

MATH 22B FINAL EXAMINATION

Instructions: Work all problems in your Blue Book. This exam will not be collected. Part I has six SHORT ANSWER questions. You need at most a sentence or two to justify your answer. (In some questions you are told you can say either “Yes” or “No”.) Each of the six questions is worth 10 points. Part II has four LONG ANSWER questions. You should show all of your work. Each of the four questions is worth 35 points.

PART I—SHORT ANSWER QUESTIONS

1. Consider the system of differential equations

$$\frac{dp_1}{dt} = -\lambda_1 p_1(t) \quad (1)$$

$$\frac{dp_2}{dt} = \lambda_1 p_1(t) - \lambda_2 p_2(t) \quad (2)$$

$$\frac{dp_3}{dt} = \lambda_2 p_2(t) - \lambda_3 p_3(t) \quad (3)$$

$$\frac{dp_4}{dt} = \lambda_3 p_3(t) \quad (4)$$

where $\lambda_i > 0$ for $i = 1, 2, 3, 4$. Define the vector

$$\mathbf{P}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{pmatrix}$$

- (a) Rewrite the above system of equations (1)–(4) as the matrix equation

$$\frac{d\mathbf{P}}{dt} = A\mathbf{P} \quad (5)$$

That is, write down the matrix A .

- (b) Now suppose that the coefficients λ_1 , λ_2 and λ_3 are functions of the independent variable t . Can you still write the system (1)–(4) as the matrix equation (5)?

2. Suppose a certain function $\wp = \wp(z)$ satisfies the ordinary differential equation

$$\left(\frac{d\wp}{dz}\right)^2 = 4\wp^3 - g_2\wp - g_3 \quad (6)$$

where g_2 and g_3 are constants.

- (a) What is the *order* of this differential equation?
 (b) Is the equation (6) *linear* or *nonlinear*?
 (c) Find a formula for $d^2\wp/dz^2$ that involves only \wp and g_2 .

3. Consider the constant coefficient matrix differential equation

$$\frac{dx}{dt} = Ax \quad (7)$$

where A is the 4×4 matrix

$$A = \begin{pmatrix} \alpha & \alpha & \beta & 0 \\ 0 & 2\alpha & 2\alpha - \beta & 0 \\ 0 & 0 & \beta & 0 \\ \alpha + \beta & -3\alpha - 2\beta & -4\alpha - 3\beta & -\beta \end{pmatrix}$$

and α and β are *real* and *positive* constants. The *eigenvalues* of A are

$$\alpha, 2\alpha, \beta, -\beta$$

with respective *eigenvectors*

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Write down the general solution to (7) in terms of the information given about the eigenvalues and eigenvectors of A . Will the general solution approach 0 as $t \rightarrow \infty$? Why or why not?

4. Consider the differential equation for $y = y(x)$:

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} + 1 = 0 \quad (8)$$

You are given that a numerical calculation finds that the roots of the polynomial equation

$$\lambda^3 + \lambda + 1 = 0$$

are approximately

$$\lambda_1 = -0.682328, \quad \lambda_2 = 0.341164 + i 1.16154, \quad \lambda_3 = 0.341164 - i 1.16154.$$

Discuss the qualitative behavior of the solutions $y = y(x)$. For example, are there oscillatory solutions? If so, does the amplitude of these oscillations, increase, decrease or stay the same with increasing x ? Are there solutions that decrease to zero? Give short reasons for your answers.

5. A simple (classical) harmonic oscillator, initially at rest, is subjected to an external driving force $F(t) = \cos \omega t$. The resulting amplitude of oscillation of our oscillator, as a function of the driving frequency ω , is displayed in Figure 1 (see next page). Assuming that the frictional forces are very small; and hence negligible, on the basis of this graph estimate the period of the oscillator? Give a reason for your answer.

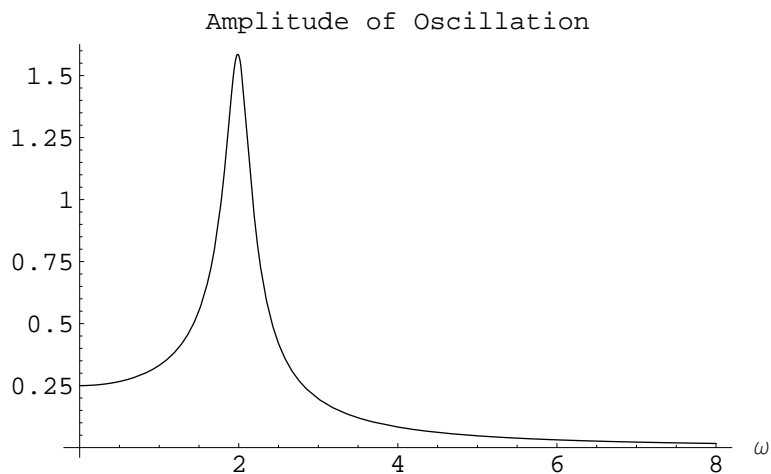


Figure 1:

6. Consider the constant coefficient matrix differential equation

$$\frac{dx}{dt} = Ax \tag{9}$$

where A is a 6×6 matrix. The characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is graphed in Figure 2 (next page). On the basis of this graph, explain in a sentence or two what you can say about any solution $x = x(t)$ to (9) as $t \rightarrow +\infty$.

PART II—LONG ANSWER QUESTIONS

1. Consider the *partial* differential equation for $u = u(x, t)$:

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = 0 \tag{10}$$

where x and $t \geq 0$ are real variables and D is a positive constant. Assuming a solution of the form (separation of variables)

$$u(x, t) = X(x)T(t)$$

derive *ordinary* differential equations for $X = X(x)$ and $T = T(t)$.

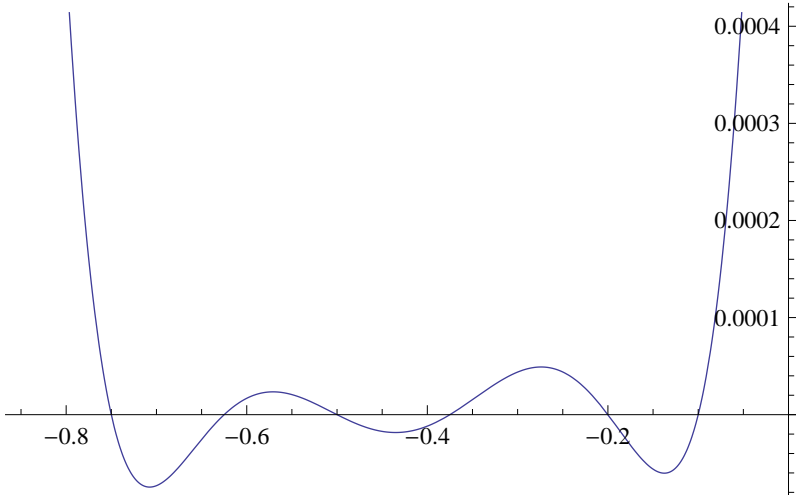


Figure 2: Plot of $p(\lambda)$ versus λ

2. Consider a *vibrating rectangular membrane*. The partial differential equation satisfied by the displacement $u = u(x, y, t)$ at position (x, y) at time t of the membrane is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (11)$$

where $v > 0$ is a constant. We assume the rectangle lies in the first quadrant in the region

$$0 \leq x \leq a, \text{ and } 0 \leq y \leq b$$

That is, the corners of the rectangle are $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b) . We assume the membrane is tied down on the edges of the rectangle:

$$u(a, y, t) = 0 \text{ for all } y, t \text{ and } u(x, b, t) = 0 \text{ for all } x, t. \quad (12)$$

Show that the frequencies of oscillation of the membrane are given by

$$\omega_{m,n} = \pi v \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^{1/2}$$

where $m, n = 1, 2, 3, \dots$

3. When we studied the quantum harmonic oscillator we were led to analyze the differential equation

$$\frac{d^2v}{dx^2} - 2x \frac{dv}{dx} + (\varepsilon - 1)v = 0 \quad (13)$$

where ε is a constant.

(a) Assume a solution of the form

$$v(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k,$$

and *show* that a_k must satisfy the recurrence relation

$$a_{k+2} = \frac{2k - \varepsilon + 1}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, \dots$$

(b) For what values of ε can we have solutions $v = v(x)$ to (13) that are *polynomials* in x ? Be sure to explain the reasoning for your answer.

4. Consider the system of ordinary differential equations

$$\begin{aligned} \frac{df_1}{dz} &= f_2 f_3 \\ \frac{df_2}{dz} &= -f_1 f_3 \\ \frac{df_3}{dz} &= -\lambda f_1 f_2 \end{aligned}$$

where $0 < \lambda < 1$. Show that

$$\mathcal{I}_1 = f_1^2 + f_2^2 \quad \text{and} \quad \mathcal{I}_2 = \lambda f_1^2 + f_3^2$$

are independent of z if f_1 , f_2 and f_3 satisfy the above differential equations.