

TRANSVERSE TIME-DEPENDENT SPIN CORRELATION FUNCTIONS FOR THE ONE-DIMENSIONAL XY MODEL AT ZERO TEMPERATURE*

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We compute exactly the transverse time-dependent spin-spin correlation functions $\langle S_i^x(0)S_{i+1}^x(t) \rangle$ and $\langle S_i^y(0)S_{i+1}^y(t) \rangle$ at zero temperature for the one-dimensional XY model that is defined by the hamiltonian

$$H_N = - \sum_{i=1}^N [(1 + \gamma)S_i^x S_{i+1}^x + (1 - \gamma)S_i^y S_{i+1}^y + hS_i^z].$$

We then analyze these correlation functions in two scaling limits: (a) γ fixed, $h \rightarrow 1$, $R \rightarrow \infty$, $t \rightarrow \infty$ such that $|(h - 1)/\gamma| [R^2 - \gamma^2 t^2]^{1/2}$ is fixed, and (b) h fixed less than one, $\gamma \rightarrow 0^+$, $R \rightarrow \infty$, $t \rightarrow \infty$ such that $\gamma [R^2 - (1 - h^2)t^2]^{1/2}$ is fixed. In these scaling regions we give both a perturbation expansion representation of the various scaling functions and we express these scaling functions in terms of a certain Painlevé transcendent of the third kind. From these representations we study both the small and large scaling variable limits in both the space-like and time-like regions.

1. Introduction and discussion of results

Dynamical properties of many-particle systems are often studied in terms of time-dependent correlation functions $\langle A(r_1, t_1)B(r_2, t_2) \rangle$.¹⁾ In general the structure of $\langle A(r_1, t_1)B(r_2, t_2) \rangle$ is complicated and depends upon the system under consideration. Because of this complexity many phenomenological approaches to time-dependent correlation functions have been given²⁾. It is therefore important to be able to study these time-dependent functions in models that are exactly solvable. The one-dimensional spin one-half XY model is such a model. We refer the reader to the two excellent reviews by deJongh and Miedema³⁾ and by Steiner et al.⁴⁾ that discuss one-dimensional spin systems. Of the various one-dimensional quantum spin systems the XY model is no doubt the simplest model to analyze.

The one-dimensional XY model is defined by the hamiltonian

$$H_N = - \sum_{i=1}^N [(1 + \gamma)S_i^x S_{i+1}^x + (1 - \gamma)S_i^y S_{i+1}^y + hS_i^z], \quad (1.1)$$

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where $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$, $\alpha = x, y, z$, and σ_i^α are the usual Pauli matrices, γ is the anisotropy parameter which we take to be non-negative, and h is the applied magnetic field in the z direction. We impose cyclic boundary conditions, i.e. $S_{N+1} \equiv S_1$ in eq. (1.1).

The antiferromagnetic ground state of (1.1) was computed exactly by Lieb et al.⁴⁾ and by Katsura⁵⁾. Since (1.1) is a quantum mechanical system, the ground state is nontrivial (it is BCS-like) and hence the behavior of both the static and time-dependent correlation functions at zero temperature can be expected to be nontrivial. In particular, McCoy⁶⁾ and later Barouch and McCoy⁷⁾ showed that for $\gamma > 0$ and $h < 1$ there exists spontaneous magnetization in the x -direction which goes continuously to zero as $h \rightarrow 1^-$ (with a $\beta = \frac{1}{8}$) for fixed $\gamma > 0$ and goes continuously to zero as $\gamma \rightarrow 0^+$ (with a $\beta = \frac{1}{4}$) for fixed $h < 1$.

A study of the correlation functions

$$\rho_{xx}(R, t) = \langle S_1^x(0) S_{R+1}^x(t) \rangle. \quad (1.2)$$

and

$$\rho_{yy}(R, t) = \langle S_1^y(0) S_{R+1}^y(t) \rangle \quad (1.3)$$

(here the brackets denote the ground state average) was begun (for $t = 0$ and $h = 0$) by McCoy⁶⁾ and later extended to $h \neq 0$ by Barouch and McCoy⁷⁾. The extension to the $t \neq 0$ case was begun by McCoy et al.⁸⁾. All of these authors study the correlation functions for large R and t and compute the leading term in the asymptotic expansion of these correlation functions. At infinite temperature some interesting results for $\rho_{xx}(R, t)$ have been given by Sur et al.⁹⁾ by Brandt and Jacobi¹⁰⁾, and by Capel and Perk¹¹⁾.

The techniques used in refs. 6–8 are techniques first used in studying the asymptotic behavior of the spin-spin correlation function of the two-dimensional Ising model^{12,13)}. This similarity of the two models was made precise by Suzuki¹⁴⁾ who proved the sum rule

$$\langle \sigma_{nm} \sigma_{nm} \rangle_{\text{IS}} = \cosh^2 K_1^* \langle \sigma_m^x(0) \sigma_n^x(0) \rangle_{XY} - \sinh^2 K_1^* \langle \sigma_m^y(0) \sigma_n^y(0) \rangle_{XY} \quad (1.4)$$

relating Ising model correlation functions [the LHS of (1.4)] to XY model correlation functions [the RHS of (1.4)] where we identify

$$\tanh 2K_1 = (1 - \gamma^2)^{1/2} h^{-1}, \quad \cosh 2K_2^* = \gamma^{-1},$$

with $\tanh K_j^* = \exp(-2K_j)$, $K_j = E_j/k_B T$, and E_j are the usual Ising model interaction energy constants. The relation (1.4) is valid only for

$$h^2 + \gamma^2 \geq 1 \quad (1.5)$$

(see fig. 1).

Krinsky¹⁵⁾ has studied the generalization of (1.4) to the triangular Ising lattice. Given the two-point scaling functions of the two-dimensional Ising model^{16–20)}, it follows from (1.4) and the fact $\langle \sigma_m^y(0) \sigma_m^y(0) \rangle_{XY} = 0 (|m - m'|^{-9/4})$ at

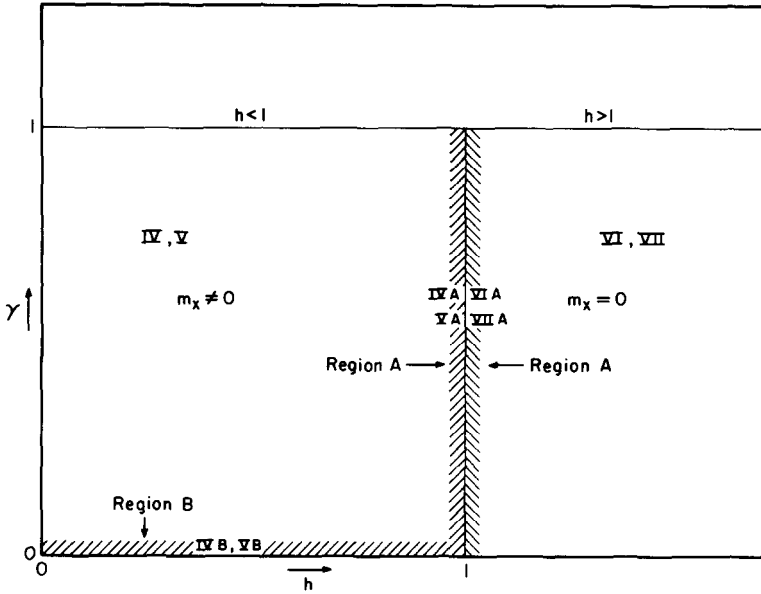


Fig. 1. Schematic representation of the model in (γ, h) space. Labels refer to the appropriate section numbers of the paper. The spontaneous magnetization in the x direction $m_x \rightarrow 0$ as $h \rightarrow 1^-$ and $m_x = 0$ for $\gamma = 0$. This shaded parts represent the two scaling regions A and B.

$h = 1$ that the static scaling functions for $\langle \sigma_m^x(0) \sigma_m^x(0) \rangle_{XY}$ in region A (see fig. 1) are essentially the Ising model scaling functions.

In this paper we extend the work of MBA and compute the zero temperature correlation functions (1.2) and (1.3) for all R and t . Much of this work derives heavily from the perturbation expansion developed by Cheng and Wu¹³⁾ and further generalized in the work of Wu et al.¹⁷⁾ [see also McCoy et al.¹⁸⁾]. Furthermore, using the Painlevé function results of McCoy et al.¹⁹⁾ (which is equivalent to a certain solution to the hyperbolic sine-Gordon equation) we express the various scaling functions in terms of these new transcendental functions (this makes a study of the short distance behavior straightforward). Fig. 1 shows schematically the regions in (γ, h) space studied with the labels referring to the appropriate sections in the paper.

Of particular interest are the scaling regions A and B. The scaling region A is defined for fixed $\gamma > 0$ by

$$h \rightarrow 1^\pm; \quad R \rightarrow \infty; \quad t \rightarrow \infty, \quad (1.6a)$$

such that the scaling variable,

$$s = \left| \frac{h-1}{\gamma} \right| (R^2 - \gamma^2 t^2)^{1/2}, \quad (1.6b)$$

is fixed. We note that s can be either real ($R > \gamma t$, space-like region) or purely imaginary ($R < \gamma t$, time-like region). In this scaling limit the correlation

functions $\rho_{xx}(R, t)$ and $\rho_{yy}(R, t)$ can be written in the scaling form

$$\rho_{xx}(R, t) = \rho_{xx}(\infty) \hat{F}_{\pm}^x(s) + o([\gamma^2 |1 - h^2|]^{1/4}) \quad (1.7)$$

and

$$\rho_{yy}(R, t) = \rho_{yy}(\infty) \hat{F}_{\pm}^y(s) + o\left(\left|\frac{h-1}{\gamma}\right|^{9/4}\right), \quad (1.8)$$

where

$$\rho_{xx}(\infty) = [2(1 + \gamma)]^{-1} [\gamma^2 |1 - h^2|]^{1/4} \quad (1.9a)$$

and

$$\rho_{yy}(\infty) = (1 + \gamma) 2^{-11/4} \gamma^{-5/4} \left[\left| \frac{h-1}{\gamma} \right| \right]^{9/4}, \quad (1.9b)$$

and the $+$ ($-$) sign in (1.7) and (1.8) refers to $h \rightarrow 1^+$ ($h \rightarrow 1^-$).

Figs. 2 and 3 show the dependence of the scaling functions $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$

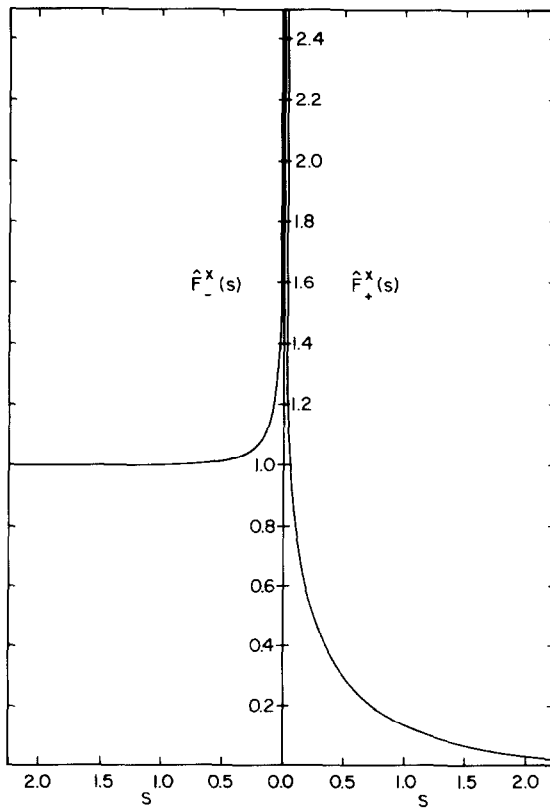


Fig. 2. Scaling functions $\hat{F}_{\pm}^x(s)$ as functions of the scaling variable s (space-like region).

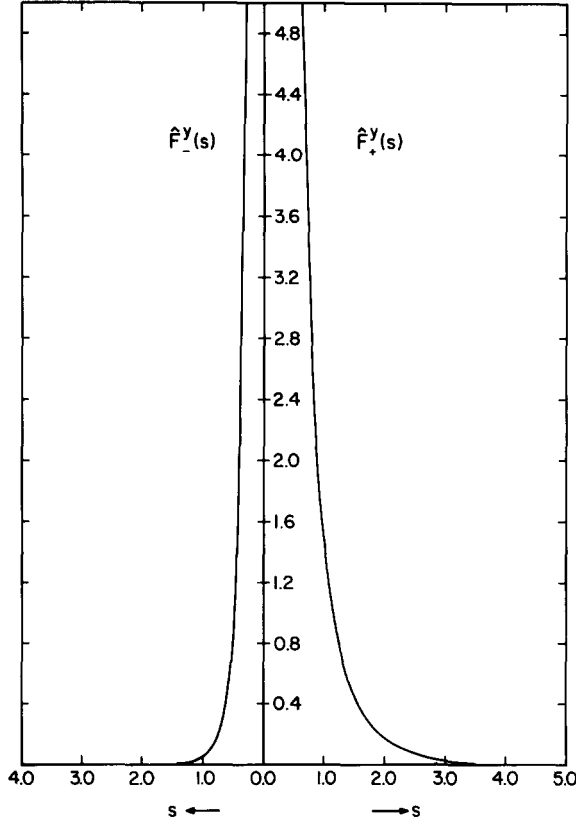


Fig. 3. Scaling functions $\hat{F}_{\pm}^y(s)$ as functions of the scaling variable s (space-like region).

on the scaling variable s in the space-like region. The scaling function $\hat{F}_{\pm}^x(s)$ is essentially the two-point scaling function of the two-dimensional Ising model¹⁶⁻¹⁹). For small s (s real) the behavior of $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$ is

$$\hat{F}_{\pm}^x(s) = C_x s^{-1/4} \{1 \pm \frac{1}{2}s\Omega + \frac{1}{16}s^2 \pm \frac{1}{32}s^2\Omega + \mathcal{O}(s^4\Omega^2)\} \quad (1.10)$$

and

$$\hat{F}_{\pm}^y(s) = C_y s^{-9/4} \{1 \mp \frac{1}{2}s(3\Omega + 4) - \frac{7}{16}s^2 \mp \frac{1}{32}s^3(11\Omega + 4) + \mathcal{O}(s^4\Omega^3)\}, \quad (1.11)$$

respectively, and where

$$\Omega = \ln(s/8) + \gamma_E, \quad (1.12a)$$

and

$$C_x = 2^{-1/6} e^{1/4} A^{-3}, \quad (1.12b)$$

$$C_y = \frac{1}{4} C_x, \quad (1.12c)$$

with $A = 1.282427 \dots$ being Glaisher's constant¹⁷), and $\gamma_E = 0.577 \dots$ being Euler's constant.

