One particle reduced density matrix of impenetrable bosons in one dimension at zero temperature

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We compute exactly the one particle reduced density matrix $\rho(r)$ of a system of impenetrable bosons in one dimension at zero temperature. We do this by relating $\rho(r)$ to a certain double scaling limit of the transverse correlation function of the one-dimensional spin 1/2 $X-Y$ model. We study the asymptotic behavior of $\rho(r)$ for large r. This expansion contains oscillatory terms which arise due to the intrinsic quantum mechanical nature of the problem. We use these results to discuss the analytic structure of the momentum density function $n(k)$.

I. INTRODUCTION

One of the model systems that has generated considerable interest is the system of bosons in one dimension interacting with the potential $c\delta(x_i - x_j)$. In particular, the limit $c\to\infty$ corresponds to a gas of impenetrable bosons. The ground state many particle wave function of this system of impenetrable bosons was first derived by Girardeau.[1] The study of the one particle reduced density matrix (which we refer to simply as the density matrix) was initiated by Schultz[2] and by Lenard.[3]

In this paper, we report an exact calculation of the reduced density matrix for this system at zero temperature. Let $\psi_{N,L}(x_1,x_2,\ldots,x_N,\tau)$ be the normalized ground state wave function of N impenetrable bosons on a chain of length L at time τ. The density matrix $\rho_{N,L}(x-x',\tau)$ is defined by

$$\rho_{N,L}(x-x',\tau) = N\int_0^L dx_1 \cdots \int_0^L dx_{N-1} \times \psi_{N,L}(x_1,x_2,\ldots,x_{N-1},x,\tau) \times \psi^*_{N,L}(x_1,x_2,\ldots,x_{N-1},x',0). \tag{1.1}$$

In particular, we study $\rho_{N,L}(x-x',\tau)$ in the thermodynamic limit: $N\to\infty$, $L\to\infty$ such that $\rho = N/L$ is the constant particle density, and we write

$$\rho(x-x',\tau) = \lim_{N\to\infty, L\to\infty} \rho_{N,L}(x-x',\tau). \tag{1.2}$$

In the following sections we will derive an exact answer for $\rho(r)\equiv\rho(r,0)$. In the remainder of this section we will describe the main features of the result.

Let $x = k_F r$ (k_F = Fermi wave vector). (Note that in our paper $k_F = 1$, while in Ref. 3, $k_F = \pi$.) Then $\rho(x)$ has the following asymptotic expansion:

$$\rho(x) = \frac{\rho_\infty}{|x|^{1/2}}\left\{1 + \frac{1}{8x^2}\left(\cos 2x - \frac{1}{4}\right) + \frac{3\sin 2x}{16x^3} + \frac{3}{256x^4}\left(\frac{11}{8} - 31\cos 2x\right) + O(x^{-5})\right\} \tag{1.3}$$

where

$$\rho_\infty = \pi e^{1/2} 2^{-1/3} A^{-6} = 0.92418\ldots \tag{1.4}$$

[in Eq. (1.4), $A = 1.2824\ldots$ is Glaisher's constant].

Lenard[3] derived an expansion of $\rho(x)$ for small x [see Eq. (56) and (57) of Ref. 3]. We have used these results to extend the expansion to order x^9 [Lenard expanded $\rho(x)$ to order x^4]. The result is

$$\rho(x) = 1 - \frac{x^2}{6} + \frac{|x|^3}{9\pi} + \frac{x^4}{120} - \frac{11|x|^5}{1350\pi} - \frac{x^6}{5040} + \frac{122|x|^7}{105\pi\times 7!} + \left(\frac{1}{24\,300\pi^2} + \frac{1}{9!}\right)x^8 - \frac{253|x|^9}{98\,000\times 27^2\times\pi} + O(x^{10}). \tag{1.5}$$

