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# SPIN-SPIN CORRELATION FUNCTIONS FOR THE TWO-DIMENSIONAL ISING MODEL*+

BARRY M. McCOY‡ AND CRAIG A. TRACY

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, L.I., New York

and

TAI TSUN WU

Division of Engineering and Applied Physics
Harvard University
Cambridge, Massachusetts

We give a simplified derivation of an exact expression for the spin-spin correlation function of the two-dimensional Ising model suitable for studying large spin separation for $T<T_c$.

## I. INTRODUCTION

Some time ago, Montroll, Potts and Ward [*1*] derived a representation of the multi-spin correlation functions of

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the two-dimensional Ising model in terms of determinants. These determinants are of small size when the spins can be grouped into a set of pairs such that the members of each pair are close together. However, if all the spins are widely separated the size of the determinant grows with the separation and the behavior of the correlation function is no longer manifest.

The process of converting the determinants of MPW into a form useful for studying widely separated spins was initiated for the 2-point function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ (where M specifies the row and N specifies the column) by Wu [2] who considered $M=0$ and soon continued by Cheng and Wu [3] who considered $M\neq 0$. These authors derived the leading terms in $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for widely separated spins for both $T<T_c$ and $T>T_c$. Recently the complete expansion of the 2-point function in the form suitable for widely separated spins has been computed by Wu, McCoy, Tracy and Barouch [4,5] and for suitably large separation was shown to be convergent. With this complete expansion it was then possible to give a detailed description of the 2-point scaling functions. In particular, the scaling functions were shown to be expressible in terms of a Painlevé function of the third kind [5-7].

Here we give a simplified derivation of the results of section III of reference 5 (these results are summarized by equations (2.9) - (2.13) of reference 5). We assume that the reader is familar with the Pfaffian approach to the two-dimensional Ising model [8-11]. As background we refer the reader to Kasteleyn [12] and MPW or to any one of a number of review papers [13-16].

## II. PERTURBATION EXPANSION FOR $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ FOR $T<T_c$ AND LARGE $M^2+N^2$

### A. Preliminary Notation

The two-dimensional Ising model on a square lattice is specified by the interaction energy

$$\mathcal{E} = -E_1 \sum_{j,k} \sigma_{j,k}\sigma_{j,k+1} - E_2 \sum_{j,k} \sigma_{j,k}\sigma_{j+1,k} \qquad (2.1)$$

where the first (second) index of $\sigma_{j,k}$ specifies the row (column) of the lattice and $\sigma_{j,k} = \pm 1$. We define

$$z_1 = \tanh(\beta E_1) , \qquad z_2 = \tanh(\beta E_2) \tag{2.2}$$

where $\beta = (k_B T)^{-1}$. At $T = T_c$ we have

$$\sinh 2\beta_c E_1 \ \sinh 2\beta_c E_2 = 1 \tag{2.3}$$

or, equivalently

$$z_{1c} z_{2c} + z_{1c} + z_{2c} - 1 = 0 \tag{2.4}$$

## B. Method of MPW

In the Pfaffian approach to the calculation of correlation functions for the Ising model defined by (2.1), the problem is equivalent to the calculation of the partition function of an Ising model defined on a "defective" square lattice [1]. To be precise, to compute $<\sigma_{0,0}\sigma_{M,N}>$ we first join the lattice points (0,0) and (M,N) by a line drawn on the lattice. If the line is drawn on a horizontal (vertical) bond, the bond strength is changed from $z_1 (z_2)$ to $z_1^{-1}$ $(z_2^{-1})$. This defines the defective lattice. The partition function for the Ising model on this defective lattice is the Pfaffian of some antisymmetric matrix. The correlation function becomes the ratio of the partition function for the defective lattice to the partition function for the "pure" lattice. Since the square of a Pfaffian is a determinant, $<\sigma_{0,0}\sigma_{M,N}>^2$ is expressible as the ratio of two determinats. The determinant arising from the defective lattice can be written as the product of two determinants with one determinant being precisely the determinant appearing in the denominator. Hence the method of MPW reduces the computation of $<\sigma_{0,0}\sigma_{M,N}$ to the evaluation of a single determinant. The size of this determinant is proportional to the length of the line drawn.

