tag{1.2}$$

where m, γ and k are positive real numbers. What is the condition that implies there are NO OSCILLATIONS; that is, the system simply relaxes back to its equilibrium position?

(a) $\gamma + 4mk > 0$
(b) $\gamma - 4mk < 0$
(c) $\gamma^2 - 4mk < 0$
(d) $\gamma^2 - 4mk > 0$
(e) $\gamma = \sqrt[3]{4mk}$

4. In DE (1.1) let $f(t) = 0$ and suppose the matrix A is given by

$$A = \begin{pmatrix} 2 & -6a & -1-12a & 1+6a \\ 1 & 1/2-2a & -3-4a & 3/2+2a \\ 0 & -3/2 & -1 & 3/2 \\ 1 & -3/2-2a & -3-4a & 7/2+2a \end{pmatrix} \tag{1.3}$$

$$= \begin{pmatrix} 1 & 0 & -1 & 4a \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 4a \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \tag{1.4}$$

where a is a postive number. Write down the general solution $x(t)$ to (1.1) for this choice of A and f.

NOTE:(1.4) is the diagonalization of the matrix A defined by (1.3). There is enough information given for you to write down immediately the answer with no computations involved.

5. Consider the DE (1.1) for $n = 3$ where $f(t) = 0$ and

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

where α is a real number. A linear algebra computation shows that for this matrix

A we have the following results:

$$Af_1 = f_1 \qquad \text{where } f_1 = \begin{pmatrix} 2\alpha \\ -\alpha \\ 1 \end{pmatrix}$$

$$Af_2 = -\sqrt{\alpha}\, f_2 \qquad \text{where } f_2 = \begin{pmatrix} 1-\sqrt{\alpha} \\ -1 \\ 1 \end{pmatrix}$$

$$Af_3 = \sqrt{\alpha}\, f_3 \qquad \text{where } f_3 = \begin{pmatrix} 1+\sqrt{\alpha} \\ -1 \\ 1 \end{pmatrix}$$

QUESTION: For what values of α will we have *oscillatory* solutions?

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2 Detailed answers required in this section. You must show all your work in your Blue Book to receive full credit.

#1 (45 pts): Consider the small vibrations of a thin membrane spread out over a rectangular region (*vibrating rectangular membrane*). The partial differential equation satisfied by the displacement $U = U(x, y, t)$ at position (x, y) at time t of the membrane is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0 \tag{2.1}$$

where $v > 0$ is a constant (and has the dimensions of velocity). We assume the rectangle lies in the first quadrant region

$$0 \le x \le a \text{ and } 0 \le y \le b.$$

That is, the corners of the rectangle are $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b). We assume the membrane is tied down on the edges of the rectangle:

$$U(a, y, t) = 0 \text{ for all } y, t \text{ and } U(x, b, t) = 0 \text{ for all } x, t. \tag{2.2}$$

PROBLEM: Find the *frequencies* at which the membrane can oscillate.

#2 (40 pts): Consider the second-order (possibly nonlinear) differential equation

$$\frac{d^2x}{dt^2} = f(x), \;\; x \in \mathbb{R}. \tag{2.3}$$

Introduce the *energy function*[1]

$$E(x,\dot{x}) = \frac{1}{2}\,\dot{x}^2 + V(x) \tag{2.4}$$

where $V(x)$ is the potential energy defined by

$$V(x) = -\int_{x_0}^{x} f(\xi)\,d\xi,\ x_0 \text{ is a fixed point.}$$

1. Prove the law of conservation of energy: *The total energy of points moving according to the equation (2.3) is conserved: $E(x(t),\dot{x}(t))$ is independent of t.*

2. Show that the time it takes to go from x_1 to x_2 (in one direction) is equal to

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - V(x)}}.$$

#3 (40 pts): First some useful information: The time-independent Schrödinger equation for the harmonic oscillator in *dimensionless* variables is the second order linear differential equation

$$H\psi(x) = -\frac{d^2\psi}{dx^2} + x^2\psi = E\psi(x),\ \ x \in \mathbb{R}. \tag{2.5}$$

We proved in class, and you may assume as given, that if we want square-integrable solutions[2] to (2.5) then E is quantized; that is, takes on the discrete values

$$E_n = 2n+1,\ \ n = 0,1,2,\ldots$$

The corresponding eigenfunctions,

$$H\psi_n = E_n\psi_n$$

are given by

$$\psi_n(x) = \left[\sqrt{\pi}n!2^n\right]^{-1/2} H_n(x)\mathrm{e}^{-x^2/2},\ \ n = 0,1,2,\ldots$$

where H_n are the Hermite polynomials. Recall the Hermite polynomials satisfy $H_n(-x) = (-1)^n H_n(x)$. The functions ψ_n are *orthonormal*, i.e.

$$(\psi_m,\psi_n) = \int_{-\infty}^{\infty} \psi_m(x)\overline{\psi_n(x)}\,dx = \delta_{m,n}$$

where $\delta_{m,n}$ is the Kronecker delta function

$$\delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

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[1]Recall $\dot{x} = dx/dt$.

[2]A function $\psi(x)$ is called square-integrable if $\int_{-\infty}^{\infty} |\psi(x)|^2\,dx < \infty$.

We also proved that ψ_n satisfy the important relations[3]

$$x\,\psi_n(x) = \sqrt{\frac{n}{2}}\,\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\,\psi_{n+1}(x) \tag{2.6}$$

$$\frac{d\psi_n(x)}{dx} = \sqrt{\frac{n}{2}}\,\psi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\,\psi_{n+1}(x) \tag{2.7}$$

You may assume as given all of these facts.

Questions:

1. Prove that $\langle x\rangle_n := (x\psi_n, \psi_n) = 0$.
2. Prove that $\langle x^2\rangle_n := \big(x^2\psi_n, \psi_n\big) = n + \frac{1}{2}$.
3. If $\hat{p} = -i\frac{d}{dx}$, prove that $\langle \hat{p}\rangle_n := (\hat{p}\psi_n, \psi_n) = 0$.
4. Prove that $\langle \hat{p}^2\rangle_n := \big(\hat{p}^2\psi_n, \psi_n\big) = n + \frac{1}{2}$.
5. Let $\Delta x := \sqrt{\langle x^2\rangle_n - (\langle x\rangle_n)^2}$ and $\Delta p := \sqrt{\langle \hat{p}^2\rangle_n - (\langle \hat{p}\rangle_n)^2}$. Prove that for all $n = 0, 1, \ldots$

$$\Delta x \cdot \Delta p \geq \frac{1}{2}.$$

[3] We take $\psi_{-1} = 0$.