CONNECTION BETWEEN THE KdV EQUATION AND THE TWO-DIMENSIONAL ISING MODEL *

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We construct the one-parameter family of solutions to $d^2w/dz^2 = zw + 2w^3$ that tend to zero for $z \to +\infty$ by specializing an equation previously solved in connection with the two-dimensional Ising model. These solutions are intimately related to the KdV equation.

Among the various profound but solvable models of physics, two of the most studied are the Korteweg–de Vries (KdV) equation for nonlinear wave phenomenon [1] and the two-dimensional Ising model [2] for statistical mechanics. It has been known [3] for several years that a Painlevé transcendent [4] of the third kind plays an important role in the two-dimensional Ising model. Recently Ablowitz and Segur [5] have shown there exists an important relation between the long time asymptotic solution of the modified KdV equation (and hence indirectly the KdV equation) and a certain Painlevé transcendent of the second kind. It is the purpose to show here that this transcendent of the second kind may be derived from the previously studied transcendent of the third kind. Thus there is an unexpected connection between these two models.

The most general Painlevé equation of the second kind is [4]

$$d^2w/dz^2 = zw + 2w^3 + \alpha' \tag{1}$$

where α' is a constant (in ref. [5] $\alpha' = 0$ is the relevant case). As was shown by Painlevé and is discussed in the book by Ince, the second Painlevé equation can be obtained as a limiting case of the third Painlevé equation [4]

$$\frac{d^2v}{dy^2} = \frac{1}{v}\left(\frac{dv}{dy}\right)^2 - \frac{1}{y}\frac{dv}{dy} + \frac{1}{y}(\alpha v^2 + \beta) + \gamma v^3 + \frac{\delta}{v} \tag{2}$$

where y is the independent variable and α, β, γ, and δ are constants. In fact, if one sets

$$\alpha = -\frac{1}{2\epsilon^6}, \quad \beta = \frac{1}{2\epsilon^6} + \frac{2\alpha'}{\epsilon^3}, \quad \gamma = \frac{1}{4\epsilon^6}, \quad \delta = -\frac{1}{4\epsilon^6},$$
$$y = 1 + \epsilon^2 z, \quad \text{and } v = 1 + 2\epsilon w \tag{3}$$

in (2), then (1) is obtained in the limit $\epsilon \to 0$.

The third Painlevé equation was first introduced into the physics literature by Myers [6], where a particular case of (2) arose in the investigation of scattering from a finite strip. Recently the present authors [7] explicitly constructed and analyzed the one-parameter family of Painlevé transcendents of the third kind that remain bounded as $y \to +\infty$ when the constants α, β, γ, and δ in (2) satisfy the restriction

$$\alpha\sqrt{-\delta} + \beta\sqrt{\gamma} = 0\,. \tag{4}$$

This restriction is satisfied by the choice of α, β, γ, and δ in (3) if we set $\alpha' = 0$, i.e. the $\alpha' = 0$ case of (1). Hence it follows that the one-parameter family of Painlevé transcendents of the second kind satisfying (1) with $\alpha' = 0$ and tending to zero as $z \to +\infty$ can be obtained as a special case of the Painlevé transcendents of ref. [7]. It is precisely this one-parameter family,

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which we denote by $w(z;r)$, that is needed in the work of Ablowitz and Segur [5].

The results of performing the limit (3) on the solutions of ref. [7] is that $w(z;r)$ is given by

$$w(z;r) = \sum_{n=0}^{\infty} r^{2n+1} w_{2n+1}(z) \qquad (5)$$

where

$$w_1(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \mathrm{d}\zeta_1 \exp[\tfrac{1}{3}\zeta_1^3 - z\zeta_1] = \mathrm{Ai}(z) \qquad (6a)$$

and for $n \geqslant 1$

$$w_{2n+1}(z) = \int_{\mathcal{L}} \frac{\mathrm{d}\zeta_1}{2\pi \mathrm{i}} \cdots \frac{\mathrm{d}\zeta_{2n+1}}{2\pi \mathrm{i}}$$
$$\times \prod_{j=1}^{2n+1} \exp[\tfrac{1}{3}\zeta_j^3 - z\zeta_j] \prod_{j=1}^{2n} (\zeta_j + \zeta_{j+1})^{-1} \qquad (6b)$$

where $\mathcal{L}$ is any contour in the right half-plane that begins at a point at infinity in the sector $-\frac{1}{2}\pi < \arg \zeta \leqslant -\frac{1}{6}\pi$ and ends at infinity in the conjugate sector. From the work of Painlevé [4] we know $w(z;r)$ is a meromorphic function of z.

Ablowitz and Segur [8] by use of inverse scattering methods have also constructed the solution we denote here by $w(z;r)$. However, our representation is simpler because it involves integrals over elementary functions whereas the representation of Ablowitz and Segur involves integrals over Airy functions.

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