CROSSOVER SCALING FUNCTION FOR THE ONE-DIMENSIONAL *XY* MODEL AT ZERO TEMPERATURE

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We compute the time-dependent spin–spin correlation function $\langle \sigma_1^x(0)\, \sigma_{R+1}^x(\tau)\rangle$ at zero temperature for the one-dimensional *XY* model in the double scaling limit $R\to\infty$, $h\to 1^-$, $\gamma\to 0^+$, $\tau\to\infty$, such that the scaling variables $(1-h^2)^{1/2}R$, γR, and $\frac{1}{2}(1-h^2)\tau$ are fixed. Here γ is the anisotropy parameter in the spin hamiltonian and h is the external magnetic field in the z-direction.

Of all the one-dimensional spin systems that have been studied [1], the simplest and best understood quantum model is the spin $\frac{1}{2}$ *XY* model [2]. This spin system for N spins located on a line is defined by the hamiltonian

$$H_N = -\sum_{j=1}^{N}\left((1+\gamma)\, S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y + h S_j^z\right) \tag{1}$$

where $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$, $\alpha = x, y, z$, and σ_i^α are the usual Pauli matrices, γ is the anisotropy parameter which we take to be $0 \leqslant \gamma \leqslant 1$, and h is the magnetic field in the z-direction. We consider the case of periodic boundary conditions and thus take $S_{N+1}^\alpha \equiv S_1^\alpha$. At zero temperature it is known [3] that there exists spontaneous magnetization in the x-direction:

$$M_x = [2(1+\gamma)]^{-1/2}\,\gamma^{1/4}(1-h^2)^{1/8} \tag{2}$$

which clearly vanishes for either $\gamma\to 0^+$ or $h\to 1^-$.

The large R and large τ behavior of $\langle S_1^x(0) S_{1+R}^x(\tau)\rangle$ was studied in refs. [3] and [4], and a description of the scaling behavior near the critical points $\gamma = 0$ and $h = 1$ was given in ref. [5]. To be precise, the correlation function $\langle S_1^x(0)\, S_{1+R}^x(\tau)\rangle$ was computed [5] in the two distinct scaling limits:

Region A: For fixed $\gamma > 0$ we let $R\to\infty$, $h\to 1^-$, $\tau\to\infty$ such that $r_{\mathrm{A}} = (1-h)\gamma^{-1}R$ and $t_{\mathrm{A}} = (1-h)\tau$ are fixed and are of order one.

Region B: For fixed $h < 1$ we let $R\to\infty$, $\gamma\to 0^+$, $\tau\to\infty$ such that $r_{\mathrm{B}} = \gamma R$ and $t_{\mathrm{B}} = (1-h^2)^{1/2}\gamma\tau$ are fixed and are of order one.

In region A (B) $\langle S_1^x(0)\, S_{1+R}^x(\tau)\rangle$ is expressible in terms of a scaling function of the variables $r_{\mathrm{A}} (r_{\mathrm{B}})$ and $t_{\mathrm{A}} (t_{\mathrm{B}})$ [5] [‡1].

However, this is not the complete description of the scaling behavior of $\langle S_1^x(0)\, S_{1+R}^x(\tau)\rangle$. In particular, one cannot go continuously from the scaling function of Region A to the scaling function of Region B (or vice versa). To be able to study the transition from one region to the other, we must analyze $\langle S_1^x(0)\, S_{1+R}^x(\tau)\rangle$ in the double scaling limit; that is

Region C: We let $R\to\infty$, $h\to 1^-$, $\gamma\to 0^+$, $\tau\to\infty$ such that $r = (1-h^2)^{1/2}R$, $g = \gamma(1-h^2)^{-1/2}$, and $t = \frac{1}{2}(1-h^2)\tau$ are fixed and are of order one.

In Region C the resulting scaling function, sometimes referred to as a crossover scaling function [6], will be a function of the three variables r, g, and t. Note that g is expressible in terms of the interaction constants of the hamiltonian (1). Region A will be recovered in the limit $r\to\infty$, $g\to\infty$ such that $r_{\mathrm{A}} = \frac{1}{2}r/g$ and $t_{\mathrm{A}} = t$ are fixed; and Region B will be recovered in the limit $r\to\infty$, $g\to 0$, $t\to\infty$ such that $r_{\mathrm{B}} = rg$ and $t_{\mathrm{B}} = 2gt$ are fixed.

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[‡1] As shown in ref. [5], the scaling variables r_{A} and t_{A} (r_{B} and t_{B}) of Region A (B) occur only in the combination $(r_{\mathrm{A}}^2 - t_{\mathrm{A}}^2)^{1/2}$ $[(r_{\mathrm{B}}^2 - t_{\mathrm{B}}^2)^{1/2}]$.

If we denote by $\lim_C$ the scaling limit in Region C, then the result [‡2] we find is

$$\lim_{C} M_x^{-2} \langle S_1^x(0)\, S_{1+R}^x(\tau)\rangle = \hat{F}_2(r,g,t)$$
$$= \exp[-F_2(r,g,t)] \tag{3}$$

with

$$F_2(r,g,t) = \sum_{n=1}^{\infty} h_{2n}(r,g,t) \tag{4}$$

where

$$h_{2n}(r,g,t) = (2n)^{-1}(2\pi)^{-2n} \int_{-\infty}^{\infty} \mathrm{d}k_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}k_{2n}$$
$$\times \prod_{j=1}^{2n} \left\{ \frac{\exp(-\mathrm{i}rk_j - \mathrm{i}t\epsilon_j)}{\epsilon_j} \left(\frac{\epsilon_j - \epsilon_{j+1}}{k_j + k_{j+1}}\right)\right\} \tag{5}$$

and we take $k_{2n+1} \equiv k_1$. The quantity ϵ_j is given by

$$\epsilon_j = \epsilon(k_j, g) = [(k_j^2 + \mu^2)(k_j^2 + \mu^{-2})]^{1/2} \tag{6}$$

where

$$\mu = \begin{cases} g + \mathrm{i}\sqrt{1-g^2} & \text{for } 0 < g \leqslant 1 \\ g + \sqrt{g^2 - 1} & \text{for } g > 1 \end{cases}$$

and (7)

$$\bar{\mu} = \begin{cases} g - \mathrm{i}\sqrt{1-g^2} & \text{for } 0 < g \leqslant 1 \\ g - \sqrt{g^2 - 1} & \text{for } g > 1\,. \end{cases}$$