Clearly, there are many ways to draw a line on the lattice that connects the lattice points (0,0) and (M,N). In MPW the correlation function $<\sigma_{0,0}\sigma_{0,N}>$ was expressed as a Toeplitz [*17*] determinant by connecting the lattice points (0,0) and (0,N) by the horizontal line passing through the points (0,0), (0,1), ..., (0,N). By an application of

Szegö's theorem [*18*,*19*] the square of the spontaneous magnetization [*10*,*20*] was computed by evaluation $\lim_{N\to\infty} \langle\sigma_{0,0}\sigma_{0,N}\rangle$. Wu [*2*] studied this Toeplitz determinant for fixed T and large N. Though these results concern the large distance behavior of the spin-spin correlation function, the drawing of lines that contain a minimal number of bonds between the lattice points (0,0) and (M,N) is best suited for studying the short-range order of the correlation function. To study the behavior of $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for large $M^2 + N^2$ it is better, as was first shown by Cheng and Wu [*3*], to work with determinants which arise from drawing the line through infinity. That is to say, we may connect the lattice points (0,0) and (M,N) by a horizontal line passing through the points (0,0), (0,-1), (0,-2), ··· and the points (M,N), (M,N+1), (M,N+2), ···. Note that the resulting determinant is of infinite size. This method results (as was shown in reference 5) in an exact expression for the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for both $T<T_c$ and $T>T_c$.

We now describe our simplification of the analysis of reference 5 for the case $T<T_c$. We again connect the lattice points (0,0) and (M,N) by horizontal lines extending to infinity. Thus we have in the notation of reference 5 (M and N are hereafter assumed positive).

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle^2 = \det(C) \tag{2.5}$$

$$C = \begin{vmatrix} 0 & -S & T & U \\ S^T & 0 & -U & V \\ -T & U & 0 & S \\ -U & -V & -S^T & 0 \end{vmatrix} \tag{2.6}$$

S T U, and V are infinite matrices defined for m,n = 0,1,2... by

$$S_{mn} = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi \; e^{-i(n-m)\phi} \; S(e^{i\phi}) \; , \tag{2.7}$$

$$T_{mn} = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi \; e^{-i(m+n)\phi} \; T(e^{i\phi}) \; , \tag{2.8}$$

$$U_{mn} = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi \; e^{-i(m+n)} \; U(e^{i\phi}) \; , \tag{2.9}$$

and

$$V_{mn} = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi \; e^{-i(m+n)} \; V(e^{i\phi}) \; , \tag{2.10}$$

where

$$S(e^{i\phi}) = \left[\frac{(1-\alpha_1 e^{i\phi})(1-\alpha_2 e^{-i\phi})}{(1-\alpha_1 e^{-i\phi})(1-\alpha_2 e^{i\phi})}\right]^{\frac{1}{2}} \tag{2.11}$$

$$T(e^{i\phi_2}) = -(1-z_1^2)(2\pi)^{-1} \int_{-\pi}^{\pi} d\phi_1 \; \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1,\phi_2)} \; \lambda(\phi_1,\phi_2) \tag{2.12}$$

$$U(e^{i\phi_2}) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi_1 \; \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1,\phi_2)} \left[-2iz_2(1-z_1^2)e^{-i\phi_2}\sin\phi_1\right] \tag{2.13}$$

$$V(e^{i\phi_2}) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi_1 \; \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1,\phi_2)} \; e^{-2i\phi_2} \; (1-z_1^2)\lambda(\phi_1,-\phi_2) \tag{2.14}$$

and with $z_1$ and $z_2$ given by (2.2),

$$\Delta(\phi_1,\phi_2) = (1+z_1^2)(1+z_2^2) - 2z_2(1-z_1^2)\cos\phi_1 - 2z_1(1-z_2^2)\cos\phi_2 \tag{2.15}$$

$$\lambda(\phi_1,\phi_2) = 1 - z_2^2 - z_1(1+z_2^2+2z_2\cos\phi_1)e^{-i\phi_2} \tag{2.16}$$

and

$$\alpha_1 = z_1 \frac{1-z_2}{1+z_2}$$
$$\alpha_2 = z_1^{-1} \frac{1-z_2}{1+z_2} \tag{2.17}$$

The square root in (2.11) is defined so that $S(e^{i\pi})>0$. Following MPW it is convenient to define

$$a = (1+z_1^2)(1+z_2^2)$$
$$\gamma_1 = 2z_2(1-z_1^2) \tag{2.18}$$
$$\gamma_2 = 2z_1(1-z_2^2)$$

We also note that for $T<T_c$ $(0<\alpha_1<\alpha_2<1)$ the index [*21*] of the generating function (2.11) is zero.

