# PAINLEVÉ TRANSCENDENTS AND SCALING FUNCTIONS OF THE TWO-DIMENSIONAL ISING MODEL†‡

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## 1. INTRODUCTION

In this lecture I should like to report on the work I have done in collaboration with Barry McCoy and Tai Tsun Wu (and on certain aspects with Eytan Barouch) on the correlation functions of the two-dimensional Ising model. In particular, I wish to demonstrate how a particular solution of the two-dimensional hyperbolic sine-Gordon equation (also known as the two-dimensional non-linear Debye-Hückel equation),

$$\Delta\phi = \sinh \phi, \quad (r > 0), \tag{1.1}$$

plays a fundamental role in the scaled two-point function in both the one-phase and two-phase regions. Before I discuss these results, I would like first to review the definition of the 2-d. Ising model[1)-4)] and related quantities (Section 2); and then briefly recall the scaling theory hypothesis[5)-7)] for correlation functions (Section 3).

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*A. O. Barut (ed.), Nonlinear Equations in Physics and Mathematics, 221–237.* 

## 2. TWO-DIMENSIONAL ISING MODEL

Consider a two-dimensional square lattice of $\mathcal{M}$ rows and $\mathcal{N}$ columns and at each lattice site, which we label with the pair of indices (i,j), we define a variable $\sigma_{i,j}$ that can assume the values of ±1. The two-dimensional Ising model with nearest neighbor interactions on this square lattice is specified by the energy of interaction

$$\mathcal{E} = -E_1 \sum_{j,k} \sigma_{j,k}\sigma_{j,k+1} - E_2 \sum_{j,k} \sigma_{j,k}\sigma_{j+1,k}. \tag{2.1}$$

The partition function $Z_{\mathcal{M}\mathcal{N}}(\beta)$ is

$$Z_{\mathcal{M}\mathcal{N}}(\beta) = \sum_{\{\sigma_{i,j}=\pm\}} \exp(-\beta\mathcal{E}) \tag{2.2}$$

and is related to the free energy per lattice site by

$$-\beta f(\beta) = \lim_{\mathcal{M},\mathcal{N}\to\infty} \frac{1}{\mathcal{M}\cdot\mathcal{N}} \log Z_{\mathcal{M}\mathcal{N}}(\beta) \tag{2.3}$$

where $\beta = (k_B T)^{-1}$, T = temperature and $k_B$ is Boltzmann's constant.

This free energy, $f(\beta)$, was computed exactly by Onsager[2] who observed that there is a singularity at a temperature $T_c$ determined from the equation

$$\sinh 2\beta E_1 \ \sinh 2\beta E_2 = 1. \tag{2.4}$$

Strictly speaking, to interpret this temperature $T_c$ as the critical temperature, the spontaneous magnetization $\mathcal{M}_s(T)$ must be analyzed. This quantity was known to Onsager,[8] but the first published derivation of $\mathcal{M}_s(T)$ was given by Yang[3] who showed

$$\mathcal{M}_s(T) = \left[1 - (\sinh 2\beta E_1 \ \sinh 2\beta E_2)^{-2}\right]^{1/8}. \tag{2.5}$$

Actually, Yang considered the special case of the symmetric lattice $E_1 = E_2 = E$ (as we do from here on) and the general case $E_1 \neq E_2$ was derived by Chang.[9] The important point to note is $\mathcal{M}_s(T)$ goes continuously to zero as T approaches the temperature $T_c$ determined by (2.4). This approach to zero is proportional to

$(1-T/T_c)^{1/8}$ and the 1/8 defines one of the critical exponents[5)-7)] of this model.

The correlation functions are defined as follows: Consider n points in the lattice

$$M_1, N_1; M_2, N_2; \cdots; M_n, N_n, \tag{2.6}$$

then the n-point function (in the thermodynamic limit) is

$$\langle \sigma_{M_1N_1}\sigma_{M_2N_2} \cdots \sigma_{M_nN_n} \rangle =$$

$$= \lim_{\substack{\mathcal{M} \to \infty \\ \mathfrak{n} \to \infty}} \frac{\Sigma\, \sigma_{M_1N_1}\sigma_{M_2N_2} \cdots \sigma_{M_nN_n} e^{-\beta\mathcal{E}}}{\Sigma\, e^{-\beta\mathcal{E}}}$$

where the sums are over all configurations, i.e., $\{\sigma_{i,j} = \pm 1\}$.