Figs. 4 and 5 show the dependence of $\hat{F}_{\pm}^x(-i\tau)$ and $\hat{F}_{\pm}^y(-i\tau)$ on τ ($s = -i\tau$) in the time-like region. For small τ (τ real) the behavior of $\hat{F}_{\pm}^x(-i\tau)$ and $\hat{F}_{\pm}^y(-i\tau)$ is

$$\hat{F}_{\pm}^x(-i\tau) = C'_x \tau^{-1/4} \left[1 \mp \frac{i\tau}{2} \Omega_1 - \frac{1}{16} \tau^2 \mp \frac{i}{32} \tau^3 \Omega_1 + \mathcal{O}(\tau^4 \Omega_1^3) \right] \quad (1.13)$$

and

$$\hat{F}_{\pm}^y(-i\tau) = C'_y \tau^{-9/4} \left[1 \pm \frac{i\tau}{2} (3\Omega_1 + 4) + \frac{7}{16} \tau^2 \pm \frac{i}{32} \tau^3 (11\Omega_1 + 4) + \mathcal{O}(\tau^4 \Omega_1^3) \right], \quad (1.14)$$

where

$$\Omega_1 = \ln(\tau/8) + \gamma_E - i\pi/2, \quad (1.15a)$$

$$C'_x = e^{i\pi/8} C_x, \quad (1.15b)$$

$$C'_y = e^{i\pi/8} C_y. \quad (1.15c)$$

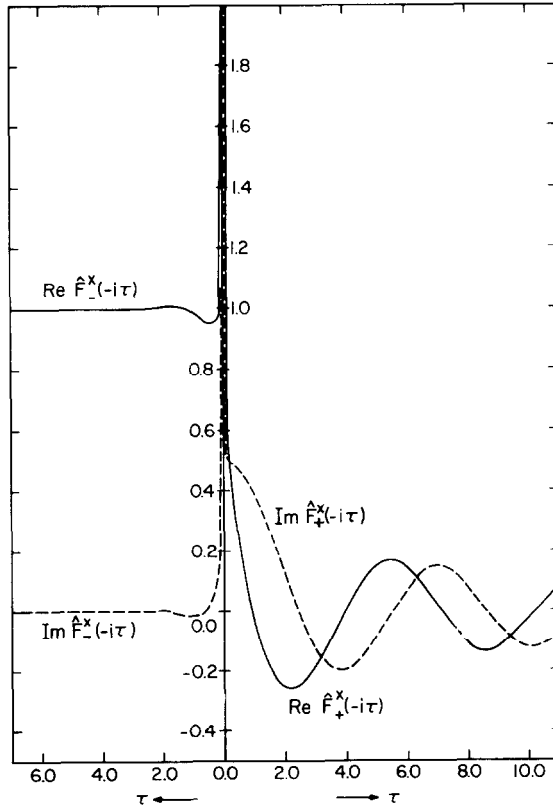


Fig. 4. scaling functions $\hat{F}_{\pm}^x(s)$ on the imaginary axis ($s = -i\tau$) as functions of τ (time-like region).

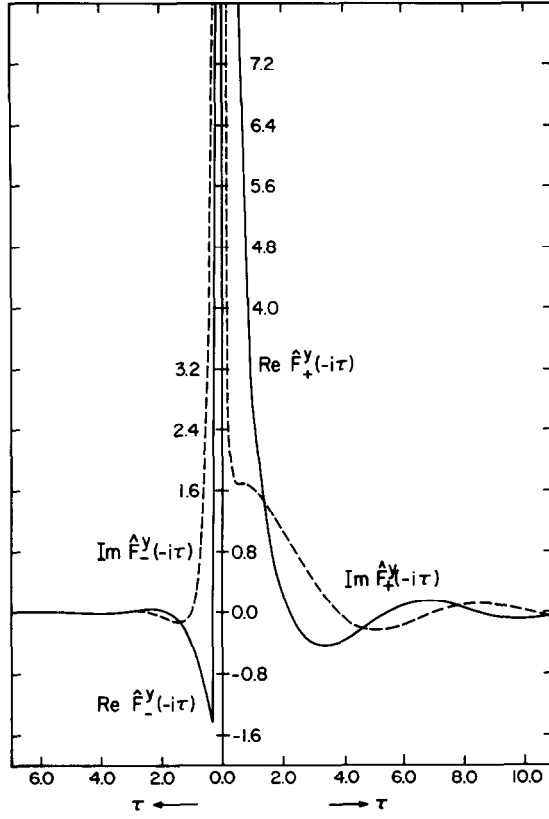


Fig. 5. Scaling functions $\hat{F}_{\pm}^y(s)$ on the imaginary axis ($s = -i\tau$) as functions of τ (time-like region).

Thus, in the space-like region the scaling functions $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$ are real and decreasing functions, whereas in the time-like region the functions are complex and both the real and imaginary parts show oscillatory behavior. This is consistent with the picture of a spin-wave traveling down the chain at a velocity γ such that at a time $t = R/\gamma$, when the spin-wave reaches the point R , the oscillations begin.

The xx scaling functions can be expressed as

$$\hat{F}_{+}^x(s) = \sum_{k=0}^{\infty} g_{2k+1}(s) \exp\left(-\sum_{n=1}^{\infty} f^{(2n)}(s)\right) \quad (1.16)$$

and

$$\hat{F}_{-}^x(s) = \exp\left(-\sum_{n=1}^{\infty} f^{(2n)}(s)\right), \quad (1.17)$$

where the functions $g_{2k+1}(s)$ and $f^{(2n)}(s)$ are given explicitly in (2.15) and (2.16), respectively. From these expressions for $g_{2k+1}(s)$ and $f^{(2n)}(s)$ we find that $\hat{F}_{+}^x(s)$

has only odd-number spin-wave excitations and $\hat{F}_+^x(s)$ has only even-number spin-wave excitations. For $h > 1$ the lowest lying excitation is the single spin-wave excitation and its contribution to $\hat{F}_+^x(s)$ is $g_1(s)$ in (1.16). That is to say, the one-spin wave approximation to $\hat{F}_+^x(s)$ is

$$\hat{F}_+^x(s) \approx g_1(s), \quad (1.18)$$

where

$$g_1(s) = \frac{1}{\pi} K_0(s) \quad (\text{space-like region}) \quad (1.19a)$$

and

$$g_1(-i\tau) = \frac{i}{2} H_0^{(1)}(\tau) \quad (\text{time-like region}) \quad (1.19b)$$

in which $K_0(s)$ and $H_0^{(1)}(\tau)$ are Bessel functions²¹). The one-spin wave approximation to $\hat{F}_+^x(s)$ is good to 1% for all $s \geq 0.1$ ($\tau \geq 0.1$) in the space-like region (time-like region). The high order spin-wave excitations (3, 5, 7, ...) contribute significantly to $\hat{F}_+^x(s)$ only in the region $s \leq 0.1$. In particular, the location of the zeros of $\hat{F}_+^x(-i\tau)$ (see fig. 4) are given to a good approximation by the one-spin wave approximation. On the other hand, the correct short-distance behavior of $\hat{F}_+^x(s)$ in both the space-like and time-like regions [eqs. (1.10) and (1.13) above] results only when *all spin-wave excitations are kept* in (1.16).

Concerning the scaling function $\hat{F}_\pm^y(s)$ in region A (see figs. 3 and 5) we make the following remarks: (1) $\hat{F}_\pm^y(s)$ approaches zero in the space-like (time-like) region in a monotonic (oscillatory) manner as $s \rightarrow \infty$ ($\tau \rightarrow \infty$), is just the statement that there is no long-range order in the y -direction; (2) if the spins were classical, then $\rho_{yy}(R, t) \equiv 0$; hence the nonzero value of $\rho_{yy}(R, t)$ represents a quantum effect; (3) the lowest lying spin-wave excitation again dominates the scaling function for $s(\tau) \geq 0.8$ within 1% accuracy; and (4) if one plots the function $s^{9/4} \hat{F}_\pm^y(s)$ for space-like s , then the maximum value of $s^{9/4} \hat{F}_\pm^y(s)$ occurs at $s = 0.452 \dots$, which is a 64% increase over its value at $s = 0$.

The scaling region B is described by

$$h < 1; \quad \gamma \rightarrow 0; \quad R \rightarrow \infty; \quad t \rightarrow \infty \quad (1.20a)$$

such that

$$s_1 = \gamma(R^2 - (1 - h^2)t^2)^{1/2} \quad (1.20b)$$

is fixed. In this region (see fig. 1) the correlation functions take the scaling form

$$\rho_{xx}(R, t) \sim \rho_{xx}(\infty) [\hat{F}_-^x(s_1)]^2 \quad (1.21)$$

and

$$\rho_{yy}(R, t) \sim \rho_{yy}(\infty) [\hat{F}_+^x(s_1)]^2. \quad (1.22)$$

Figs. 6 and 7 show the dependence of these scaling functions on the scaling variable $s_1(-i\tau_1)$ in the space-like (time-like) region. When $s_1 = 0$ (equivalently $\gamma = 0$) in (1.21) and (1.22), the leading order behavior of $\rho_{xx}(R, t)$ and $\rho_{yy}(R, t)$ reduces to

$$\rho_{xx}(R, t) = \rho_{yy}(R, t) \sim \frac{1}{4} e^{1/2} 2^{2/3} A^{-6} (1 - h^2)^{1/4} [R^2 - (1 - h^2)t^2]^{-1/4}, \quad (1.23)$$

where A is Glaisher's constant.

The time-dependent correlation functions $\rho_{xx}(R, t)$ and $\rho_{yy}(R, t)$ can be Fourier transformed in R and t , i.e.

$$\tilde{\rho}_{xx}(k, \omega) = \sum_{R=-\infty}^{\infty} e^{ikR} \int_{-\infty}^{\infty} dt e^{i\omega t} [\rho_{xx}(R, t) - \rho_{xx}(\infty)], \quad (1.24)$$

and the resulting transformed functions can be analyzed as functions of k and ω . In the scaling region A of fig. 1 these transformed scaling functions take

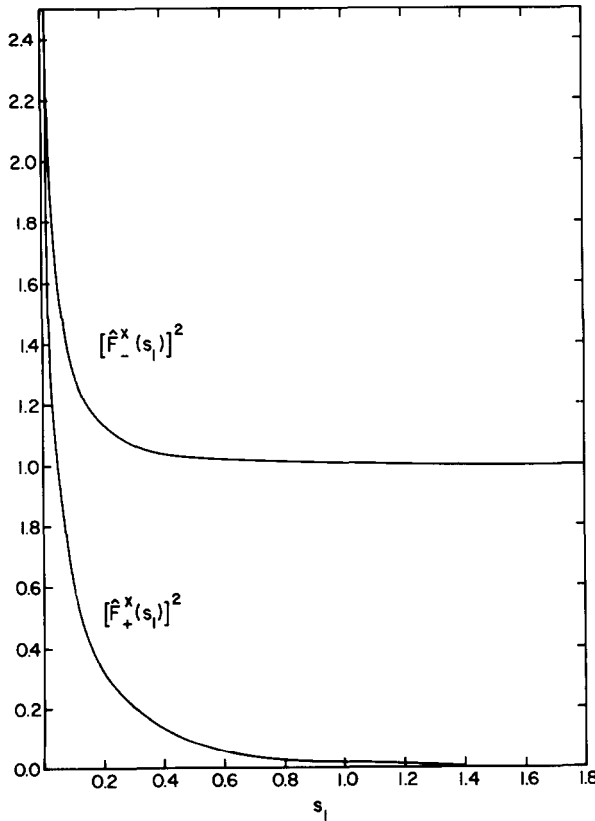


Fig. 6. Scaling functions $[\hat{F}_-^x(s_1)]^2$ and $[\hat{F}_+^x(s_1)]^2$ as functions of s_1 (space-like region).

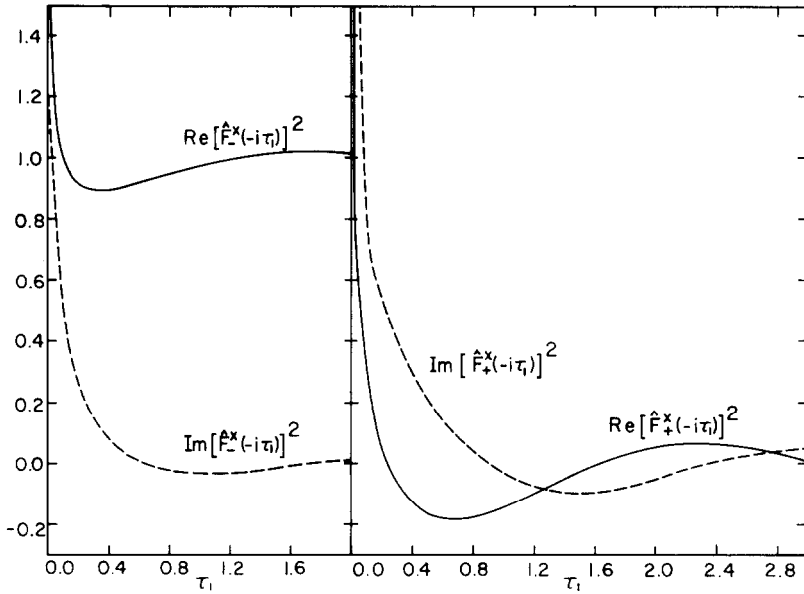


Fig. 7. Scaling functions $[\hat{F}_-^x(s_1)]^2$ and $[\hat{F}_+^x(s_1)]^2$ on the imaginary axis ($s_1 = -i\tau_1$) as functions of τ_1 (time-like region).

the scaling form

$$\tilde{\rho}_{xx}(k, \omega) = \rho_{\pm}^x(\epsilon), \quad (1.25)$$

where

$$\epsilon = |h - 1|^{-1}(\omega^2 - \gamma^2 k^2)^{1/2} \quad (1.26)$$

and $\rho_{\pm}^x(\epsilon)$ are the spectral weight functions in the Källén-Lehman representation of the scaling two-point functions $G_{\pm}^{(2)}(p^2)$ of the two-dimensional Ising model (see Appendix B). We decompose $\rho_{\pm}^x(\epsilon)$:

$$\rho_{+}^x(\epsilon) = \sum_{n=0}^{\infty} \rho_{2n+1}^x(\epsilon), \quad (1.27)$$

$$\rho_{-}^x(\epsilon) = \sum_{n=1}^{\infty} \rho_{2n}^x(\epsilon), \quad (1.28)$$

where $\rho_{2n+1}^x(\epsilon) \equiv 0$ ($\rho_{2n}^x(\epsilon) \equiv 0$) for $\epsilon < 2n + 1$ ($\epsilon < 2n$) and is nonzero and has no singularities for $\epsilon > 2n + 1$ ($\epsilon > 2n$). Representations similar to (1.27) and (1.28) hold for $\tilde{\rho}_{yy}(k, \omega)$ and for both $\tilde{\rho}_{xx}(k, \omega)$ and $\tilde{\rho}_{yy}(k, \omega)$ in region B.