We define

$$A = \begin{bmatrix} 0 & S \\ -S^T & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix} \tag{2.19}$$

and

$$B = \begin{bmatrix} T & U \\ -U & V \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix} \tag{2.20}$$

so that $C$ becomes

$$C = \begin{bmatrix} -A & B \\ -B^T & A \end{bmatrix} = \mathcal{A} + \mathcal{B} \tag{2.21}$$

## C. Evaluating det($C$)

Now from (2.15) and (2.21) we obtain [22]

$$\langle \sigma_{0,0}\sigma_{M,N}\rangle^2 = \det(C)$$

$$= \det(\mathcal{A})\ \det(1 + \mathcal{A}^{-1}\mathcal{B})$$

$$= M^4 \exp[\mathrm{Tr}\ \log(1 + \mathcal{A}^{-1}\mathcal{B})]$$

$$= M^4 \exp\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\ \mathrm{Tr}[(\mathcal{A}^{-1}\mathcal{B})^k]\right] \tag{2.22}$$

In (2.22) we used

$$\det(\mathcal{A}) = M^4 \tag{2.23}$$

where $M$ is the spontaneous magnetization [10,20]

$$M = [1 - (\sinh 2\beta E_1\ \sinh 2\beta E_2)^{-2}]^{1/8} \tag{2.24}$$

The evaluation of det($\mathcal{A}$) is discussed in MPW and Chapter X of reference 11.

If we define $F_2$ by

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = M^2\exp(-F_2) \tag{2.25}$$

then it follows from (2.22) that

$$F_2 = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} \mathrm{Tr}[(A^{-1}B)^k] \tag{2.26}$$

From (2.19) and (2.20)

$$\mathrm{Tr}[(A^{-1}B)^{2j+1}] = 0 \quad , \quad j = 0, 1, 2, \ldots \tag{2.27}$$

and

$$(A^{-1}B)^2 = \begin{bmatrix} A^{-1}BA^{-1}B^T & 0 \\ 0 & A^{-1}B^TA \;\; B \end{bmatrix} \tag{2.28}$$

Thus if we write $F_2$ as

$$F_2 = \sum_{k=1}^{\infty} F_2^{(2k)} \quad , \tag{2.29}$$

we have for $k = 1, 2, 3, \ldots$

$$F_2^{(2k)} = \frac{1}{2k} \mathrm{Tr}[(A^{-1}BA^{-1}B^T) \ldots (A^{-1}BA^{-1}B^T)] \tag{2.30}$$

where the factor $(A^{-1}BA^{-1}B^T)$ occurs $k$ times in (2.30). Note that in going from (2.26) to (2.30) we used (2.28) and the cyclic property of the trace. We now evaluate the trace appearing in (2.30).

We define for $|\xi|$ and $|\xi'| < 1$

$$S^{-1}(\xi,\xi') = \sum_{m,n=0}^{\infty} \xi^m \xi'^n (S^{-1})_{mn} \tag{2.31}$$

where S is the Wiener-Hopf operator [21] defined by (2.7). The generating function $S(\xi)$, defined by (2.11), has the factorization

$$[S(\xi)]^{-1} = P(\xi)Q(\xi^{-1}) \tag{2.32}$$

where $P(\xi)$ and $Q(\xi)$ are analytic inside the unit circle $|\xi|=1$ and are given by

$$P(\xi) = \left[\frac{1-\alpha_2\xi}{1-\alpha_1\xi}\right]^{\frac{1}{2}} \tag{2.33a}$$

and

$$Q(\xi) = \left[\frac{1-\alpha_1\xi}{1-\alpha_2\xi}\right]^{\frac{1}{2}} \tag{2.33b}$$

We note

$$P(\xi)Q(\xi) = 1. \tag{2.34}$$

It follows from (2.31), (2.32) and the method of Wiener-Hopf [21] that

$$S^{-1}(\xi,\xi') = Q(\xi)P(\xi')(1-\xi\xi')^{-1} . \tag{2.35}$$

We define

$$A^{-1}(\xi,\xi') = \begin{bmatrix} 0 & -S^{-1}(\xi',\xi) \\ S^{-1}(\xi,\xi') & 0 \end{bmatrix} \tag{2.36}$$

so that $A^{-1}(\xi,\xi')$ is the matrix generating function of the function of the inverse matrix elements of A [defined by

(2.19)].