## 3. SCALING LIMIT AND SCALING FUNCTIONS

The scaling limit[5)-7)] of the pair correlation function is the limit

$$R \to \infty, \quad \xi \to \infty \quad (T \to T_c), \quad \text{such that } x = R/\xi \text{ is fixed} \tag{3.1}$$

where R is the radial distance $[=(M^2+N^2)^{\frac{1}{2}}$ for the symmetric case $E_1 = E_2$ of the previous section], $\xi = \xi(T)$ is the correlation length which goes to infinity as $T \to T_c^{\pm}$ [for the 2-d. Ising model[4)] $\xi(T)$ diverges as $(1-T/T_c)^{-1}$] and the variable x is called a scaling variable. The scaling theory hypothesis[5)-7)] is the assertion that the pair correlation function $\langle \sigma_{0,0}\sigma_{M,N} \rangle$ (which is in general a function of M, N, and T) assumes in the scaling limit the scaling form

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle \sim \mathcal{M}^2 \hat{F}_{\pm}(x) \tag{3.2}$$

where $\mathcal{M} = |1-\sinh^{-4} 2\beta E|^{1/8}$ and the functions $\hat{F}_{\pm}(x)$ are called scaling functions. The + (–) sign denotes that $T \to T_c$ from above (below). The reader is referred to the articles by Fisher[5)] and by Kadanoff et al.[6),7)] for a complete discussion.

An alternative formulation of the scaling hypothesis can be given in momentum space: Let

$$\chi(\vec{k},T) \equiv \sum_{M=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \left[\langle \sigma_{0,0}\sigma_{M,N}\rangle - \mathcal{M}_s^2(T)\right] e^{i\vec{k}\cdot\vec{R}} \tag{3.3}$$

then the scaling limit is

$$k \to 0, \quad \xi \to \infty \quad \text{such that} \quad p = k\xi \quad \text{is fixed}$$

and the scaling theory hypothesis (again we restrict the statement of the hypothesis to the 2-d. Ising model) is the assumption that $\chi(\vec{k},T)$ in this limit becomes

$$\chi(\vec{k},T) \sim c\,\xi^{7/4} G_{\pm}(p^2) \tag{3.4}$$

where c is some lattice dependent constant.

The relation between $\hat{F}_{\pm}(x)$ and $G_{\pm}^{(2)}(p^2)$ is

$$G_{+}^{(2)}(p^2) = 2\pi \int_0^\infty dx\, x\, J_0(xp)\hat{F}_{+}(x)$$

and (3.5)

$$G_{-}^{(2)}(p^2) = 2\pi \int_0^\infty dx\, x\, J_0(xp)[\hat{F}_{-}(x) - 1]$$

where $J_0(x)$ is the zeroth order Bessel function. For a discussion of the connection of $G_{\pm}^{(2)}(p^2)$ to critical scattering experiments, see Reference 10 and the references contained therein.

## 4. EXPLICIT FORMULAS FOR $\hat{F}_{\pm}(x)$

For the case of the two-dimensional Ising model on a square lattice, the scaling hypothesis (3.2) has been verified and explicit expressions have been obtained for $\hat{F}_{\pm}(x)$.[11)-15)] The generalization of Reference 12 to the triangular lattice has been given by Vaidya.[16)] We now present these results:

### 4.1 Result No. 1

(a) $$\hat{F}_{-}(x) = \exp(f_2) \tag{4.1a}$$

$$f_2(x) = \sum_{n=1}^{\infty} f_2^{(2n)}(x)\lambda^{2n}, \qquad \lambda = \pi^{-1}, \tag{4.1b}$$

with

$$f_2^{(2n)}(x) = \frac{(-1)^{n+1}}{n} \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2n} \prod_{j=1}^{2n} \cdot \frac{e^{-xy_j}}{\sqrt{y_j^2-1}} \frac{1}{y_j+y_{j+1}} \prod_{j=1}^{n} (y_{2j}^2-1) \tag{4.1c}$$

where $y_{2n+1} \equiv y_1$.