In particular, we find

$$\rho_1^x(\epsilon) = \frac{4\pi\gamma^2}{(1+\gamma)(h-1)^2} \delta(\epsilon - 1) \quad (1.29)$$

and

$$\begin{aligned} \rho_2^x(\epsilon) &= 0, \quad \text{for } \epsilon < 2 \\ &= \frac{\gamma^2}{(1+\gamma)(h-1)^2 2\pi^2} \frac{(\epsilon^2 - 4)^{1/2}}{\epsilon^3}, \quad \text{for } \epsilon > 2. \end{aligned} \quad (1.30)$$

The δ -function behavior of $\rho_1^x(\epsilon)$ is simply the one-spin wave behavior in region A in $\hat{F}_+^x(s)$ and $\rho_2^x(\epsilon)$ shows the fact that the lowest lying excitation in region A in $\hat{F}_+^x(s)$ is the two-spin wave. Integral representations for higher $\rho_n^x(\epsilon)$ can be derived.

In particular we can express $\rho_3^x(\epsilon)$ as a single integral as follows:

$$\rho_3^x(\epsilon) = (4\pi^3)^{-1} \int_{x_-}^{x_+} \frac{dx}{x} \left[\frac{(x_+ - x)(x - x_-)}{x(a - x)} \right]^{1/2} \left[\frac{1}{\epsilon^2(a - x)} - \frac{2x}{(\epsilon^2 - 1)(1 - x)} \right], \quad (1.31)$$

where

$$x_{\pm} = \frac{3 + \epsilon^2 \pm [(\epsilon^2 - 9)(\epsilon^2 - 1)]^{1/2}}{2\epsilon^2} \quad (1.32)$$

and

$$a = (\epsilon^2 - 1)/\epsilon^2. \quad (1.33)$$

The integral (1.31) can be expressed in terms of complete elliptic integrals but we find (1.31) just as convenient. In fig. 8 we plot $\rho_2^x(\epsilon)$ and in fig. 9 we plot

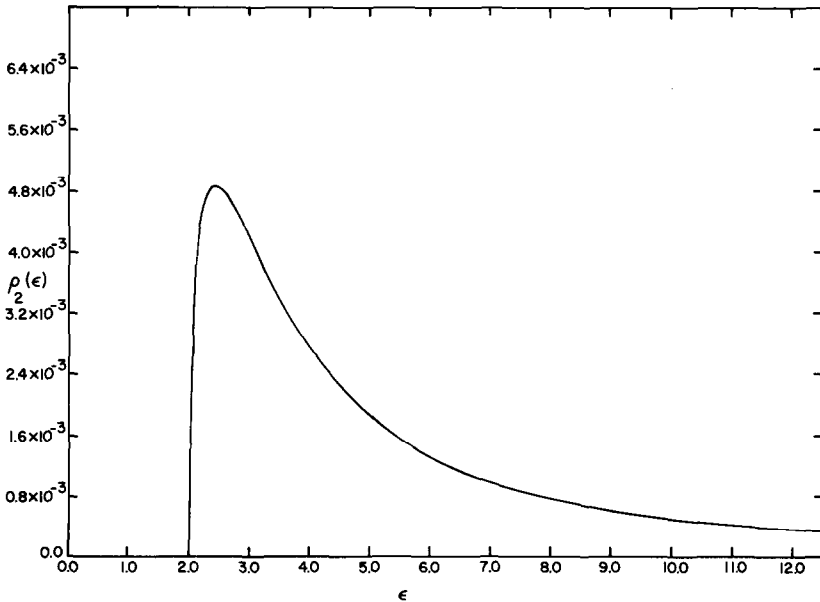


Fig. 8. Spectral function $\rho_2^x(\epsilon)$ as a function of ϵ .

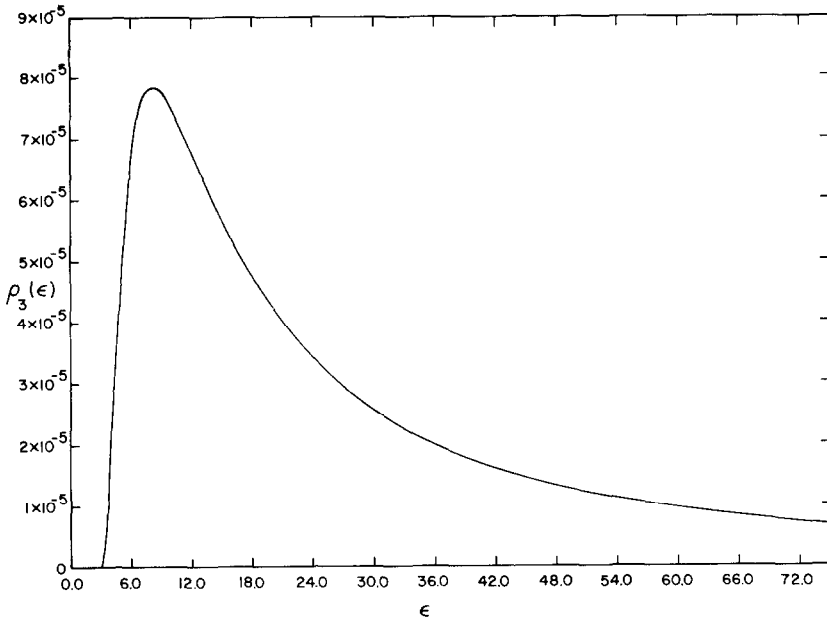


Fig. 9. Spectral function $\rho_3(\epsilon)$ as a function of ϵ .

$\rho_3^x(\epsilon)$. The slope of $\rho_2^x(\epsilon)$ at $\epsilon = 2$ is infinite whereas the slope of $\rho_3^x(\epsilon)$ at $\epsilon = 3$ is zero. It can be shown that for all $\rho_k^x(\epsilon)$, $k \geq 3$ that the slope at $\epsilon = k$ is zero.

In section 2 we collect together our principal results so that they are accessible for easy reference. In sections II to VII we elaborate in some detail on the method of computation. In particular we follow the method of Cheng and Wu¹³) as recently simplified by McCoy et al.¹⁸).

2. Summary of results

2.1. Exact results for $h < 1$ and all R and t

When $h < 1$, one has

$$\rho_{xx}^<(R, t) = \rho_{xx}(\infty) \exp \left[- \sum_{n=1}^{\infty} F_{<}^{(2n)}(R, t) \right], \quad (2.1)$$

where $\rho_{xx}(\infty)$ is given by (1.9a) and for $n = 1, 2, \dots$

$$F_{<}^{(2n)}(R, t) = (2n)^{-1} 2^{-2n} (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n} \prod_{j=1}^{2n} \left[\frac{e^{-iR\phi_j - it\Lambda_j} (\Lambda_j - \Lambda_{j+1})}{\Lambda_j \sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right], \quad (2.2)$$

with $\phi_{2n+1} \equiv \phi_1$; $\text{Im } \phi_j < 0$, $j = 1, 2, \dots, 2n$,

$$\Lambda_j = \Lambda(\phi_j) = [(\cos \phi_j - h)^2 + \gamma^2 \sin^2 \phi_j]^{1/2} \\ = \left(\frac{1+\gamma}{2} \right) [(1 - \lambda_1^{-1} e^{i\phi_j})(1 - \lambda_2^{-1} e^{i\phi_j})(1 - \lambda_1^{-1} e^{-i\phi_j})(1 - \lambda_2^{-1} e^{-i\phi_j})]^{1/2}, \quad (2.3)$$

$$\lambda_1 = \frac{h + [h^2 - (1 - \gamma^2)]^{1/2}}{1 - \gamma} \quad \text{and} \quad \lambda_2 = \frac{h - [h^2 - (1 - \gamma^2)]^{1/2}}{1 - \gamma}. \quad (2.4)$$

Also, when $h < 1$ one has for $\rho_{yy}(R, t)$ the exact expression

$$\rho_{yy}^<(R, t) = -\rho_{xx}^<(R - 2, t) \left(\frac{1+\gamma}{2} \right)^2 \det \begin{vmatrix} X_{4,0}^3 & X_{4,1}^3 - X_{2,1}^3 \\ X_{4,1}^3 + X_{2,1}^3 & X_{4,1}^4 \end{vmatrix}, \quad (2.5)$$

where

$$X_{4,0}^3 = \sum_{n=0}^{\infty} 2^{-2n} (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] \\ \times \prod_{j=1}^{2n} \left[\frac{(\Lambda_j - \Lambda_{j+1}) e^{(i/2)(\phi_j + \phi_{j+1})}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right], \quad (2.6a)$$

$$X_{4,1}^4 = \sum_{n=0}^{\infty} 2^{-2n} (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] \\ \times \prod_{j=1}^{2n} \left[\frac{(\Lambda_j - \Lambda_{j+1}) e^{(i/2)(\phi_j + \phi_{j+1})}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] e^{-i(\phi_1 + \phi_{2n+1})}, \quad (2.6b)$$

$$X_{4,1}^3 = \frac{1}{2} \sum_{n=0}^{\infty} 2^{-2n} (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] \\ \times \prod_{j=1}^{2n} \left[\frac{(\Lambda_j - \Lambda_{j+1}) e^{(i/2)(\phi_j + \phi_{j+1})}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] (e^{-i\phi_1} + e^{-i\phi_{2n+1}}), \quad (2.6c)$$

where for $n = 0$ the second product symbol in the above formulae is to be interpreted as equal to one; and finally we have

$$X_{2,1}^3 = \sum_{n=1}^{\infty} 2^{-2n+1} (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n} \prod_{j=1}^{2n} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] \\ \times \prod_{j=1}^{2n-1} \left[\frac{(\Lambda_j - \Lambda_{j+1}) e^{(i/2)(\phi_j + \phi_{j+1})}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] e^{-(i/2)(\phi_1 + \phi_{2n})} \sin \frac{1}{2}(\phi_{2n} - \phi_1). \quad (2.6d)$$

2.2. Exact results for $h > 1$ and all R and t

When $h > 1$ we have

$$\rho_{xx}^>(R, t) = \bar{\rho}_{xx}(\infty) \left[\sum_{k=1}^{\infty} x_{2k-1}(R, t) \right] \left[\exp \left(- \sum_{n=1}^{\infty} F_{>}^{(2n)}(R, t) \right) \right] \quad (2.7)$$

and

$$\rho_{yy}(R, t) = \bar{\rho}_{xx}(\infty) \left[\sum_{k=1}^{\infty} y_{2k-1}(R, t) \right] \left[\exp \left(- \sum_{n=1}^{\infty} F^{(2n)}(R, t) \right) \right], \quad (2.8)$$

where

$$\bar{\rho}_{xx}(\infty) = \frac{1}{4} [(1 - \lambda_2^2)(1 - \lambda_1^{-2})(1 - \lambda_1^{-1}\lambda_2)^{-2}]^{1/4}, \quad (2.9)$$

$$\begin{aligned} x_{2k-1}(R, t) &= 2^{-2(k-1)} (2\pi)^{-(2k-1)} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2k-1} \prod_{j=1}^{2k-1} \left[\frac{e^{-iR\phi_j - it\Lambda_j}}{\Lambda_j} \right] \\ &\times \prod_{j=1}^{2(k-1)} \left[\frac{A_j - A_{j+1}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] \cos \frac{1}{2}(\phi_{2k-1} - \phi_1), \end{aligned} \quad (2.10)$$

$$F^{(2n)}(R, t) = (2n)^{-1} 2^{-2n} (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n} \prod_{j=1}^{2n} \left[\frac{e^{-iR\phi_j - it\Lambda_j} (A_j - A_{j+1})}{A_j \sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] \quad (2.11)$$

and

$$\begin{aligned} y_{2k-1}(R, t) &= 2^{-2(k-1)} (2\pi)^{-(2k-1)} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2k-1} \prod_{j=1}^{2k-1} \left[\frac{e^{-iR\phi_j - it\Lambda_j}}{A_j} \right] \\ &\times \prod_{j=1}^{2(k-1)} \left[\frac{A_j - A_{j+1}}{\sin \frac{1}{2}(\phi_j + \phi_{j+1})} \right] A_1 A_{2k-1} \cos \frac{1}{2}(\phi_{2k-1} - \phi_1), \end{aligned} \quad (2.12)$$

with

$$A_j = A(\phi_j) = \left[\frac{(1 - \lambda_2 e^{-i\phi_j})(1 - \lambda_2 e^{i\phi_j})}{(1 - \lambda_1^{-1} e^{-i\phi_j})(1 - \lambda_1^{-1} e^{i\phi_j})} \right]^{1/2}, \quad (2.13)$$

where for $k = 1$ the second product symbol in the above formulae is to be interpreted as equal to one.

We emphasize that the above formulae for $\rho_{xx}(R, t)$ and $\rho_{yy}(R, t)$ for both $h < 1$ and $h > 1$ are exact with no approximations having been made. We now present our results when we specialize the above expressions to the scaling regions A and B. Since the scaling functions in region B can be expressed in terms of region A scaling functions [see eqs. (1.21) and (1.22)], we give only the scaling functions in region A.

2.3. Correlation functions in scaling region A

The scaling region A as defined earlier [see (1.6) and fig. 1] leads to the scaling functions $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$. Here we give explicit formulae for these scaling functions. We denote the scaling limit in region A by \lim_A .