Similarly we define

$$B(\xi) = \begin{bmatrix} T(\xi) & U(\xi) \\ -U(\xi) & V(\xi) \end{bmatrix} \qquad (2.37)$$

where the functions $T(\xi)$, $U(\xi)$ and $V(\xi)$ are given by (2.12)-(2.14), respectively. Note that the matrix generating function of the matrix $B^T$

$$B^T(\xi) = \begin{bmatrix} T(\xi) & -U(\xi) \\ U(\xi) & V(\xi) \end{bmatrix} \qquad (2.38)$$

With the above definitions we see that (2.30) is

$$F_2^{(2k)} = \frac{1}{2k}(2\pi)^{-2k}\int_{-\pi}^{\pi} d\phi_2 d\phi_4 \ldots d\phi_{4k} \mathrm{Tr}\{A^{-1}(\overline{4k},\bar{2})B(2)$$

$$A^{-1}(\bar{2},\bar{4})B^T(4)A^{-1}(\bar{4},\bar{6})B(6)\ldots A^{-1}(\overline{4k-2},\overline{4k})B^T(4k)\} \qquad (2.39)$$

where B(2) means $B(e^{i\phi_2})$ and $A^{-1}(\bar{2},\bar{4})$ means $A^{-1}(e^{-i\phi_2},e^{-i\phi_4})$, etc. In (2.39) the $\phi_{2j}$ variables satisfy the restriction $\mathrm{Im}\phi_{2j} < 0$, $j = 1,2,\ldots 2k$.

Using (2.32) in (2.36) we can factorize $A^{-1}(\xi,\xi')$:

$$A^{-1}(\xi,\xi') = (1-\xi\xi')^{-1} \begin{bmatrix} 0 & -P(\xi) \\ Q(\xi) & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & Q(\xi') \\ -P(\xi') & 0 \end{bmatrix} \qquad (2.40)$$

Using this in (2.39) we can group together (by using the cyclic property of the trace) matrices that depend upon the same $\phi_{2j}$-variable. Performing the matrix multiplications we have

$$F_2^{(2k)} = \frac{1}{2k}(2\pi)^{-2k}\int_{-\pi}^{\pi} d\phi_2 d\phi_4 \ldots d\phi_{4k} \prod_{j=1}^{2k}\left(1-e^{-i\phi_{2j}-i\phi_{2j+2}}\right)^{-1} \times$$

$$\times \mathrm{Tr}\left\{\begin{bmatrix} Q^2(\bar{2})V(2) & U(2) \\ -U(2) & P^2(\bar{2})T(2)\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} Q^2(\bar{4})V(4) & -U(4) \\ U(4) & P^2(\bar{4})T(4)\end{bmatrix} \times\right.$$

$$\times\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} Q^2(\bar{6})V(6) & U(6) \\ -U(6) & P^2(\bar{6})T(6)\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix} \ldots \times$$

$$\left.\times\begin{bmatrix} Q^2(\overline{4k})V(4k) & -U(4k) \\ U(4k) & P^2(\overline{4k})T(4k)\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\right\} \qquad (2.41)$$

In reference *5* the following two identities were proved:

$$[P(\bar{2})]^2 T(2) =$$

$$(2\pi)^{-1}\int_{-\pi}^{\pi} d\phi_1 \frac{e^{-iM\phi_1-iN\phi_2}}{\Delta(\phi_1,\phi_2)}\left[-z_1^{-1}\gamma_1 e^{-i\phi_2}(1-\cos\phi_1)\right]$$

$$(2.42)$$

and

$$[Q(\bar{2})]^2 V(2) = (2\pi)^{-1}\int_{-\pi}^{\pi} d\phi_1 \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1,\phi_2)}\left[-z_1\gamma_1 e^{-i\phi_2}(1+\cos\phi_1)\right] \tag{2.43}$$