(b) $$\hat{F}_+(x) = G(x)\hat{F}_-(x) \tag{4.2a}$$

$$G(x) = \sum_{n=0}^{\infty} g_{2n+1}(x)\lambda^{2n+1}, \qquad \lambda = \pi^{-1} \tag{4.2b}$$

with

$$g_1(x) = \int_1^\infty dy \frac{e^{-xy}}{\sqrt{y^2-1}} = K_0(x) \tag{4.2c}$$

$$g_{2n+1}(x) = (-1)^n \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2n+1} \cdot \left[\prod_{j=1}^{2n+1} \frac{e^{-xy_j}}{(y^2-1)^{\frac{1}{2}}}\right]\left[\prod_{j=1}^{2n} (y_j+y_{j+1})^{-1}\right] \cdot \left[\prod_{j=1}^{n} (y_{2j}^2-1)\right]. \tag{4.2d}$$

For large-x we have

$$f_2^{(2n)}(x) \sim c_n \frac{e^{-2nx}}{x^{2n}} \qquad (x \to \infty) \tag{4.3a}$$

and

$$g_{2n+1}(x) \sim c_n \frac{e^{-(2n+1)x}}{x^{n+\frac{1}{2}}} \qquad (x \to \infty) \tag{4.3b}$$

and for small-x we have

$$f_2^{(2n)}(x) \sim c_{2n}(\ell n x)^{2n} + c_{2n-1}(\ell n x)^{2n-1}$$

$$+ \cdots + c_1(\ell n x) + c_0 + o(1) \quad (x \to 0) \tag{4.4a}$$

and

$$g_{2n+1}(x) \sim c_{2n+1}(\ell n x)^{2n+1} + \cdots + c_1(\ell n x) + c_0 + o(1) \quad (x \to 0), \tag{4.4b}$$

where we have used the same symbol '$c_n$' to denote the various different constants.

The above representations for $\hat{F}_{\pm}(x)$ are most easily interpreted in momentum space [recall (3.5)]. If we look in the complex $p^2$-plane, then the propagator $G_+^{(2)}(p^2)$ has a single-particle pole at $p^2 = -1$ (in statistical mechanics this is referred to as the Ornstein-Zernike pole) and has continuum thresholds (which are square root type branch points) at $p^2 = -3^2, -5^2, -7^2, \cdots$. On the other hand, the propagator $G_-^{(2)}(p^2)$ in the two-phase region has only branch points which are located at $p^2 = -2^2, -4^2, -6^2, \cdots$. Thus, for example, the function $[f_4^{(2)} + \frac{1}{2}(f_2^{(2)})^2]$ when used in (3.5) gives the four-particle contribution to $G_-^{(2)}(p^2)$.

That these representations provide a rapidly convergent expansion for $G^{(2)}(p^2)$ for small $p^2$ can best be illustrated by comparing the exact value of $G_{\pm}^{(2)}(p^2)$ at $p^2 = 0$ with the contribution coming from the low-lying excitations. If we denote by $G_{+,2n+1}^{(2)}(p^2)$ the contribution to $G_+^{(2)}(p^2)$ coming from the (2n+1)-particle cut and by $G_{-,2n}^{(2)}(p^2)$ the analogous contribution to $G_-^{(2)}(p^2)$, we have

$$G_+^{(2)}(p^2) = \sum_{n=0}^{\infty} G_{+,2n+1}^{(2)}(p^2)$$

and (4.5)

$$G_-^{(2)}(p^2) = \sum_{n=1}^{\infty} G_{+,2n}^{(2)}(p^2).$$

Using (4.1) and (4.2) in (3.5), we can show[18)]

$$G^{(2)}_{+,1}(0) = 2,$$

$$G^{(2)}_{+,3}(0) = \frac{1}{\pi^2}\left\{\tfrac{1}{3}\pi^2 + 2 - 3\sqrt{3}\, C\ell_2(\pi/3)\right\}, \qquad (4.6)$$

$$G^{(2)}_{-,2}(0) = \frac{1}{6\pi},$$

and

$$G^{(2)}_{-,4}(0) = \frac{1}{8\pi^3}\left\{\frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{2}\,\zeta(3)\right\}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad C\ell_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}.$$