2.3.1. Perturbation expansion representation of $\rho_{xx}(R, t)$ in scaling region A

$$\lim_A \rho_{xx}^{-1}(\infty) \rho_{xx}(R, t) = \hat{F}_\pm^x(s) \quad (2.14a)$$

$$= \begin{cases} \exp\left[-\sum_{n=1}^{\infty} f^{(2n)}(s)\right] & (h \rightarrow 1^-), \\ G(s) \exp\left[-\sum_{n=1}^{\infty} f^{(2n)}(s)\right] & (h \rightarrow 1^+), \end{cases} \quad (2.14b)$$

$$(2.14c)$$

with

$$G(s) = \sum_{k=0}^{\infty} g_{2k+1}(s), \quad (2.14d)$$

and where $\rho_{xx}(\infty)$ is given by (1.9a) and the functions $f_{2n}(s)$ and $g_{2k+1}(s)$ have the following representations:

$$\begin{aligned} f^{(2n)}(s) &= (-1)^n n^{-1} \pi^{-2n} \int_1^{\infty} dy_1 \dots \int_1^{\infty} dy_{2n} \prod_{j=1}^{2n} \left[\frac{e^{-sy_j}}{(y_j^2 - 1)^{1/2} (y_j + y_{j+1})} \right] \\ &\times \prod_{j=1}^n (y_{2j}^2 - 1), \quad \text{for } n = 1, 2, \dots, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} g_{2k+1}(s) &= (-1)^k \pi^{-2k-1} \int_1^{\infty} dy_1 \dots \int_1^{\infty} dy_{2k+1} \prod_{j=1}^{2k+1} \left[\frac{e^{-sy_j}}{(y_j^2 - 1)^{1/2}} \right] \\ &\times \prod_{j=1}^{2k} (y_j + y_{j+1})^{-1} \prod_{j=1}^k (y_{2j}^2 - 1), \end{aligned} \quad (2.16)$$

for $k = 1, 2, \dots$, and for $k = 0$

$$g_1(s) = \frac{1}{\pi} \int_1^{\infty} dy_1 \frac{e^{-sy_1}}{(y_1^2 - 1)^{1/2}} = \frac{1}{\pi} K_0(s). \quad (2.17)$$

Sometimes it is useful to use the following representations for the functions $f^{(2n)}(s)$ and $g_{2k+1}(s)$:

$$\begin{aligned} f^{(2n)}(s) &= (2n)^{-1} (2\pi)^{-2n} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{2n} \\ &\times \prod_{j=1}^{2n} \left[\frac{e^{-i\tilde{R}x_j - i\tilde{r}(1+x_j^2)^{1/2}}}{(1+x_j^2)^{1/2} (x_j + x_{j+1})} \right] [(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}], \end{aligned} \quad (2.18)$$

for $n = 1, 2, \dots$, and

$$g_{2k+1}(s) = (2\pi)^{-2k-1} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{2k+1} \prod_{j=1}^{2k+1} \left[\frac{e^{-i\bar{R}x_j - i\bar{t}(1+x_j^2)^{1/2}}}{(1+x_j^2)^{1/2}} \right] \\ \times \prod_{j=1}^{2k} \left[\frac{(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}}{x_j + x_{j+1}} \right], \quad (2.19)$$

for $k = 1, 2, \dots$. In the above

$$\bar{R} = \gamma^{-1}|h-1|R, \quad (2.20a)$$

$$\bar{t} = |h-1|t, \quad (2.20b)$$

so that

$$s = (\bar{R}^2 - \bar{t}^2)^{1/2}. \quad (2.21)$$

2.3.2. Perturbation expansion representation of $\rho_{yy}(R, t)$ in scaling region A

$$\lim_A \rho_{yy}^{-1}(\infty) \rho_{yy}(R, t)$$

$$= \begin{cases} \hat{F}_-(s) = [G(s)\tilde{E}(s) - E^2(s)]\hat{F}_-(s) & (h \rightarrow 1^-), \end{cases} \quad (2.22a)$$

$$= \begin{cases} \hat{F}_+(s) = H(s)\hat{F}_-(s) & (h \rightarrow 1^+), \end{cases} \quad (2.22b)$$

where $\rho_{yy}(\infty)$, $\hat{F}_\pm(s)$ and $G(s)$ are given by (1.9b), (2.14b), and (2.14d) and the functions $E(s)$, $\tilde{E}(s)$, and $H(s)$ are given by

$$E(s) = \sum_{k=0}^{\infty} e_{2k+1}(s), \quad (2.23)$$

$$\tilde{E}(s) = \sum_{k=0}^{\infty} \tilde{e}_{2k+1}(s), \quad (2.24)$$

and

$$H(s) = \sum_{k=0}^{\infty} h_{2k+1}(s), \quad (2.25)$$

where

$$\tilde{e}_{2k+1}(s) = (-1)^k \pi^{-2k-1} \int_1^{\infty} dy_1 \dots \int_1^{\infty} dy_{2k+1} y_1 y_{2k+1} \prod_{k=1}^{2k+1} \frac{e^{-sy_i}}{(y_j^2 - 1)^{1/2}} \prod_{j=1}^{2k} (y_j + y_{j+1})^{-1} \\ \times \prod_{j=1}^k (y_{2j}^2 - 1), \quad \text{for } k = 1, 2, \dots, \quad (2.26a)$$

and

$$\tilde{e}_1(s) = \pi^{-1} \int_1^{\infty} dy \frac{y^2 e^{-sy}}{(y^2 - 1)^{1/2}}, \quad (2.26b)$$

$$\begin{aligned} e_{2k+1}(s) &= (-1)^k \pi^{-2k-1} \int_1^{\infty} dy_1 \dots \int_1^{\infty} dy_{2k+1} y_1 \\ &\times \prod_{j=1}^{2k+1} \frac{e^{-sy_j}}{(y_j^2 - 1)^{1/2}} \prod_{j=1}^{2k} (y_j + y_{j+1})^{-1} \prod_{j=1}^k (y_{2j}^2 - 1), \quad \text{for } k = 1, 2, \dots, \end{aligned} \quad (2.27a)$$

and

$$e_1(s) = \pi^{-1} \int_1^{\infty} dy \frac{y e^{-sy}}{(y^2 - 1)^{1/2}}, \quad (2.27b)$$

and

$$\begin{aligned} h_{2k+1}(s) &= (-1)^k \pi^{-2k-1} \int_1^{\infty} dy_1 \dots \int_1^{\infty} dy_{2k+1} \\ &\times \prod_{j=1}^{2k+1} \left[\frac{e^{-sy_j}}{(y_j^2 - 1)^{1/2}} \right] \prod_{j=1}^{2k} (y_j + y_{j+1})^{-1} \prod_{j=1}^{k+1} (y_{2j-1}^2 - 1), \quad \text{for } k = 1, 2, \dots, \end{aligned} \quad (2.28a)$$

and

$$h_1(s) = \pi^{-1} \int_1^{\infty} dy (y^2 - 1)^{1/2} e^{-sy}. \quad (2.28b)$$

The above functions can alternatively be written in the form

$$\begin{aligned} \tilde{e}_{2k+1}(s) &= -(2\pi)^{-2k-1} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{2k+1} \prod_{j=1}^{2k+1} \left[\frac{e^{-iR x_j - i\tilde{t}(1+x_j^2)^{1/2}}}{(1+x_j^2)^{1/2}} \right] \\ &\times \prod_{j=1}^{2k} \left[\frac{(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}}{(x_j + x_{j+1})} \right] x_1 x_{2k+1}, \quad \text{for } k = 1, 2, \dots, \end{aligned} \quad (2.29)$$

$$\begin{aligned} e_{2k+1}(s) &= \frac{1}{2}(2\pi)^{-2k-1} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{2k+1} \prod_{j=1}^{2k+1} \left[\frac{e^{-iR x_j - i\tilde{t}(1+x_j^2)^{1/2}}}{(1+x_j^2)^{1/2}} \right] \\ &\times \prod_{j=1}^{2k} \left[\frac{(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}}{(x_j + x_{j+1})} \right] i(x_1 + x_{2k+1}), \quad \text{for } k = 1, 2, \dots, \end{aligned} \quad (2.30)$$

and

$$h_{2k+1}(s) = (2\pi)^{-2k-1} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_{2k+1} \prod_{j=1}^{2k+1} \left[\frac{e^{-i\bar{R}x_j - i\bar{t}(1+x_j^2)^{1/2}}}{(1+x_j^2)^{1/2}} \right] \\ \times \prod_{j=1}^{2k} \left[\frac{(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}}{(x_j + x_{j+1})} \right] (1+x_1^2)^{1/2} (1+x_{2k+1}^2)^{1/2}, \\ \text{for } k = 1, 2, \dots, \quad (2.31)$$

where \bar{R} and \bar{t} are given above.

2.3.3. Painlevé function representation of $\hat{F}_{\pm}^X(s)$ and $\hat{F}_{\pm}^Y(s)$

Following refs. 17 and 19 we denote by $\eta(\theta)$ the Painlevé transcendent of the third kind^{22,23}) that satisfies the differential equation

$$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta} \left(\frac{d\eta}{d\theta} \right)^2 - \eta^{-1} + \eta^3 - \theta^{-1} \frac{d\eta}{d\theta}, \quad (2.32)$$

with the boundary conditions

$$\eta(\theta) = -\theta[\ln(\theta/4) + \gamma_E] + \mathcal{O}(\theta^5 \ln^3 \theta) \quad (\theta \rightarrow 0) \quad (2.33)$$

and

$$\eta(\theta) = 1 - 2\pi^{-1} K_0(2\theta) + \mathcal{O}(e^{-4\theta}) \quad (\theta \rightarrow \infty). \quad (2.34)$$

These boundary conditions are equivalently given by

$$\eta(-i\tau) = i\tau \left[\ln(\tau/4) + \gamma_E - i \frac{\pi}{2} \right] + \mathcal{O}(\tau^5 \ln^3 \tau) \quad (\tau \rightarrow 0) \quad (2.35)$$

and

$$\eta(-i\tau) = 1 - iH_0^{(1)}(2\tau) + \mathcal{O}(e^{-4i\tau}) \quad (\tau \rightarrow \infty) \quad (2.36)$$

along the imaginary $s(= -i\tau)$ axis. We refer the reader to ref. 19 for a more detailed discussion of these boundary conditions.

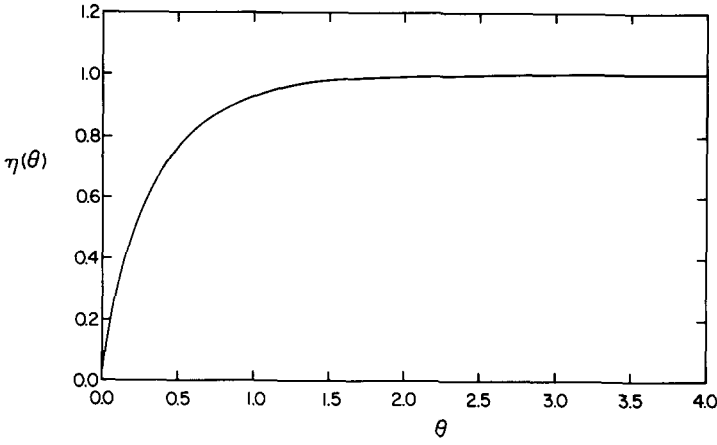
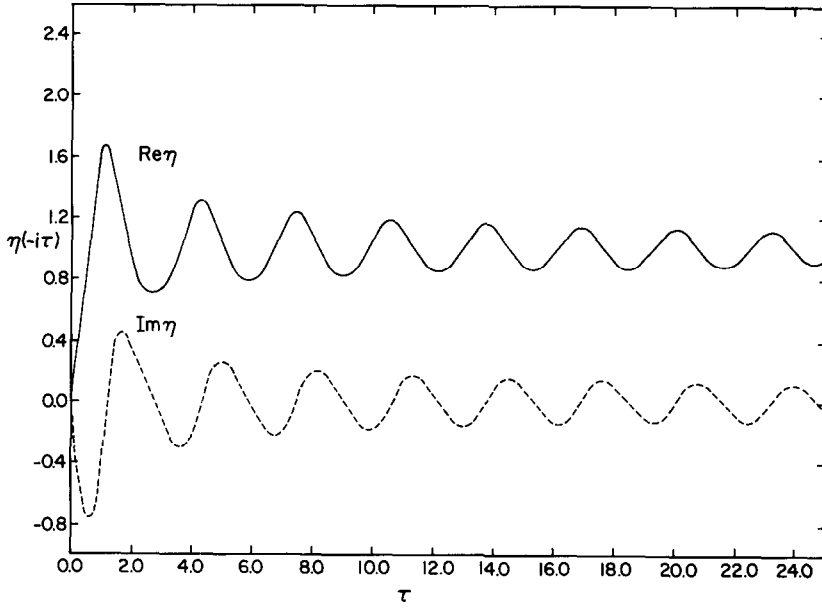
Also following ref. 19 we introduce the function $\psi(s)$ which is defined by²⁴)

$$\eta(\theta) = e^{-\psi(s)}, \quad s = 2\theta. \quad (2.37)$$

We note that $\psi(s)$ satisfies the hyperbolic sine-Gordon equation

$$\frac{d^2\psi}{ds^2} + \frac{1}{s} \frac{d\psi}{ds} = \frac{1}{2} \sinh(2\psi(s)). \quad (2.38)$$

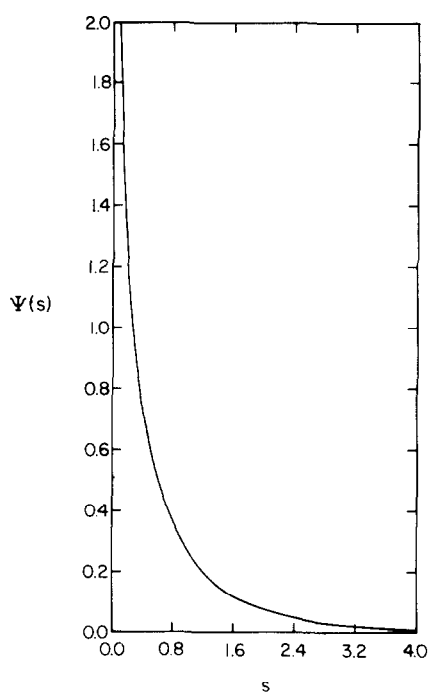
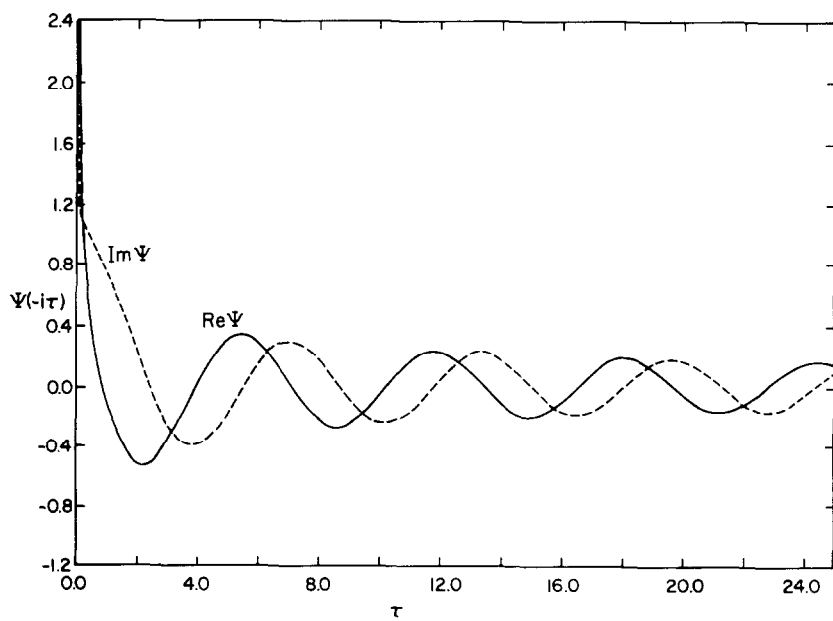
The behavior of $\eta(\theta)$ and $\psi(s)$ for both real and imaginary arguments is given in figs. 10–13.


 Fig. 10. Painlevé function of the third kind $\eta(\theta)$ as a function of θ .

 Fig. 11. Painlevé function $\eta(\theta)$ on the imaginary axis $\theta = -i\tau$ as a function of τ .