Using these identities and the definition of $U(e^{i\phi})$ we have

$$\begin{bmatrix} Q^2(\bar{2})V(2) & U(2) \\ -U(2) & P^2(\bar{2})T(2)\end{bmatrix} = (2\pi)^{-1}\int_{-\pi}^{\pi} d\phi_1 \frac{e^{-iM\phi_1 - iN\phi_2}}{\Delta(\phi_1,\phi_2)}\gamma_1 e^{-i\phi_2} \times \begin{bmatrix} -z_1(1+\cos\phi_1) & -i\sin\phi_1 \\ i\sin\phi_1 & -z_1^{-1}(1-\cos\phi_1)\end{bmatrix} \tag{2.44}$$

Note that the determinant of the matrix appearing on the RHS of (2.44) is zero. We write

$$\begin{bmatrix} -z_1(1+\cos\phi_1) & -i\sin\phi_1 \\ i\sin\phi_1 & -z_1^{-1}(1-\cos\phi_1)\end{bmatrix} = 2\begin{bmatrix} -z_1\cos^2\tfrac{1}{2}\phi_1 & -i\sin\tfrac{1}{2}\phi_1\cos\tfrac{1}{2}\phi_1 \\ i\sin\tfrac{1}{2}\phi_1\cos\tfrac{1}{2}\phi_1 & -z_1^{-1}\sin^2\tfrac{1}{2}\phi_1\end{bmatrix}$$

$$\begin{bmatrix} z_1^{\frac{1}{2}}\cos\tfrac{1}{2}\phi_1 \\ -iz_1^{-\frac{1}{2}}\sin\tfrac{1}{2}\phi_1\end{bmatrix}\begin{bmatrix} -z_1^{\frac{1}{2}}\cos\tfrac{1}{2}\phi_1 & -iz_1^{-\frac{1}{2}}\sin\tfrac{1}{2}\phi_1\end{bmatrix} \tag{2.45}$$

With (2.45) all the matrix products in (2.41) can be written as scalar products, i.e.

$$\begin{bmatrix} -z_1^{\frac{1}{2}} \cos\tfrac{1}{2}\phi_1 & -iz_1^{-\frac{1}{2}}\sin\tfrac{1}{2}\phi_1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1^{\frac{1}{2}} & \cos\tfrac{1}{2}\phi_3 \\ z_1^{-\frac{1}{2}} & i \sin\tfrac{1}{2}\phi_3 \end{bmatrix}$$

$$= -i \sin\tfrac{1}{2}(\phi_1 - \phi_3) \qquad (2.46)$$

and

$$\begin{bmatrix} -z_1^{\frac{1}{2}} \cos\tfrac{1}{2}\phi_3 & iz_1^{-\frac{1}{2}}\sin\tfrac{1}{2}\phi_3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1^{\frac{1}{2}} & \cos\tfrac{1}{2}\phi_5 \\ -i \sin\tfrac{1}{2}\phi_5 & z_1^{-\frac{1}{2}} \end{bmatrix}$$

$$= +i \sin\tfrac{1}{2}(\phi_3 - \phi_5) \qquad (2.47)$$

Thus

$$F_2^{(2k)} = \frac{(-1)^k}{2k} \gamma_1^k (2\pi)^{-4k} \int_{-\pi}^{\pi} d\phi_1 \ldots d\phi_{4k} \prod_{j=1}^{2k}$$

$$\left[\frac{\exp(-iM\phi_{2j-1} - iN\phi_{2j})}{\Delta(\phi_{2j-1}, \phi_{2j})} \frac{\sin\tfrac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{\sin\tfrac{1}{2}(\phi_{2j} + \phi_{2j+1})}\right] \qquad (2.48)$$

This in conjunction with (2.29), (2.25) and (2.24) gives the desired result.

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16. See also Chapters IV and VIII of reference 11.

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19. See also, Chapter X of reference 11 and references contained therein.

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21. A good discussion of Wiener-Hopf operators can be found in M. G. Krein, Am. Math. Soc. Transl., 22, 163 (1962). See also reference 11, Chapter IX.

22. If the reader is concerned about the steps in (2.22) because the matrices are infinite, then one can draw the line through the points (0,0), (0,-1), ..., (0,$N$) and the points (M,N), (M,N+1), ..., (M,N+$N$). One is then considering finite matrices and the steps in (2.22) are surely correct. The limit $N \to \infty$ in then performed in the final step in (2.22) and in (2.23). This limit is well-defined and the result is as given in (2.22). The reader is referred to reference 3 for further discussion of these points.