In particular, $\zeta(3) = 1.2020569031 \cdots$ and $C\ell_2(\pi/3) = 1.0149417 \cdots$ .[19] We now compare the LHS of (4.5) at $p^2 = 0$ to the first two terms of the RHS:

$$G^{(2)}_{+}(0) = 2.001\ 630\ 521$$

$$G_{+,1}(0) = 2.0$$

$$G^{(2)}_{+,1}(0) + G^{(2)}_{+,3}(0) = 2.001\ 628\ 925 \cdots \qquad (4.7)$$

$$G^{(2)}_{-}(0) = .053\ 102\ 589 \cdots$$

$$G^{(2)}_{-,2}(0) = .053\ 051\ 648 \cdots$$

$$G^{(2)}_{-,2}(0) + G^{(2)}_{-,4}(0) = .053\ 102\ 545 \cdots . \qquad (4.8)$$

The representations given above (Result No. 1) are not so useful if we wish to examine the short-distance behavior of $\hat{F}_{\pm}(x)$. That this is the case follows directly from (4.4a) and (4.4b), where we see the short distance behavior of the functions $f_2^{(2n)}(x)$ and $g_{2n+1}(x)$ have ever increasing powers of logarithms. We will see that $\hat{F}_{\pm}(x) \sim Cx^{-\frac{1}{4}}$ as $x \to 0$ so the logarithms must sum up to an algebraic power. This 'summing up' feature of logarithms is not unusual in quantum field theory. However, we point out that viewed in this language of summing logarithms, one must sum all logarithms not just leading logs.

To study the short distance behavior of $\hat{F}_{\pm}(x)$, we do not actually sum all these logarithms 'by hand.' Rather, we find that there is an underlying nonlinear differential equation whose solution, roughly speaking, sums the series representations (4.1) and (4.2). We now discuss this nonlinear differential equation. Since this workshop is on nonlinear differential equations, we present our results in their most general form and then specialize them in Section 6, when we apply these solutions to the Ising model.

## 5. PAINLEVÉ TRANSCENDENTS

The Painlevé equation of the third kind is

$$\frac{d^2w}{d\theta^2} = \frac{1}{w}\left(\frac{dw}{d\theta}\right)^2 - \frac{1}{\theta}\frac{dw}{d\theta} + \frac{1}{\theta}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \tag{5.1}$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are constants. The importance of (5.1) in in the theory of nonlinear equations is discussed in the papers by Painlevé[20] and Gambier[21] and in the book by Ince.[22] If we make the restriction

$$\alpha\sqrt{-\delta} + \beta\sqrt{\gamma} = 0 \tag{5.2}$$

on the constants appearing in (5.1), then (5.1) is easily reducible to

$$\frac{d^2w}{d\theta^2} = \frac{1}{w}\left(\frac{dw}{d\theta}\right)^2 - \frac{1}{\theta}\frac{dw}{d\theta} + \frac{2\nu}{\theta}(w^2-1) + w^3 - \frac{1}{w} \tag{5.3}$$

where $\nu$ is a constant. We call (5.3) the restricted Painlevé Equation of the third kind.