In terms of $\eta(\theta)$ and $\psi(s)$ we have

$$\hat{F}_{\pm}^x(s) = \frac{1}{2}(1 \pm \eta(\theta))\eta^{-1/2}(\theta) \exp \left[\int_{\theta}^{\infty} dx \frac{1}{x} \eta^{-2}(x) \left[(1 - \eta^2(x))^2 - \left(\frac{d\eta}{dx} \right)^2 \right] \right] \quad (2.39a)$$

$$= \left\{ \begin{array}{l} \sinh \frac{1}{2} \psi(s) \\ \cosh \frac{1}{2} \psi(s) \end{array} \right\} \exp \left[-\frac{1}{4} \int_s^{\infty} dx \left[\left(\frac{d\psi}{dx} \right)^2 - \sinh^2 \psi(x) \right] \right]. \quad (2.39b)$$

Fig. 12. Function $\psi(s)$ as a function of s .Fig. 13. Function $\psi(s)$ on the imaginary axis $s = -i\tau$ as a function of τ .

If we write the scaling functions $\hat{F}_{\pm}^y(s)$ as

$$\hat{F}_{\pm}^y(s) = Y_{\pm}(s)\hat{F}_{\pm}^x(s), \quad (2.40)$$

where from (2.22) we have

$$Y_{+}(s) = H(s)/G(s), \quad (2.41)$$

$$Y_{-}(s) = G(s)\tilde{E}(s) - E^2(s), \quad (2.42)$$

then we have the representations

$$Y_{\pm}(s) = \frac{\theta^2}{16\eta^2(\theta)} \left[(1 - \eta^2(\theta))^2 - \left(\frac{d\eta}{d\theta} \right)^2 \right] + \frac{1}{8\theta\eta(\theta)} \left(\frac{d\eta}{d\theta} \right) \left[\frac{1 \pm \eta(\theta)}{1 \mp \eta(\theta)} \right] \quad (2.43a)$$

$$= \frac{1}{4} \left[\sinh^2 \psi(s) - \left(\frac{d\psi}{ds} \right)^2 \right] - \frac{1}{2s} \left(\frac{d\psi}{ds} \right) \left\{ \coth \frac{1}{2}\psi(s) \right\} \tanh \frac{1}{2}\psi(s). \quad (2.43b)$$

In ref. 19 explicit formulae are given for $\eta(\theta)$ and $\psi(s)$.

2.3.4. Large s behavior of $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$

For $s \rightarrow \infty$ the leading behavior of $\hat{F}_{\pm}^x(s)$ and $\hat{F}_{\pm}^y(s)$ is given by the first term in the perturbation expansion results given above. More explicitly we have

$$\hat{F}_{+}^x(s) = \pi^{-1}K_0(s) + \mathcal{O}(e^{-3s}) \quad (s \rightarrow \infty), \quad (2.44)$$

$$\hat{F}_{-}^x(s) = 1 + \pi^{-2}[s^2[K_1^2(s) - K_0^2(s)] - sK_0(s)K_1(s) + \frac{1}{2}K_0^2(s)] + \mathcal{O}(e^{-4s}), \quad (2.45)$$

$$\hat{F}_{+}^y(s) = \pi^{-1}s^{-1}K_1(s) + \mathcal{O}(e^{-3s}) \quad (s \rightarrow \infty), \quad (2.46)$$

$$\hat{F}_{-}^y(s) = \pi^{-2}[K_0^2(s) - K_1^2(s) + s^{-1}K_0(s)K_1(s)] + \mathcal{O}(e^{-4s}) \quad (s \rightarrow \infty). \quad (2.47)$$

These formulae can be analytically continued into the time-like region (s imaginary) where for instance $K_0(s)$ becomes $(i\pi/2)H_0^{(1)}(\tau)$. The short distance behavior of the scaling functions is best derived from the Painlevé function representations given above. These small s expansions were given in section 1.

3. Formulation

Following MBA the two-spin correlation function is obtained by considering the four-spin correlation function

$$C_{vv}(R, t, N) = \langle S_1^v(0)S_{(N/2)-R+1}^v(0)S_{(N/2)+1}^v(t)S_{N-R+1}^v(t) \rangle \quad (v = x \text{ or } y), \quad (3.1)$$

where N is the number of spins in the chain and using the clustering property

$$\lim_{N \rightarrow \infty} C_{vv}(R, t, N) = \rho_{vv}^2(R, t), \quad (3.2)$$

where

$$\rho_{vv}(R, t) = \langle S_1^v(t)S_{R+1}^v(0) \rangle \quad (3.3)$$

By the methods of MBA we find that

$$C_{vv}^2(R, t, N) = 4^{-4} \det \begin{vmatrix} 0 & \tilde{S}^v & \tilde{T}^v & \tilde{U}^v \\ -\tilde{S}^{vT} & 0 & -\tilde{U}^v & \tilde{V}^v \\ -\tilde{T}^v & \tilde{U}^v & 0 & -\tilde{S}^v \\ -\tilde{U}^v & -\tilde{V}^v & \tilde{S}^{vT} & 0 \end{vmatrix}, \quad (3.4)$$

where each of the submatrices is of dimension $\frac{1}{2}N - R$ with $0 \leq m \leq \frac{1}{2}N - R - 1$, $0 \leq n \leq \frac{1}{2}N - R - 1$.

At zero temperature the matrix elements are given by

$$\tilde{S}_{m,n}^v = \frac{1}{N} \sum_{\phi} e^{-i(n-m)\phi} S_v(\phi). \quad (3.5a)$$

$$\tilde{T}_{m,n}^v = \frac{1}{N} \sum_{\phi} e^{-i(n+m)\phi} T_v(\phi), \quad (3.5b)$$

$$\tilde{U}_{m,n}^v = \frac{1}{N} \sum_{\phi} e^{i(n+m)\phi} U_v(\phi), \quad (3.5c)$$

and

$$\tilde{V}_{m,n}^v = \frac{1}{N} \sum_{\phi} e^{-i(n+m)\phi} V_v(\phi). \quad (3.5d)$$

Here

$$S_x(\phi) = e^{-i\phi} \Phi(-\phi); \quad S_y(\phi) = -e^{i\phi} \Phi(\phi), \quad (3.6a)$$

$$T_x(\phi) = e^{-i\phi R - i\Lambda t} \Phi(\phi); \quad T_y(\phi) = -e^{-i\phi R - i\Lambda t} \Phi(-\phi), \quad (3.6b)$$

$$U_x(\phi) = -e^{i\phi(R+1) - i\Lambda t}; \quad U_y(\phi) = e^{i\phi(R+1) - i\Lambda t}, \quad (3.6c)$$

$$V_x(\phi) = -e^{-i\phi(R+2) - i\Lambda t} \Phi(-\phi); \quad V_y(\phi) = e^{i\phi(R+2) - i\Lambda t} \Phi(\phi), \quad (3.6d)$$

with

$$\Phi(\phi) = e^{-i\phi} \left[\frac{(1 - \lambda_1^{-1} e^{i\phi})(1 - \lambda_2^{-1} e^{i\phi})}{(1 - \lambda_1^{-1} e^{-i\phi})(1 - \lambda_2^{-1} e^{-i\phi})} \right]^{1/2}. \quad (3.7)$$

It should be noted that with the definition

$$\text{Ind } S(\phi) = \frac{1}{2\pi i} [\ln S(2\pi) - \ln S(0)] \quad (3.8)$$

we have for $h < 1$

$$\text{Ind } S_x = 0 \quad \text{and} \quad \text{Ind } S_y = -2, \quad (3.9)$$

and for $h > 1$

$$\text{Ind } S_x = \text{Ind } S_y = -1. \quad (3.10)$$

In the thermodynamic limit $N \rightarrow \infty$,

$$\frac{1}{N} \sum_{\phi} \rightarrow (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi,$$

and we replace \tilde{S}_x , etc. by S^x , etc. to obtain

$$C_{vv}^2 = 4^{-4} \det \begin{bmatrix} 0 & S^v & T^v & U^v \\ -S^{vT} & 0 & -U^v & V^v \\ -T^v & U^v & 0 & -S^v \\ -U^v & -V^v & S^{vT} & 0 \end{bmatrix}. \quad (3.11)$$

Here, as in (3.4), the superscript T denotes the transpose operation.

4. Evaluation of $\rho_{xx}(R, t)$ for $h < 1$

The evaluation of the determinants in (3.11) is based on the method developed in ref. 18. Since these techniques are not widely known we reproduce below the details of the derivation.

Let

$$C_{xx}^2 = 4^{-4} \det \mathcal{C}. \quad (4.1)$$

In what follows we will drop the superscripts on S, T, etc. for simplicity. We write \mathcal{C} as

$$\mathcal{C} = \mathcal{A} + \mathcal{B}$$

where

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}; \quad \mathcal{B} = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}, \quad (4.2a)$$

and

$$A = \begin{bmatrix} 0 & S \\ -S^T & 0 \end{bmatrix}; \quad B = \begin{bmatrix} T & U \\ -U & V \end{bmatrix}. \quad (4.2b)$$

Then using an elementary formula of matrix theory we have

$$C_{xx}^2 = 4^{-4} \det \mathcal{A} \det(1 + \mathcal{A}^{-1} \mathcal{B}) \quad (4.3)$$

$$= \rho_{xx}^4(\infty) \exp \left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{Tr}(\mathcal{A}^{-1} \mathcal{B})^k \right], \quad (4.4)$$

where we have used the fact that^{7,8}) $\det \mathcal{A} = 4^4 \rho_{xx}^4(\infty)$ with $\rho_{xx}(\infty)$ given by (1.9a).

Define $F_<(R, t)$ by

$$\rho_{xx}(R, t) = \rho_{xx}(\infty) \exp[-F_<(R, t)]. \quad (4.5)$$

We have

$$\text{Tr}[(\mathcal{A}^{-1} \mathcal{B})^{2j+1}] = 0, \quad \text{for } j = 0, 1, 2, \dots, \quad (4.6)$$

and

$$(\mathcal{A}^{-1} \mathcal{B})^2 = \begin{bmatrix} A^{-1} B A^{-1} B^T & 0 \\ 0 & A^{-1} B^T A^{-1} B \end{bmatrix}. \quad (4.7)$$

Hence, if we write $F_{<}$ as

$$F_{<} = \sum_{k=1}^{\infty} F_{<}^{(2k)}, \quad (4.8)$$

then

$$F_{<}^{(2k)} = \frac{1}{2k} \text{Tr}[(A^{-1}BA^{-1}B^T)^k]. \quad (4.9)$$

We define for $|\xi|$ and $|\xi'| < 1$

$$S_x^{-1}(\xi, \xi') = \sum_{m,n=0}^{\infty} \xi^m \xi'^n (S_x^{-1})_{m,n}. \quad (4.10)$$

The generating function $S_x(\xi)$ has the factorization

$$[S_x(\xi)]^{-1} = P_x(\xi)Q_x(\xi^{-1}), \quad (4.11)$$

where $P_x(\xi)$ and $Q_x(\xi)$ are analytic inside the unit circle $|\xi| = 1$ and are given by

$$P_x(\xi) = [(1 - \lambda_1^{-1}\xi)(1 - \lambda_2^{-1}\xi)]^{1/2} \quad (4.12a)$$

and

$$Q_x(\xi) = [(1 - \lambda_1^{-1}\xi)(1 - \lambda_2^{-1}\xi)]^{-1/2}. \quad (4.12b)$$

Note that

$$P_x(\xi)Q_x(\xi) = 1. \quad (4.13)$$

It follows from the method of Wiener–Hopf²⁵ that

$$S_x^{-1}(\xi, \xi') = Q_x(\xi)P_x(\xi')(1 - \xi\xi')^{-1}. \quad (4.14)$$

We define

$$A^{-1}(\xi, \xi') = \begin{bmatrix} 0 & -S_x^{-1}(\xi', \xi) \\ S_x^{-1}(\xi, \xi') & 0 \end{bmatrix} \quad (4.15)$$

so that $A^{-1}(\xi, \xi')$ is the matrix generating function of the inverse matrix elements of A . Similarly,

$$B(\xi) = \begin{bmatrix} T(\xi) & U(\xi) \\ -U(\xi) & V(\xi) \end{bmatrix}. \quad (4.16)$$

Note that

$$B^T(\xi) = \begin{bmatrix} T(\xi) & -U(\xi) \\ U(\xi) & V(\xi) \end{bmatrix}. \quad (4.17)$$

With these definitions we have

$$F_{<}^{(2k)} = \frac{1}{2k} (2\pi)^{-2k} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2k} \\ \times \text{Tr}\{A^{-1}(\bar{2k}, \bar{1})B(1)A^{-1}(\bar{1}, \bar{2})B^T(2) \dots A^{-1}(\bar{2k}-1, \bar{2k})B^T(2k)\}, \quad (4.18)$$

where $B(1) \equiv B(e^{i\phi_1})$ and $A^{-1}(\bar{1}, \bar{2}) \equiv A^{-1}(e^{-i\phi_1}, e^{-i\phi_2})$, etc. In (4.18) $\text{Im } \phi_1 < \text{Im } \phi_2 < \dots < \text{Im } \phi_{2k}$.

We can factorize $A^{-1}(\xi, \xi')$ as

$$A^{-1}(\xi, \xi') = (1 - \xi\xi')^{-1} \begin{bmatrix} 0 & -P(\xi) \\ Q(\xi) & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & Q(\xi') \\ -P(\xi') & 0 \end{bmatrix}. \quad (4.19)$$

Using the cyclic property of the trace we group together matrices that depend on the same ϕ_i variables. Performing the matrix multiplications we have

$$F_{<}^{(2k)} = \frac{1}{2k} (2\pi)^{-2k} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2k} \prod_{j=1}^{2k} (1 - e^{-i\phi_j - i\phi_{j+1}})^{-1} \\ \times \text{Tr} \left\{ \begin{bmatrix} V_x(1)Q_x^2(\bar{1}) & U_x(1) \\ -U_x(1) & T_x(1)P_x^2(\bar{1}) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right. \\ \times \begin{bmatrix} V_x(2)Q_x^2(\bar{2}) & -U_x(2) \\ U_x(2) & T_x(2)P_x^2(\bar{2}) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \times \dots \left. \begin{bmatrix} V_x(2k)Q_x^2(\bar{2k}) & -U_x(2k) \\ U_x(2k) & T_x(2k)P_x^2(\bar{2k}) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}. \quad (4.20)$$

Now when used in the above integrals the following identities are valid (see discussion in refs. 17 and 18):

$$V_x(1)Q_x^2(\bar{1}) = -\frac{1+\gamma}{2} e^{-i\phi_1(R+1)-i\Lambda_1 t} \Lambda^{-1}(1), \quad (4.21)$$

$$T_x(1)P_x^2(\bar{1}) = \frac{2}{1+\gamma} e^{-i\phi_1(R+1)-i\Lambda_1 t} \Lambda(1). \quad (4.22)$$

Hence,

$$\begin{bmatrix} V_x(1)Q_x^2(\bar{1}) & U_x(1) \\ -U_x(1) & T_x(1)P_x^2(\bar{1}) \end{bmatrix} = \frac{e^{-i\phi_1(R+1)-i\Lambda_1 t}}{\Lambda_1} \begin{bmatrix} -\left(\frac{1+\gamma}{2}\right)^{1/2} \\ \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_1 \end{bmatrix} \\ \times \left[\left(\frac{1+\gamma}{2}\right)^{1/2} \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_1 \right], \quad (4.23)$$

and

$$\begin{bmatrix} V_x(2)Q_x^2(\bar{2}) & -U_x(2) \\ U_x(2) & T_x(2)P_x^2(\bar{2}) \end{bmatrix} = \frac{e^{-i\phi_2(R+1)-i\Lambda_2 t}}{2} \begin{bmatrix} -\left(\frac{1+\gamma}{2}\right)^{1/2} \\ -\left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_2 \end{bmatrix} \\ \times \left[\left(\frac{1+\gamma}{2}\right)^{1/2} - \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_2 \right]. \quad (4.24)$$

Using these all the matrix products can be written as scalar products, i.e.

$$\left[\left(\frac{1+\gamma}{2}\right)^{1/2} \quad \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_2 \right] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\left(\frac{1+\gamma}{2}\right)^{1/2} \\ -\left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_2 \end{bmatrix} = \Lambda_2 - \Lambda_1 \quad (4.25)$$

and

$$\left[\left(\frac{1+\gamma}{2}\right)^{1/2} - \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_2 \right] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\left(\frac{1+\gamma}{2}\right)^{1/2} \\ \left(\frac{2}{1+\gamma}\right)^{1/2} \Lambda_3 \end{bmatrix} = \Lambda_2 - \Lambda_3. \quad (4.26)$$

This gives the result (2.2).