We conclude with the following remarks [‡2]:

(1) Examination of

$$\int_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\, [\hat{F}_2(r,g,t) - 1]\, \mathrm{d}t$$

shows that the quantity $\epsilon(k,g)$, plotted in fig. 1 as a function of k for various values of g, is the dispersion curve for the elementary excitations (spin waves) [‡3]. Furthermore, any excitation consists of an even number of these elementary excitations.

(2) For fixed t and large r the behavior of the functions $h_{2n}(r,g,t)$ depends qualitatively on the parameter g appearing in $\epsilon(k,g)$. For the case $g < 1$ the behavior is oscillatory and for $g > 1$ the behavior is non-oscillatory (for $g = 1$, $F_2(r,g,t) = 0$) [‡3]. Specifically, $h_{2n}(r,g,t)$ is asymptotically equal to $\exp(-2n \,\mathrm{Re}\, \bar{\mu}\, r)$ for all g, and for $g < 1$ this decreasing exponential factor is multiplied by a sum of oscillatory terms of the form $\exp(\pm \mathrm{i}2m\, \mathrm{Im}\, \mu r)$, $m = 0, 1, 2, ..., n$. In both cases these exponential factors are multiplied by some power of r, depending upon n, and times some constant which depends upon n and g.

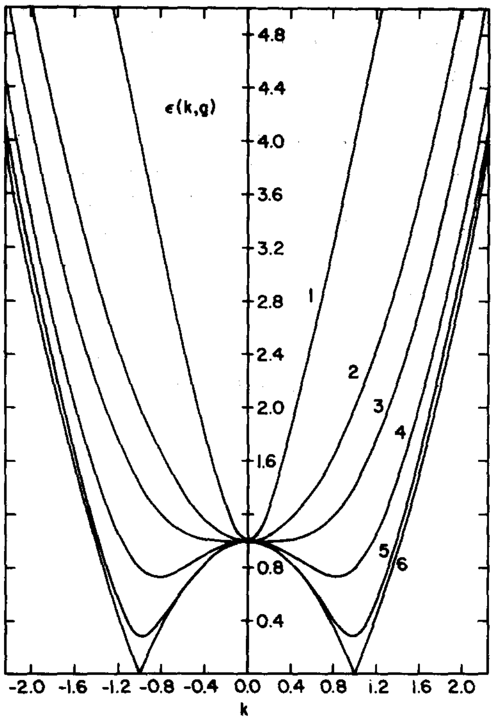

Fig. 1. $\epsilon(k,g)$ as a function of k for different values of g. (1) $g = 2.0$; (2) $g = 1.0$; (3) $g = 1/\sqrt{2}$; (4) $g = 0.4$; (5) $g = 0.2$ and (6) $g = 0.0$.

(3) From fig. 1 we see that for $g < 2^{-1/2}$, the minimum excitation energy, $\epsilon_{\min} = 4g^2(1-g^2)$, occurs at $k = k_{\min} = (1-2g^2)^{1/2}$; and for $g \geqslant 2^{-1/2}$ the only minimum is at $k = 0$.

Hence, in the complex ω-plane the closest singularity, corresponding to the 2-spin wave state, occurs for a given g at $\min(2, 2\epsilon_{\min})$.

(4) The limit $g \to 0$ for fixed r is of interest, since in this limit the quantity

$$\rho(r,t) = \lim_{g\to 0} \pi g^{1/2} \hat{F}_2(r,g,t) \tag{8}$$

is the time-dependent, one-particle reduced density matrix for a system of impenetrable bosons in one

[‡2] Details to be published elsewhere.

[‡3] This was first discussed in refs. [3] and [4].

dimension [7]. The factor π was chosen so that $\rho(0, 0) = 1$. The evaluation of this limit will be the subject of future work.

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References

[1] For recent review articles see L.J. de Jongh and A.R. Miedema, Adv. Phys. 23 (1974) 1;
M. Steiner, J. Villain and C.G. Windsor, Adv. Phys. 25 (1976) 87;
J.C. Bonner, J. Appl. Phys. 49 (1978) 1299.

[2] E. Lieb, T. Schultz and D. Mattis, Ann. Phys. (NY) 16 (1961) 406;
S. Katsura, Phys. Rev. 127 (1962) 1508.

[3] B.M. McCoy, Phys. Rev. 173 (1968) 531;
E. Barouch and B.M. McCoy, Phys. Rev. A3 (1971) 786.

[4] B.M. McCoy, E. Barouch and D. Abraham, Phys. Rev. A4 (1971) 2331.

[5] H.G. Vaidya and C.A. Tracy, Physica 92A (1978) 1.

[6] See, e.g., M.E. Fisher, Rev. Mod. Phys. 46 (1974) 597, and references contained therein.

[7] T.D. Schultz, J. Math. Phys. 4 (1963) 666;
A. Lenard, J. Math. Phys. 5 (1964) 930.