Let us denote by $\eta(\theta;\nu,\lambda)$ the one-parameter family of solutions of (5.3) that remains bounded as $\theta \to \infty$ along the positive real $\theta$-axis. Then we have[14]

### 5.1 Result No. 2

For sufficiently large positive $\theta$ and $\mathrm{Re}\,\nu > -\frac{1}{2}$, the function $\eta(\theta;\nu,\lambda)$ defined above has the representation

$$\frac{1-\eta(\theta;\nu,\lambda)}{1+\eta(\theta;\nu,\lambda)} = G(x;\nu,\lambda) = \sum_{n=0}^{\infty} \lambda^{2n+1} g_{2n+1}(x;\nu) \tag{5.4}$$

where $2\theta = x$

$$g_1(x;\nu) = \int_1^{\infty} dy \, \frac{e^{-xy}}{(y^2-1)^{\frac{1}{2}}} \left(\frac{y-1}{y+1}\right)^{\nu}, \tag{5.5a}$$

and for $n \geqq 1$

$$g_{2n+1}(x;\nu) = (-1)^n \int_1^{\infty} dy_1 \cdots \int_1^{\infty} dy_{2n+1} \left[ \prod_{j=1}^{2n+1} \frac{e^{-xy_j}}{(y_j^2-1)^{\frac{1}{2}}} \cdot \left(\frac{y_j-1}{y_j+1}\right)^{\nu} \right] \left[ \prod_{j=1}^{2n} (y_j+y_{j+1})^{-1} \right] \left[ \prod_{j=1}^{n} (y_{2j}^2-1) \right]. \tag{5.5b}$$

If we define $\psi(x;\nu,\lambda)$ by

(i) $\eta(\theta;\nu,\lambda) = e^{-\psi(x;\nu,\lambda)}, \quad x = 2\theta,$

(ii) $\psi(x;\nu,\lambda) \to 0 \quad \text{as} \quad x \to +\infty,$

then $\psi(x;\nu,\lambda)$ satisfies the differential equation

$$\psi'' + x^{-1}\psi' = \tfrac{1}{2}\sinh(2\psi) + 2\nu x^{-1}\sinh(\psi). \tag{5.6}$$

Furthermore, $\psi(x;\nu,\lambda)$ has the representation:

5.2 Result No. 3

$$\psi(x;\nu,\lambda) = \sum_{n=0}^{\infty} \lambda^{2n+1} \psi_{2n+1}(x;\nu) \tag{5.7}$$

with

$$\psi_1(x;\nu) = 2g_1(x;\nu) \tag{5.8a}$$

$$\psi_{2n+1}(x;\nu) = \frac{2}{2n+1}\int_1^\infty dy_1 \cdots \int_1^\infty dy_{2n+1}\left[\prod_{j=1}^{2n+1} \cdot \frac{e^{-xy_j}}{y_j+y_{j+1}}\right]\left[\prod_{j=1}^{2n+1}\left(\frac{y_j-1}{y_{j+1}}\right)^{\nu-\frac{1}{2}} + \prod_{j=1}^{2n+1}\left(\frac{y_j-1}{y_j+1}\right)^{\nu+\frac{1}{2}}\right] \tag{5.8b}$$

Notice that both $g_{2n+1}(x;\nu)$ and $\psi_{2n+1}(x;\nu)$ are in the form of iterated integrals: Define the linear operator $\mathcal{K}$

$$(\mathcal{K}f)(x) = \int_1^\infty d\sigma_\pm(y)\, e^{-\theta(x+y)}(x+y)^{-1} f(y) \tag{5.9a}$$

where the measure $d\sigma_\pm$ is

$$d\sigma_\pm = d\sigma_\pm(y) = \left(\frac{y-1}{y+1}\right)^{\nu\pm\frac{1}{2}} dy. \tag{5.9b}$$

Consider the eigenfunctions and eigenvalues $\phi_j$ and $\lambda_j$,

$$(\mathcal{K}\phi_j^\pm)(x) = \lambda_j^\pm(\theta,\nu)\phi_j^\pm(x;\theta,\nu). \tag{5.10}$$