4.1. Scaling limit $h \rightarrow 1^-$

Now we consider the scaling limit in region A as defined above in (1.6). For $h = 1$, $\Lambda(\phi) = 0$ at $\phi = 0$. Thus, the leading contribution to the integral for $h \rightarrow 1^-$ comes from the behavior of the integrand around $\phi_j = 0$, $j = 1, 2, \dots, 2n$. Expanding $\Lambda(\phi_j)$ around $\phi_j = 0$

$$\Lambda(\phi_j) \sim \left(\frac{1+\gamma}{2}\right)(1 - \lambda_1^{-1})(1 - \lambda_2^{-1}) \left[1 + \frac{\lambda_1^{-1}}{(1 - \lambda_1^{-1})^2} \phi_j^2 + \frac{\lambda_2^{-1}}{(1 - \lambda_2^{-1})^2} \phi_j^2 \right]^{1/2} \quad (4.27)$$

$$\sim \left(\frac{1+\gamma}{2}\right)(1 - \lambda_1^{-1})(1 - \lambda_2^{-1})[1 + x_j^2], \quad (4.28)$$

where

$$x_j = \lambda_2^{-1/2}(1 - \lambda_2^{-1})^{-1} \phi_j. \quad (4.29)$$

Let

$$R\phi_j = \lambda_2^{1/2}(1 - \lambda_2^{-1})R x_j = \bar{R}x_j, \quad (4.30)$$

where

$$\bar{R} = \lambda_2^{1/2}(1 - \lambda_2^{-1})R. \quad (4.31)$$

Similarly,

$$t\Lambda_j = \bar{t}[1 + x_j^2]^{1/2}, \quad (4.32)$$

where

$$\bar{t} = \left(\frac{1+\gamma}{2} \right) (1 - \lambda_1^{-1})(1 - \lambda_2^{-1})t. \quad (4.33)$$

Similarly,

$$\sin \frac{1}{2}(\phi_j + \phi_{j+1}) \sim \frac{1}{2} \lambda_2^{1/2} (1 - \lambda_2^{-1})(x_j + x_{j+1}) \quad (4.34)$$

and

$$\frac{d\phi_j(\Lambda_j - \Lambda_{j+1})}{\Lambda_j \sin \frac{1}{2}(\phi_j + \phi_{j+1})} \sim \frac{2 dx_j [(1+x_j^2)^{1/2} - (1+x_{j+1}^2)^{1/2}]}{(1+x_j^2)^{1/2}(x_j + x_{j+1})}. \quad (4.35)$$

Applying the above scaling limits to the integral $F_{\leq}^{(2n)}(R, t)$ [see (2.2)] we obtain the value in the scaling limit which we denote by $f^{(2n)}(s)$ given by (2.18). Note that as $h \rightarrow 1^-$

$$\bar{R} \rightarrow \frac{1-h}{\gamma} R \quad \text{and} \quad \bar{t} \rightarrow (1-h)t. \quad (4.36)$$

To show that $f^{(2n)}(s)$ depends only upon s , we proceed as follows. Define θ by

$$R = (R^2 - \gamma^2 t^2)^{1/2} \cosh \theta; \quad \gamma t = (R^2 - \gamma^2 t^2)^{1/2} \sinh \theta, \quad (4.37)$$

and let

$$x_j = \sinh(\psi_j - \theta), \quad (4.38)$$

then

$$\bar{R}x_j + \bar{t}(1+x_j^2)^{1/2} = (R^2 - \gamma^2 t^2)^{1/2} u_j \quad (4.39)$$

with

$$u_j = \sinh \psi_j. \quad (4.40)$$

Hence, in the new scaling variable s as defined by (1.6b) we have

$$f^{(2n)}(s) = (2n)^{-1} (2\pi)^{-2n} \int_{-\infty}^{\infty} du_1 \dots \int_{-\infty}^{\infty} du_{2n} \prod_{j=1}^{2n} \left[\frac{e^{-is u_j} [(1+u_j^2)^{1/2} - (1+u_{j+1}^2)^{1/2}]}{(1+u_j^2)^{1/2}(u_j + u_{j+1})} \right]. \quad (4.41)$$

Comparing with the two-dimensional Ising model correlation functions derived in ref. 17 [eqs. (3.141) and (2.26)] we see that

$$\rho_{xx}(R, t) \cong \rho_{xx}(\infty) \hat{F}_{\pm}^x(s). \quad (4.42)$$

4.2. Scaling limit in region B

We now consider the scaling limit in region B as defined above in (1.20). In this region λ_1 and λ_2 are complex and $\lambda_1 = \lambda_2^*$.

Let

$$\lambda_1 = |\lambda| e^{i\psi} \quad \text{and} \quad \lambda_2 = |\lambda| e^{-i\psi}, \quad (4.43)$$

where

$$|\lambda| = \frac{1}{\rho} = \left(\frac{1+\gamma}{1-\gamma} \right)^{1/2} \text{ defines } \rho \quad (4.44)$$

and

$$\cos \psi = h(1-\gamma^2)^{-1/2} \text{ defines } \psi, \quad (4.45)$$

then

$$\Lambda(\phi_j) = \left(\frac{1+\gamma}{2} \right) [(1-2\rho \cos(\phi_j + \psi) + \rho^2)(1-2\rho \cos(\phi_j - \psi) + \rho^2)]^{1/2}. \quad (4.46)$$

As $\gamma \rightarrow 0$, $\rho \rightarrow 1$ and the leading contribution to the integrals in (2.2) comes from the zeroes of $\Lambda(\phi_j)$ at $\phi_j = \pm \psi$. To leading order we require contributions such that

$$\phi_i + \phi_{i+1} \simeq 0, \quad i = 1, 2, \dots, 2n,$$

that is, from the regions in the neighborhood of

$$(a) \quad \phi_1 = \psi; \quad \phi_2 = -\psi; \quad \phi_3 = \psi, \dots, \phi_{2n} = -\psi \quad (4.47a)$$

and

$$(b) \quad \phi_1 = -\psi; \quad \phi_2 = \psi; \quad \phi_3 = -\psi, \dots, \phi_{2n} = \psi. \quad (4.47b)$$

Expanding $\Lambda(\phi)$ around $\phi + \psi = 0$ we have

$$\Lambda(\phi) \sim \left(\frac{1+\gamma}{2} \right) (1-\rho)(2-2\cos\psi)^{1/2} \left[1 + \frac{\rho}{(1-\rho)^2} (\phi + \psi)^2 \right]^{1/2}. \quad (4.48)$$

Similarly, the expansion of $\Lambda(\phi)$ around $\phi - \psi = 0$ gives

$$\Lambda(\phi) \sim \left(\frac{1+\gamma}{2} \right) (1-\rho)(2-2\cos\psi)^{1/2} \left[1 + \frac{\rho}{(1-\rho)^2} (\phi - \psi)^2 \right]^{1/2}. \quad (4.49)$$

(a) In the neighborhood of poles at

$$\phi_{2j+1} = \psi \quad \text{and} \quad \phi_{2j} = -\psi,$$

define

$$x_{2j+1} = \rho^{1/2} (1-\rho)^{-1} (\phi_{2j+1} - \psi) \quad (4.50a)$$

and

$$x_{2j} = \rho^{1/2} (1-\rho)^{-1} (\phi_{2j} + \psi). \quad (4.50b)$$

(b) In the neighborhood of poles at

$$\phi_{2j+1} = -\psi \quad \text{and} \quad \phi_{2j} = \psi,$$

define

$$x_{2j+1} = \rho^{1/2} (1-\rho)^{-1} (\phi_{2j+1} + \psi) \quad (4.51a)$$

and

$$x_{2j} = \rho^{1/2}(1 - \rho)^{-1}(\phi_{2j} - \psi). \quad (4.51b)$$

Applying these scaling limits to the integral $F_{<}^{(2n)}(R, t)$ [eq. (2.2)] we obtain

$$F_{<}^{(2n)}(R, t) \sim 2f^{(2n)}(s_1) \quad (4.52)$$

in the scaling limit B , where $f^{(2n)}(s_1)$ is given by (2.18) as a function of the variables \bar{R}_1 and \bar{t}_1 defined by

$$\bar{R}_1 = \gamma R \quad \text{and} \quad \bar{t}_1 = \gamma(1 - h^2)^{1/2}t. \quad (4.53)$$

As in section 4.1 above we can now reduce the dependence in \bar{R}_1, \bar{t}_1 to a dependence on the combined scaling variable s_1 [see eq. (1.20b)] to get the result in (1.21).

5. Evaluation of $\rho_{yy}(R, t)$ for $h < 1$

Here $\text{Ind } S_y = -2$. To work with a generating function of index zero we consider the shifted matrix

$$D_y^2(R, t, N) = \det C_y, \quad (5.1)$$

$$C_y = \begin{pmatrix} \overline{0} & \overline{S^y} & \overline{T^y} & \overline{U^y} \\ -\overline{S^{yT}} & \overline{0} & -\overline{U^y} & \overline{V^y} \\ -\overline{T^y} & \overline{U^y} & \overline{0} & -\overline{S^y} \\ -\overline{U^y} & -\overline{V^y} & \overline{S^{yT}} & \overline{0} \end{pmatrix}, \quad (5.2)$$

where the horizontal (vertical) bar represents addition of two rows (columns). Thus, the new generating functions are

$$\bar{\bar{S}}_y(\phi) = e^{2i\phi} S_y(\phi), \quad (5.3a)$$

$$\bar{\bar{T}}_y(\phi) = T_y(\phi), \quad (5.3b)$$

$$\bar{\bar{U}}_y(\phi) = e^{2i\phi} U_y(\phi), \quad (5.3c)$$

$$\bar{\bar{V}}_y(\phi) = e^{4i\phi} V_y(\phi). \quad (5.3d)$$

The evaluation of D_y^2 is carried out in a manner analogous to section 4. It can be immediately seen that

$$D_y^{1/2}(R, t) = \rho_{xx}^{<}(R - 2, t). \quad (5.4)$$

Now consider the ratio

$$r_y^2(R, t, N) = [D_y^2(R, t, N)]^{-1} \begin{vmatrix} 0 & \bar{\bar{S}}^y & \bar{\bar{T}}^y & \bar{\bar{U}}^y \\ -\bar{\bar{S}}^{yT} & 0 & -\bar{\bar{U}}^y & \bar{\bar{V}}^y \\ -\bar{\bar{T}}^y & \bar{\bar{U}}^y & 0 & -\bar{\bar{S}}^y \\ -\bar{\bar{U}}^y & -\bar{\bar{V}}^y & \bar{\bar{S}}^{yT} & 0 \end{vmatrix}. \quad (5.5)$$

Using Jacobi's theorem, this ratio can be obtained from the 4×8 matrix of $\frac{1}{2}N - R + 2$ component vector X_b^a which are the solutions of the linear equations

$$C_y \begin{bmatrix} X_1^i \\ X_2^i \\ X_3^i \\ X_4^i \end{bmatrix} = \delta^i, \quad (5.6)$$

where δ^i are the eight columns of

$$\begin{bmatrix} \delta_D^0 & \delta_D^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_U^0 & \delta_U^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_U^0 & \delta_U^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_U^0 & \delta_U^1 \end{bmatrix}, \quad (5.7)$$

and the $\frac{1}{2}N - R + 2$ component vectors $\delta_{U,D}^i$ are

$$\delta_U^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad \delta_U^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad \delta_D^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \delta_D^1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (5.8)$$

The ratio r_y^2 is expressible as an 8×8 determinant of X_b^a . Because of the antisymmetry of C and the fact that a number of elements are exponentially small in N , we have in the thermodynamic limit

$$\begin{aligned} r_y(R, t) &= \lim_{N \rightarrow \infty} r_y(R, t, N) \\ &= \det \begin{bmatrix} 0 & -X_{2,1}^3 & -X_{4,0}^3 & -X_{4,1}^3 \\ X_{2,1}^3 & 0 & -X_{4,0}^4 & -X_{4,1}^4 \\ X_{4,0}^3 & X_{4,0}^4 & 0 & -X_{4,1}^7 \\ X_{4,1}^3 & X_{4,1}^4 & X_{4,1}^7 & 0 \end{bmatrix}, \end{aligned} \quad (5.9)$$

where the X_b^a are infinite component vectors obtained from the $N = \infty$ set of equations corresponding to (5.6). In this set of equations the right-hand side is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \delta^0 & \delta^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^0 & \delta^1 \end{bmatrix},$$

where

$$\delta^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}; \quad \delta^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}. \quad (5.10)$$

We use the methods of section 4 in the evaluation of various terms in the

above determinant. One typical example is given below. We also find that

$$X_{4,1}^7 = -X_{2,1}^3 \quad \text{and} \quad X_{4,0}^4 = X_{4,1}^3 \quad (5.11a,b)$$

Thus,

$$r_y^{1/2}(R, t) = -\det \begin{vmatrix} X_{4,0}^3 & X_{4,1}^3 - X_{2,1}^3 \\ X_{4,1}^3 + X_{2,1}^3 & X_{4,1}^4 \end{vmatrix}. \quad (5.12)$$

5.1. *Evaluation of $X_{4,1}^3$* (note: all of S, T , etc. represent \bar{S}, \bar{T} , etc.)