Then we can rewrite (5.8b) as

$$\psi_{2n+1}(x;\nu) = \frac{2}{2n+1}\int_0^\infty d\zeta\left[(e,\mathcal{K}^{2n}e)_+ + (e,\mathcal{K}^{2n}e)_-\right] \tag{5.11}$$

where the vector $|e\rangle$ is

$$\langle y|e\rangle = e^{-(\zeta+\theta)y}$$

and the scalar product $(\ ,\ )_\pm$ is

$$(g,f)_\pm = \int_1^\infty d\sigma_\pm(y)\,\overline{g(y)}\,f(y).$$

Using (5.11) in (5.7) and recalling $\eta = e^{-\psi}$, we have

$$\eta(\theta;\nu,\lambda) = \prod_{j=1}^{\infty}\left(\frac{1-\lambda_j^+\lambda}{1+\lambda_j^+\lambda}\right)^{a_j^+}\prod_{j=1}^{\infty}\left(\frac{1-\lambda_j^-\lambda}{1+\lambda_j^-\lambda}\right)^{a_j^-} \tag{5.12}$$

where

$$a_j^{\pm} = a_j^{\pm}(\theta,\nu) = (\lambda_j^{\pm})^{-1}\int_\theta^\infty d\zeta \left|\int_1^\infty d\sigma_{\pm}(y)e^{-\zeta y}\phi_j^{\pm}(y;\theta,\nu)\right|^2 . \tag{5.13}$$

The representation (5.12) clearly displays the behavior of $\eta(\theta;\nu,\lambda)$ in the complex $\lambda$-plane (recall $\lambda$ is the integration constant parameter). It is an open problem to find explicit formulas for $\lambda_j^{\pm}$ and $\phi_j^{\pm}(x)$.

If we examine the small distance behavior of the functions $\psi_{2n+1}(x;\nu)$ as given by (5.8), we find[14)]

$$\psi_{2n+1}(x;\nu) = \sigma_{2n+1}\ln\left(\frac{1}{x}\right) + B_{2n+1} + o(1) \quad (x\to 0^+) \tag{5.14}$$

That is to say, the logarithms do not increase in order when we go to the $\psi$-representation. Thus

$$\begin{aligned}\psi(x;\nu,\lambda) &= \sum_{n=0}^{\infty}\lambda^{2n+1}\psi_{2n+1}(x;\nu)\\ &= \sigma\ln\left(\frac{1}{x}\right) + B + o(1) \quad (x\to 0^+)\end{aligned} \tag{5.15}$$

with

$$\sigma = \sum_{n=0}^{\infty}\lambda^{2n+1}\sigma_{2n+1}$$

and

$$\ln B = -\sum_{n=0}^{\infty}\lambda^{2n+1}B_{2n+1} . \tag{5.16}$$

We expect (5.15) to be valid whenever the series expansions for $\sigma$ and B converge. There is a very simple physical argument for the case $\nu=0$ that gives (5.15). For $\nu=0$ the nonlinear differential equation (5.6) is essentially the spherically symmetric version of (1.1), the nonlinear Debye-Hückel equation. In Debye-Hückel theory $\phi(x)$ is the electrostatic potential of the field of the ion cloud surrounding the test charge at the origin. As $x \to 0$, we must see the bare test charge that is located at the origin. Hence the factor $\sigma \ln(1/x)$. The quantity B represents the potential due to all other ions of the cloud at the position of the test charge.

The mathematical problem is to extract from (5.8) the coefficients $\sigma_{2n+1}$ and $B_{2n+1}$ and then sum the resulting series (5.16). We have[14)]

5.3 Result No. 4

$$\sigma = \sigma(\lambda) = \frac{2}{\pi} \arcsin(\pi\lambda)$$

$$B = B(\sigma, \nu)$$

$$= 2^{-3\sigma} \frac{\Gamma^2((1-\sigma)/2)}{\Gamma^2((1+\sigma)/2)} \frac{\Gamma(((1+\sigma)/2)+\nu)}{\Gamma(((1-\sigma)/2)+\nu)}. \tag{5.17}$$

We notice that $\sigma$ does not depend upon $\nu$. This is easy to understand since the logarithm in (5.14) comes from the region of integration variables $y_j$ large in (5.8). In this region $((y_j-1)/(y_j+1))^\nu \sim 1$. Also we see that $\lambda = 1/\pi$ $(\sigma = 1)$ plays a distinguished role in these formulas. It is precisely $\lambda = 1/\pi$ that is needed in the Ising model.

Using Result No. 4, we can determine the behavior of $\eta(x/2;\nu,\lambda)$ at $x=0$:

5.3.1 $0 < \lambda < 1/\pi$

$$\eta(x/2;\nu,\lambda) = Bx^\sigma \Big\{ 1 - \nu B^{-1}(1-\sigma)^{-2} x^{1-\sigma} + B\nu(1+\sigma)^{-2} x^{1+\sigma} + \left[\frac{1}{4}\nu^2 B^{-2}(1-\sigma)^{-4} - \frac{1}{16} B^{-2}(1-\sigma)^{-2}\right] \cdot x^{2-2\sigma} + 0(x^2) \Big\} . \tag{5.18}$$

5.3.2 $\lambda = 1/\pi$

$$\eta(x/2;\nu,\pi^{-1}) \sim \frac{1}{2}x\left[\nu \ln^2 x - C(\nu)\ln x + \frac{1}{4\nu}(C^2(\nu)-1)\right]$$

where

$$C(\nu) = 1 + 2\nu\left[3\ln 2 - 2\gamma - \psi(\nu+1)\right] \tag{5.19}$$

and $\psi(x) = (d/dx)\ln\Gamma(x)$, $\Gamma(x)$ being the gamma function.

5.3.3 $\lambda > 1/\pi$ (for simplicity we set $\nu = 0$)

$$\eta(x/2;0,\lambda) \sim -\frac{1}{4\mu}x\sin[2\mu \ln(x/8) + 2\phi(\mu)]. \tag{5.20}$$

with $\sigma = 1+2i\mu$ and

$$\Gamma(iy) = |\Gamma(iy)|e^{i\phi(y)}$$
$$= \left(\frac{\pi}{y\sinh y}\right)^{\frac{1}{2}} e^{i\phi(y)}.$$

5.3.4 For $\lambda < 0$ use

$$\eta(x/2;\nu,-\lambda) = \frac{1}{\eta(x\ 2;\nu,\lambda)}. \tag{5.21}$$

In Case 5.3.3, the origin $x=0$ is the limit point of a set of zeros and in Case 5.3.4 for $\lambda < -(1/\pi)$, the origin is the limit point of a set of poles on the positive real axis. The asymptotic spacing of these zeros and poles follows from the above formulas. These zeros and poles are intimately connected with the eigenvalues $\lambda_j^{\pm}(\theta;\nu)$ as can be seen by recalling (5.12).

## 6. $\hat{F}_{\pm}(x)$ IN TERMS OF $\psi(x;0,\pi^{-1})$

Comparing Result No. 1 with Result No. 2, we see

$$\frac{\hat{F}_{+}(x)}{\hat{F}_{1}(x)} = \frac{1-\eta(x/2;0,\pi^{-1})}{1+\eta(x/2;0,\pi^{-1})} = \tanh[\tfrac{1}{2}\psi(x;0,\pi^{-1})]. \tag{6.1}$$

This relates the ratio of $\hat{F}_+(x)$ to $\hat{F}_-(x)$ to the Painlevé function $\eta(x/2;0,\pi^{-1})$. We still need a relation of $\hat{F}_\pm(x)$ to $\eta(x/2;0,\pi^{-1})$ [or $\psi(x;0,\pi^{-1})$]. This we have in the next result.[14]

6.1 Result No. 5

$$\hat{F}_-(x) = \exp\left(\sum_{n=1}^{\infty} \lambda^{2n} f_2^{(2n)}(x)\right)$$

$$= \cosh \tfrac{1}{2}\psi(x,0,\lambda) \exp\left[-\frac{1}{4}\int_x^\infty dr\, r\, \mathcal{L}(r)\right] \tag{6.2}$$

where

$$\mathcal{L}(r) = \left(\frac{d\psi}{dr}\right)^2 - \sinh^2\psi \tag{6.3}$$

the $\psi$ being $\psi(r;0,\lambda)$.