Now

$$X_4^3 = (C_y^{-1})_{42} \delta^0 \quad (5.13)$$

$$= - \sum_{n=0}^{\infty} S^{-1} \{ T X_{11}^n - U X_{21}^n \} S^{T^{-1}} \delta^0, \quad (5.14)$$

where $X = A^{-1} B A^{-1} B^T$, so that

$$\begin{aligned} X_{4,1}^3 &= -S_{1m_1}^{-1} \sum_{n=0}^{\infty} \{ T_{m_1 m_2} (X_{11}^n)_{m_2 m_3} - U_{m_1 m_2} (X_{21}^n)_{m_2 m_3} \} S_{0m_3}^{-1} \\ &= - \sum_{n=0}^{\infty} (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \frac{\partial}{\partial \xi} S^{-1}(\xi, \bar{1})|_{\xi=0} \\ &\quad \times \{ T(1) [A^{-1}(\bar{1}, \bar{2}) B(2) \dots A^{-1}(\bar{2n}, \overline{2n+1}) B^T(2n+1)]_{11} \\ &\quad - U(1) [A^{-1}(\bar{1}, \bar{2}) B(2) \dots A^{-1}(\bar{2n}, \overline{2n+1}) B^T(2n+1)]_{21} \} S^{-1}(0, \overline{2n+1}). \end{aligned} \quad (5.16)$$

We know from eq. (4.14) that

$$S^{-1}(\xi, \bar{1}) = (1 - \xi e^{-i\phi_1})^{-1} Q(\xi) P(\bar{1}), \quad (5.17)$$

where $P(\xi)[Q(\xi)]$ is given by eq. (4.12a) [(4.12b)]. Hence,

$$\frac{\partial}{\partial \xi} S^{-1}(\xi, \bar{1})|_{\xi=0} = [e^{-i\phi_1} + Q'(0)] P(\bar{1}). \quad (5.18)$$

Factorizing the matrices $A^{-1}(\bar{1}, \bar{2})$, etc. and carrying out the scalar products as in section 4, we have

$$\begin{aligned} X_{4,1}^3 &= \left(\frac{1+\gamma}{2} \right) \sum_{n=0}^{\infty} (-1)^n (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n} \left[\frac{\Lambda_j - \Lambda_{j+1}}{1 - e^{-i(\phi_j + \phi_{j+1})}} \right] \\ &\quad \times \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - i\Lambda_j}}{\Lambda_j} \right] [e^{-i\phi_1} + Q'(0)]. \end{aligned} \quad (5.19)$$

A similar analysis yields

$$X_{2,1}^3 = -\left(\frac{1+\gamma}{2}\right) \sum_{n=1}^{\infty} (-1)^n (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n} \prod_{j=1}^{2n-1} (1 - e^{-i\phi_j - i\phi_{j+1}})^{-1} \\ \times (\Lambda_j - \Lambda_{j+1}) \prod_{j=1}^{2n} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] e^{-i\phi_1}, \quad (5.20)$$

$$X_{4,0}^3 = \left(\frac{1+\gamma}{2}\right) \sum_{n=0}^{\infty} (-1)^n (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n} \frac{\Lambda_j - \Lambda_{j+1}}{1 - e^{-i\phi_j - i\phi_{j+1}}} \\ \times \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] \quad (5.21)$$

and

$$X_{4,1}^4 = \left(\frac{1+\gamma}{2}\right) \sum_{n=0}^{\infty} (-1)^n (2\pi)^{-2n-1} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} \prod_{j=1}^{2n} \left[\frac{\Lambda_j - \Lambda_{j+1}}{1 - e^{-i\phi_j - i\phi_{j+1}}} \right] \\ \times \prod_{j=1}^{2n+1} \left[\frac{e^{-i(R-1)\phi_j - it\Lambda_j}}{\Lambda_j} \right] [e^{-i\phi_1} + Q'(0)][e^{-i\phi_{2n+1}} + Q'(0)]. \quad (5.22)$$

All terms containing $Q'(0)$ can be removed by the following operations on the determinant in eq. (5.12) which does not change its value:

- (a) row $2 - Q'(0)$ [row 1],
- (b) col. $2 - Q'(0)$ [col. 1].

Thus, we get the final result given in eqs. (2.5) and (2.6).

5.2. Scaling limit $h \rightarrow 1^-$

Now we consider the scaling limit in region A. We first note that $X_{2,1}^3 \rightarrow 0$ in the scaling limit. We rearrange the determinant by subtracting rows and columns to take the form

$$\begin{vmatrix} X_{4,0}^3 - X_{4,1}^3 & X_{4,1}^3 \\ 2X_{4,1}^3 - X_{4,0}^3 - X_{4,1}^4 & X_{4,1}^4 - X_{4,1}^3 \end{vmatrix}, \quad (5.23)$$

and then take the scaling limit as in section 4.1. Some of the exponentials have to be expanded to higher orders before a nonzero contribution is obtained. On doing this we get the results quoted in section 2, eq. (2.22).

5.3. Scaling limit $\gamma \rightarrow 0$

Here again $X_{2,1}^3 \rightarrow 0$ in the scaling limit. on expanding the $\Lambda(\phi_j)$ values around (1) $\phi_{2j+1} = \psi$, $\phi_{2j} = -\psi$ and (2) $\phi_{2j+1} = -\psi$, $\phi_{2j} = \psi$, as in section 4.2, we

obtain

$$X_{4,1}^3 \sim \frac{e^{-iR\psi} + e^{iR\psi}}{(2 - 2 \cos \psi)^{1/2}} G(s_1), \quad (5.24)$$

$$X_{4,0}^3 \sim \frac{e^{-i(R-1)\psi} + e^{i(R-1)\psi}}{(2 - 2 \cos \psi)^{1/2}} G(s_1) \quad (5.25)$$

and

$$X_{4,1}^4 \sim \frac{e^{-i(R+1)\psi} + e^{i(R-1)\psi}}{(2 - 2 \cos \psi)^{1/2}} G(s_1). \quad (5.26)$$

On solving the determinant in eq. (5.12) we obtain the result quoted in eq. (1.22).

6. Evaluation of $\rho_{xx}(R, t)$ for $h > 1$

In this case $\text{Ind } S_x = -1$. Hence, we work with a new generating function $\bar{S}_x(\phi) = e^{i\phi} S_x(\phi)$ which has index zero. This is achieved by considering the shifted matrix

$$D_x^2(R, t, N) = \det C, \quad (6.1)$$

$$C = \begin{vmatrix} \overline{0} & \overline{S^x} & \overline{T^x} & \overline{U^x} \\ -\overline{S^{xT}} & \overline{0} & -\overline{U^x} & \overline{V^x} \\ -\overline{T^x} & \overline{U^x} & \overline{0} & -\overline{S^x} \\ -\overline{U^x} & -\overline{V^x} & \overline{S^{xT}} & \overline{0} \end{vmatrix}, \quad (6.2)$$

where a horizontal (vertical) bar indicates the addition of a row (column). Now

$$\bar{S}_x(\phi) = e^{i\phi} S_x(\phi), \quad (6.3a)$$

$$\bar{T}_x(\phi) = T_x(\phi), \quad (6.3b)$$

$$\bar{U}_x(\phi) = e^{i\phi} U_x(\phi), \quad (6.3c)$$

$$\bar{V}_x(\phi) = e^{2i\phi} V_x(\phi). \quad (6.3d)$$

The evaluation of D_x^2 can be carried out in a manner analogous to section 4, and we obtain the expression for $F_{>}(R, t)$ given in eq. (2.11).

Consider the ratio

$$r_x^2(R, t, N) = [D_x^2(R, t, N)]^{-1} \begin{vmatrix} 0 & \bar{S}^x & \bar{T}^x & \bar{U}^x \\ -\bar{S}^{xT} & 0 & -\bar{U}^x & \bar{V}^x \\ -\bar{T}^x & \bar{U}^x & 0 & -\bar{S}^x \\ -\bar{U}^x & -\bar{V}^x & \bar{S}^{xT} & 0 \end{vmatrix}. \quad (6.4)$$

Define the 4×4 matrix of $\frac{1}{2}N - R + 1$ components \tilde{X}_a^b by

$$C \begin{bmatrix} \tilde{X}_1^1 & \tilde{X}_1^2 & \tilde{X}_1^3 & \tilde{X}_1^4 \\ \tilde{X}_2^1 & \tilde{X}_2^2 & \tilde{X}_2^3 & \tilde{X}_2^4 \\ \tilde{X}_3^1 & \tilde{X}_3^2 & \tilde{X}_3^3 & \tilde{X}_3^4 \\ \tilde{X}_4^1 & \tilde{X}_4^2 & \tilde{X}_4^3 & \tilde{X}_4^4 \end{bmatrix} = \begin{bmatrix} \tilde{\delta}_D & 0 & 0 & 0 \\ 0 & \tilde{\delta}_U & 0 & 0 \\ 0 & 0 & \tilde{\delta}_D & 0 \\ 0 & 0 & 0 & \tilde{\delta}_U \end{bmatrix}, \quad (6.5)$$

where

$$\tilde{\delta}_D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}; \quad \tilde{\delta}_U = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \quad (6.6)$$

By using Jacobi's theorem we have

$$r_x^2(R, t, N) = \det \begin{bmatrix} \tilde{X}_{1,N/2-R}^1 & \tilde{X}_{1,N/2-R}^2 & \tilde{X}_{1,N/2-R}^3 & \tilde{X}_{1,N/2-R}^4 \\ \tilde{X}_{2,0}^1 & \tilde{X}_{2,0}^2 & \tilde{X}_{2,0}^3 & \tilde{X}_{2,0}^4 \\ \tilde{X}_{3,N/2-R}^1 & \tilde{X}_{3,N/2-R}^2 & \tilde{X}_{3,N/2-R}^3 & \tilde{X}_{3,N/2-R}^4 \\ \tilde{X}_{4,0}^1 & \tilde{X}_{4,0}^2 & \tilde{X}_{4,0}^3 & \tilde{X}_{4,0}^4 \end{bmatrix}. \quad (6.7)$$

Since $C^T = -C$ the determinant in r_x^2 has to be antisymmetric. Also, some of the elements are exponentially small in N and hence vanish in the limit $N \rightarrow \infty$. Using these facts we obtain

$$r_x^2(R, t) = \lim_{N \rightarrow \infty} r_x^2(R, t, N) = (X_{4,0}^2)^4. \quad (6.8)$$

Hence

$$\rho_{xx}(R, t) = \bar{\rho}_{xx}(\infty) X_{4,0}^2 \exp[-F_>(R, t)]. \quad (6.9)$$

Let

$$X_{4,0}^2 = \sum_{k=0}^{\infty} x_{2k+1} = (C_{4,2}^{-1} \delta_U)_0. \quad (6.10)$$

Now

$$C^{-1} = \sum_{k=0}^{\infty} (-1)^k (\mathcal{A}^{-1} \mathcal{B})^k \mathcal{A}^{-1}, \quad (6.11)$$

so that

$$C_{4,2}^{-1} = - \sum_{n=0}^{\infty} (\mathcal{A}^{-1} \mathcal{B})_{41}^{2n+1} S^{T^{-1}} \quad (6.12)$$

$$= (A^{-1} B^T)_{21} X_{11}^n S^{T^{-1}} + (A^{-1} B^T)_{22} X_{21}^n S^{T^{-1}} \quad (6.13)$$

$$= \bar{S}^{-1} \bar{T}(X)_{11}^n \bar{S}^{T^{-1}} - \bar{S}^{-1} \bar{U}(X)_{21}^n \bar{S}^{T^{-1}}, \quad (6.14)$$

where

$$X = A^{-1} B A^{-1} B^T. \quad (6.15)$$

Thus,

$$x_{2n+1} = (\bar{S}^{-1} \bar{T})_{0m} [X_{mp}^n]_{11} \bar{S}_{0p}^{-1} - (\bar{S}^{-1} \bar{U})_{0m} [X_{mp}^n]_{21} \bar{S}_{0p}^{-1}. \quad (6.16)$$

Now

$$\begin{aligned} X_{mp}^n &= (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_2 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} A_{mm_1}^{-1} e^{-i(m_1+m_2)\phi_2} B(2) A_{m_2m_3}^{-1} \\ &\quad \times e^{-i(m_3+m_4)\phi_3} B^T(3) \dots A_{rs}^{-1} e^{-i(s+p)\phi_{2n+1}} B^T(2n+1) \end{aligned} \quad (6.17)$$

$$\begin{aligned} &= (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_2 \dots \int_{-\pi}^{\pi} d\phi_{2n+1} A_{mm_1}^{-1} e^{-im_1\phi_2} B(2) A^{-1}(\bar{2}, \bar{3}) B^T(3) \\ &\quad \times \dots A^{-1}(\bar{2n}, \bar{2n+1}) B^T(2n+1). \end{aligned} \quad (6.18)$$

We can carry out the 2×2 matrix products as in section 4 to derive the final result for $x_{2n+1}(R, t)$ given in eq. (2.10). Some additional manipulation is required before the results can be put into the final form $F_>(R, t)$, $x_>(R, t)$, where $F_>(R, t)$ is identical to $F_<(R, t)$ except for the change from Λ_j to A_j . These are identical to the ones in ref. 17 and will not be reproduced here.

6.1. Scaling limit $h \rightarrow 1^+$

We now consider the scaling limit A . The analysis of this scaling limit is similar to that of section 4.1 except that here we expand $A(\phi)$ [instead of $\Lambda(\phi)$] about $\phi = 0$. Now the scaling variables are

$$\bar{R} = \left(\frac{h-1}{\gamma} \right) R; \quad \bar{t} = (h-1)t \quad \text{and} \quad s = \frac{h-1}{\gamma} (R^2 - \gamma^2 t^2)^{1/2}. \quad (6.19)$$

On carrying out the necessary algebra we derive the results quoted in section 2. Here again the scaling function is the same as that of the Ising model and is expressed in terms of Painlevé function of the third kind^{17,19}.

7. Evaluation of $\rho_{yy}(R, t)$ for $h > 1$

Since $\text{Ind } S_y = -1$ the case is treated in a manner similar to section 6. We have the generating functions

$$\bar{S}_y(\phi) = e^{i\phi} S_y(\phi), \quad (7.1a)$$

$$\bar{T}_y(\phi) = T_y(\phi), \quad (7.1b)$$

$$\bar{U}_y(\phi) = e^{i\phi} U_y(\phi) \quad (7.1c)$$

and

$$\bar{V}_y(\phi) = e^{2i\phi} V_y(\phi). \quad (7.1d)$$

Comparing with \bar{S}_x , etc. we note that \bar{S}_y, \bar{T}_y , etc. can be obtained from \bar{S}_x, \bar{T}_x , etc. by the simple transformation $\gamma \leftrightarrow -\gamma$, i.e. $\lambda_1^{-1} \leftrightarrow \lambda_2$. Thus, the results for $\rho_{xx}(h > 1)$ can be directly transformed accordingly to obtain ρ_{yy} for $h > 1$. These are written out explicitly in section 2. We also verify the Suzuki sum rule⁴⁾ relating the Ising model and XY model correlation function. This is done in Appendix B.

7.1. Scaling limit $h \rightarrow 1^+$

The analysis in this case is similar to that of section 6.1. We note that our function $H(s)$ is the same as that of ref. 19 (except for a change of sign) and consequently can be expressed in terms of Painlevé functions of the third kind. On carrying out this scaling analysis we get the result in eq. (2.22).

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Appendix A

Suzuki sum rule

Suzuki¹⁴⁾ derived the following sum rule relating the Ising model correlation function to the XY model correlation functions:

$$\langle \sigma_{nm} \sigma_{nm} \rangle_{IS} = \cosh^2 K_1^* \langle \sigma_m^x(0) \sigma_m^x(0) \rangle_{XY} - \sinh^2 K_1^* \langle \sigma_m^y(0) \sigma_m^y(0) \rangle_{XY}, \quad (A.1)$$

where the Ising model hamiltonian is

$$\mathcal{H}^I = -E_1 \sum \sigma_{n,m} \sigma_{n,m+1} - E_2 \sum \sigma_{n,m} \sigma_{n+1,m} \quad (A.2)$$

and

$$\tanh 2K_1 = (1 - \gamma^2)^{1/2} h^{-1}; \quad \cosh 2K_2^* = \gamma^{-1}, \quad (A.3)$$

where

$$K_i = E_i / k_B T \quad \text{and} \quad \tanh K_2^* = \exp(-2K_2), \quad i = 1, 2. \quad (A.4)$$

Hence, the identity (A.1) translates to

$$\begin{aligned} \langle \sigma_{00} \sigma_{0R} \rangle_{IS} = & \frac{1}{2} \gamma^{-1} [\langle \sigma_0^x(0) \sigma_R^x(0) \rangle_{XY} - \langle \sigma_0^y(0) \sigma_R^y(0) \rangle_{XY}] \\ & + \frac{1}{2} [\langle \sigma_0^x(0) \sigma_R^x(0) \rangle_{XY} + \langle \sigma_0^y(0) \sigma_R^y(0) \rangle_{XY}], \end{aligned} \quad (A.5)$$

where

$$\langle \sigma_{00} \sigma_{0R} \rangle_{IS} = S_{\infty}^{\geq} \left[\sum_{k=0}^{\infty} z_{2k+1}(R) \right] \exp \left[- \sum_{n=1} F_2^{(2n)}(R) \right], \quad (\text{A.6})$$

with

$$S_{\infty}^{\geq} = [4z_1 z_2 (1 - z_1^2)(1 - z_2^2)]^{1/2} [\sinh 2\beta E_1 \sinh 2\beta E_2]^{-2} - 1]^{1/4}, \quad (\text{A.7})$$

$$z_i = \tanh \beta E_i; \quad \beta = 1/kT; \quad i = 1, 2, \quad (\text{A.8})$$

$$\begin{aligned} z_{2k+1}(R) = & (-1)^k \gamma_1^{2k} (2\pi)^{-2(2k+1)} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{4k+2} \prod_{j=1}^{2k+1} \frac{e^{-iR\phi_{2j}}}{\Delta(\phi_{2j-1}, \phi_{2j})} \\ & \times \prod_{j=1}^{2k} \left[\frac{\sin \frac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{\sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right] \cos \frac{1}{2}(\phi_{4k+1} - \phi_1) \cos \frac{1}{2}(\phi_{4k+2} - \phi_2), \end{aligned} \quad (\text{A.9})$$

with

$$\gamma_1 = 2z_2(1 - z_1^2); \quad \gamma_2 = 2z_1(1 - z_2^2); \quad a = (1 + z_1^2)(1 + z_2^2), \quad (\text{A.10})$$

$$\Delta(\phi_{2j-1}, \phi_{2j}) = a - \gamma_1 \cos \phi_{2j-1} - \gamma_2 \cos \phi_{2j}, \quad (\text{A.11})$$

and

$$\begin{aligned} F_2^{(2n)}(R) = & (-1)^n \gamma_1^{2n} (2n)^{-1} (2\pi)^{-4n} \int_{-\pi}^{\pi} d\phi_1 \dots \int_{-\pi}^{\pi} d\phi_{4n} \\ & \prod_{j=1}^{2n} \left[\frac{e^{-iR\phi_{2j}} \sin \frac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{\Delta(\phi_{2j-1}, \phi_{2j}) \sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right]. \end{aligned} \quad (\text{A.12})$$

$\langle \sigma_0^x(0) \sigma_R^x(0) \rangle_{XY}$ and $\langle \sigma_0^y(0) \sigma_0^y(0) \rangle_{XY}$ are given by $\rho_{xx}^{\geq}(R, 0)$ and $\rho_{yy}^{\geq}(R, 0)$ of eqs. (2.7) and (2.8), respectively. We can express $F_2^{(2n)}(R, 0)$ in the following form:

$$\begin{aligned} F_2^{(2n)}(R, 0) = & (2n)^{-1} 2^{-2n} (2\pi)^{-2n} \int_{-\pi}^{\pi} d\phi_2 \int_{-\pi}^{\pi} d\phi_4 \dots \int_{-\pi}^{\pi} d\phi_{4n} \\ & \prod_{j=1}^{2n} \left[\frac{e^{-iR\phi_{2j}} (A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-)}{\Lambda_{2j} \sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right], \end{aligned} \quad (\text{A.13})$$

where

$$A_{2j}^+ = [(1 - \lambda_2 e^{i\phi_{2j}})(1 - \lambda_2 e^{-i\phi_{2j}})]^{1/2}, \quad (\text{A.14a})$$

$$A_{2j}^- = [(1 - \lambda_1^{-1} e^{i\phi_{2j}})(1 - \lambda_1^{-1} e^{-i\phi_{2j}})]^{1/2} \quad (\text{A.14b})$$

and

$$\Lambda_{2j} = A_{2j}^+ A_{2j}^-. \quad (\text{A.15})$$

Similarly,

$$x_{2k+1}(R, 0) = 2^{-2k} (2\pi)^{-(2k+1)} \int_{-\pi}^{\pi} d\phi_2 \int_{-\pi}^{\pi} d\phi_4 \dots \int_{-\pi}^{\pi} d\phi_{4k+2} \prod_{j=1}^{2k+1} \frac{e^{-iR\phi_{2j}}}{\Lambda_{2j}} \\ \times \prod_{j=1}^{2k} \left[\frac{A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-}{\sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right] \cos \frac{1}{2}(\phi_{4k+2} - \phi_2) A_2^- A_{4k+2}^- \quad (\text{A.16})$$

and

$$y_{2k+1}(R, 0) = 2^{-2k} (2\pi)^{-(2k+1)} \int_{-\pi}^{\pi} d\phi_2 \int_{-\pi}^{\pi} d\phi_4 \dots \int_{-\pi}^{\pi} d\phi_{4k+2} \prod_{j=1}^{2k+1} \frac{e^{-iR\phi_{2j}}}{\Lambda_{2j}} \\ \times \prod_{j=1}^{2k} \left[\frac{A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-}{\sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right] \cos \frac{1}{2}(\phi_{4k+2} - \phi_2) A_2^+ A_{4k+2}^+. \quad (\text{A.17})$$

To prove identity (A.5) we integrate the expression for $\langle \sigma_{00} \sigma_{0R} \rangle_{\text{IS}}$ over the odd-angle variable by evaluating them at the poles $\Delta(\phi_{2j-1}, \phi_{2j}) = 0$. We note that

$$\int_{-\pi}^{\pi} \frac{d\phi_1 f(\phi_1)}{\Delta(\phi_1, \phi_2)} = \frac{2\pi}{(1 - z_2^2) \Lambda_2} [f(\phi_1)]_{\Delta=0}, \quad (\text{A.18})$$

where $\Delta(\phi_1, \phi_2) = 0$ corresponds to the relations

$$e^{\pm i\phi_1} = \frac{1}{\gamma_1} \left\{ a - \frac{\gamma_2}{2} (e^{i\phi_2} + e^{-i\phi_2}) \pm (1 - z_2^2) \Lambda_2 \right\} \quad (\text{A.19a})$$

and

$$\cos \phi_1 = \frac{1}{\gamma_1} (a - \gamma_2 \cos \phi_2), \quad (\text{A.19b})$$

$$\sin \phi_1 = \frac{(1 - z_2^2) \Lambda_2}{i\gamma_1}. \quad (\text{A.19c})$$

We can also prove the following relations under the transformations (A.3) and the pole identities (A.19)

$$\frac{\gamma_1 \sin \frac{1}{2}(\phi_{2j-1} - \phi_{2j+1})}{1 - z_2^2} = \frac{i}{2} [A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-] \quad (\text{A.20a})$$

and

$$\frac{S_{\infty}^> \cos \frac{1}{2}(\phi_{4k+1} - \phi_1)}{1 - z_2^2} = \frac{1}{2} \bar{\rho}_{xx}(\infty) \left[\left(\frac{1 + \gamma}{\gamma} \right) A_2^- A_{4k+2}^- - \left(\frac{1 + \gamma}{\gamma} \right) A_2^+ A_{4k+2}^+ \right]. \quad (\text{A.20b})$$

Thus, integrating over the odd-angle variable ϕ_{2j+1} we have

$$F_2^{(2n)}(R) = (-1)^n \gamma_1^{2n} (2n)^{-1} (2\pi)^{-4n} \int_{-\pi}^{\pi} d\phi_2 \dots \int_{-\pi}^{\pi} d\phi_{4n} \frac{(2\pi)^{2n}}{(1 - z_2^2)^{2n}}$$

$$\times \prod_{j=1}^{2n} \left[\frac{e^{-iR\phi_{2j}}}{\Lambda_{2j} \sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \right] \left[\frac{i(1-z_2^2)}{2\gamma_1} \right]^{2n} \prod_{j=1}^{2n} [A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-] \quad (\text{A.21})$$

which, on comparing with eq. (A.15), shows that

$$F_2^{(2n)}(R) = F_2^{(2n)}(R, 0). \quad (\text{A.22})$$

Similarly,

$$\begin{aligned} z_{2k+1}(R) &= (-1)^k \gamma_1^{2k} (2\pi)^{-2(2k+1)} \int_{-\pi}^{\pi} d\phi_2 \dots \int_{-\pi}^{\pi} d\phi_{4k+2} \left(\frac{2\pi}{1-z_2^2} \right)^{2k+1} \prod_{j=1}^{2k+1} \frac{e^{-iR\phi_{2j}}}{\Lambda_{2j}} \\ &\times \prod_{j=1}^{2k} \frac{1}{\sin \frac{1}{2}(\phi_{2j} + \phi_{2j+2})} \cos \frac{1}{2}(\phi_{4k+2} - \phi_2) \left[\frac{i(1-z_2^2)}{2\gamma_1} \right]^{2k} \\ &\times \prod_{j=1}^{2k} [A_{2j}^+ A_{2j+2}^- - A_{2j+2}^+ A_{2j}^-] \frac{(1-z_2^2) \bar{\rho}_{xx}(\infty)}{2S_\infty^>} \\ &\times \left[\left(\frac{1+\gamma}{\gamma} \right) A_2^- A_{4k+2}^- - \left(\frac{1-\gamma}{\gamma} \right) A_2^+ A_{4k+2}^+ \right] \end{aligned} \quad (\text{A.23})$$

which, on comparing with eqs. (A.16) and (A.17), shows that

$$S_\infty^> z_{2k+1}(R) = \frac{1}{2} \bar{\rho}_{xx}(\infty) \{ \gamma^{-1} [x_{2k+1}(R, 0) - y(R, 0)] + [x_{2k+1}(R, 0) + y_{2k+1}(R, 0)] \}. \quad (\text{A.24})$$

From eqs. (A.22) and (A.24) the sum rule (A.5) follows.

Appendix B

Define

$$\hat{\rho}(k, z) = \int_{-\infty}^{\infty} d\bar{R} \int_0^{\infty} d\bar{t} e^{ik\bar{R} + iz\bar{t}} \hat{F}_x(s), \quad s = (\bar{R}^2 - \bar{t}^2)^{1/2}. \quad (\text{B.1})$$

Then,

$$\bar{\rho}(k, z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{\tilde{\rho}(k, \omega)}{\omega - z}; \quad \text{Im } z > 0. \quad (\text{B.2})$$

Now

$$\hat{\rho}(k, z) = \int_{-\infty}^{\infty} d\bar{R} \int_0^{i\infty} dM e^{ik\bar{R} + izM} \hat{F}_x(s) \quad (\text{B.3})$$

$$= i \int_{-\infty}^{\infty} d\bar{R} \int_0^{\infty} dM e^{ik\bar{R} - zM} \hat{F}_x(r), \quad r = (\bar{R}^2 + M^2)^{1/2}. \quad (\text{B.4})$$

Hence,

$$\hat{\rho}(k, iz) + \hat{\rho}(k, -iz) = i \int_{-\infty}^{\infty} d\bar{R} \int_{-\infty}^{\infty} dM e^{ik\bar{R} - izM} \hat{F}_x(r). \quad (B.5)$$

Setting

$$\bar{R} = r \cos \phi; \quad M = r \sin \phi; \quad k = \epsilon \cos \psi; \quad z = \epsilon \sin \psi,$$

with $\epsilon = (k^2 + z^2)^{1/2}$,

$$\hat{\rho}(k, iz) + \hat{\rho}(k, -iz) = i \int_0^{\infty} dr r \int_0^{2\pi} d\phi e^{ire \cos \phi} \hat{F}_x(r) \quad (B.6)$$

$$= 2\pi i \int_0^{\infty} dr r J_0(r\epsilon) \hat{F}_x(r). \quad (B.7)$$

The right-hand side is the Fourier transform of the two dimensional Ising model two-point function as a function of the wave number ϵ . It has the spectral representation¹⁶⁾

$$\int_0^{\infty} dr r J_0(r\epsilon) \hat{F}_x(r) = \int_0^{\infty} \frac{\rho_{\pm}^{\text{IS}}(x) dx}{x + \epsilon^2}, \quad (B.8)$$

where $+$ ($-$) refers to $T > T_c$ ($T < T_c$) [in our case $h > 1$ ($h < 1$)]. Using eq. (B.2) and after some algebraic manipulations we see that

$$\tilde{\rho}_{XY}(\epsilon) = \rho_{\pm}^{\text{IS}}(\epsilon). \quad (B.9)$$

Note added in proof

Lajzerowicz and Pfeuty²⁶⁾ have shown that

$$\rho_{yy}(R, t) = -\frac{1}{h^2} \frac{\partial^2}{\partial t^2} \rho_{xx}(R, t)$$

for the transverse Ising model [$\gamma = 1$ in eq. (1.1)]. In the scaling region A, the $\gamma = 1$ restriction can be removed; and hence, the above formula can be used to derive $\rho_{yy}(\bar{R}, \bar{t})$ in scaling region A.

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