From this representation and the short-distance behavior of $\psi(x;0,\lambda)$ of the previous section, we have[12] for $\lambda = 1/\pi$

$$\hat{F}_\pm(x) = Cx^{-\frac{1}{4}}\left\{1 \pm x\Omega + \frac{1}{16}x^2 \pm \frac{1}{32}x^3\Omega + \frac{1}{256}x^4(-\Omega^2+\Omega+\tfrac{1}{8}) + 0(x^5\Omega^4)\right\} \tag{6.4}$$

with $\Omega = \ell n(x/8)+\gamma$, $\gamma$ = Euler's constant.

Some comments:

(1) The RHS of (6.2) looks almost like an action. If one varies the $\mathcal{L}(r)$ in (6.3), one gets

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = -\tfrac{1}{2}\sinh(2\psi) \tag{6.5}$$

which disagrees with (5.6) by a minus sign.
What is the correct interpretation of the RHS of (6.2)?

(2) As Result No. 5 is stated the identity holds for all $\lambda$ though its application to the Ising model is only for $\lambda = 1/\pi$. Does the variable $\lambda$ have a physical interpretation? If we keep $\lambda$ as a variable and compute the short distance behavior of $\hat{F}_\pm(x;\lambda)$, we find that the anamolous dimension is now a function of $\lambda$. This suggests the Baxter model.[23]

(3) In Reference 14, (6.2) is generalized to $\nu \neq 0$ in which case $\mathcal{L}(r)$ of (6.3) becomes

$$\mathcal{L}(r) = \left(\frac{d\psi}{dr}\right)^2 - \sinh^2\psi - \frac{4\nu}{r}\sinh^2\tfrac{1}{2}\psi. \tag{6.6}$$

Recently Ablowitz and Segur[24] have shown a deep connection of the Painlevé transcendent of second kind

$$\frac{d^2W}{dz^2} = zW + 2W^3 \tag{6.7}$$

and the long-time behavior of the modified KdV equation. Equation (6.7) is a special case[25] of Result No. 2 (or Result No. 3) and thus the solution that is needed in the modified KdV analysis is known.[25] Ablowitz and Segur[26] have also obtained this same solution to (6.7) using the inverse spectral transform method. Does the generalized $\nu \neq 0$ identity (6.2) when analyzed in the KdV limit have any applications?

## 7. n-POINT FUNCTIONS

The generalization of Result No. 1 to the n-point functions has been recently given by a number of authors.[27)-29] We will not write down the formulas here but merely note the form of the answer in the scaling limit: First for $T \to T_c^-$

$$\lim \mathcal{M}^{-n}\langle\sigma_{M_1N_1}\cdots\sigma_{M_nN_m}\rangle = \exp(f_n)$$

$$f_n = \sum_{k=2}^{\infty} f_n^{(k)}$$

where $f_n^{(k)}$ is a 2k-dimensional integral. For $T \to T_c^+$

$$\lim \mathcal{M}^{-n}\langle\sigma_{M_1N_1}\cdots\sigma_{M_nN_n}\rangle = g_n\exp(f_n)$$

$$g_n = |\det g_{(n)ij}|^{\frac{1}{2}}$$

$$g_{(n)ij} = \sum_{k=1}^{\infty} g_{(n)ij}^{(k)},$$

where $g^{(k)}_{(n)ij}$ are 2k-dimensional integrals.

We point out that the integrals $f^{(k)}_n$ and $g^{(k)}_{(n)ij}$ require special care in treating singularities of the integrand. This point is discussed in References 27 and 28, but the integrals appearing in Reference 29 are ambiguous since no prescription for interpreting the singular integrals is given.

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Note Added in Proof

For some new results, see B. M. McCoy and T. T. Wu, Phys. Lett. 72B, 219 (1977); R. Z. Bariev, 64A, 169 (1977); D. Wilkinson, to appear in Phys. Rev. D; and R. Haberman, Stud. Appl. Math. 57, 247 (1977).