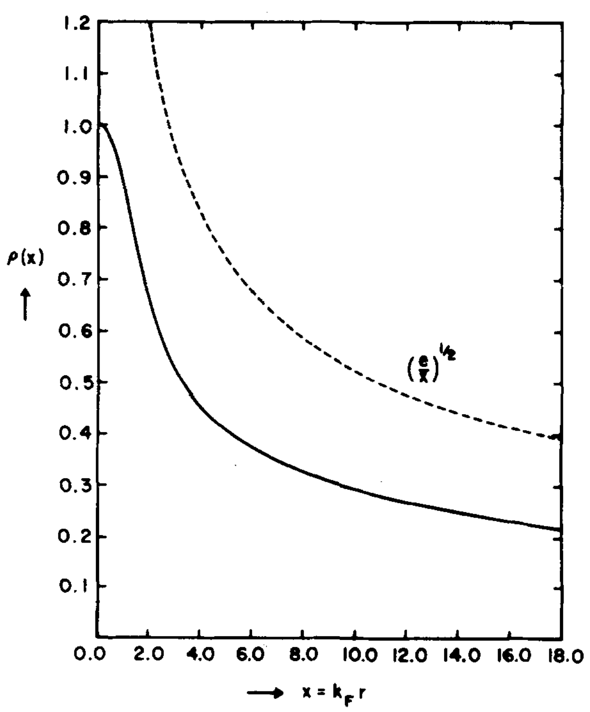

FIG. 1. $\rho(x = k_F r)$ as a function of x. The dotted line is a plot of Lenard's upper bound.

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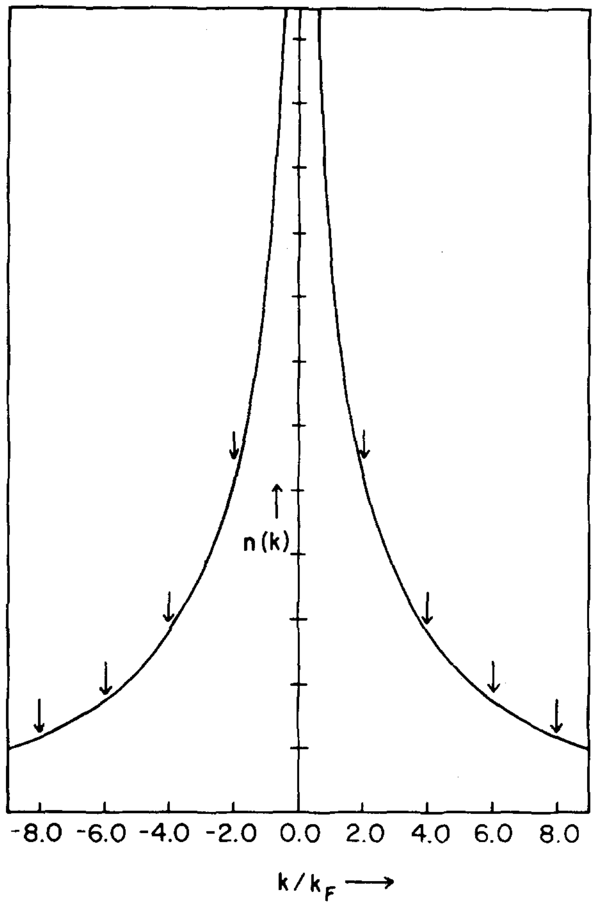

FIG. 2. Schematic plot of $n(k)$ as a function of (k/k_f), showing the $k^{-1/2}$ singularity at the origin. The arrows mark the points of nonanalyticity of $n(k)$ where a derivative of $n(k)$ diverges.

We use the expansions (1.3) and (1.5) to plot $\rho(x)$ as a function of x in Fig. 1. We also plot Lenard's upper bound $\rho(x) \leqslant (e/x)^{1/2}$.

The expansion of $\rho(x)$ for large x has the following general structure:

$$\rho(x) = \frac{\rho_\infty}{|x|^{1/2}} \left[1 + \sum_{n=1}^{\infty} \frac{c_{2n}}{x^{2n}} + \sum_{m=1}^{\infty} \frac{\cos 2mx}{x^{2m}} \times \left(\sum_{n=0}^{\infty} \frac{c'_{2n,m}}{x^{2n}} \right) + \sum_{m=1}^{\infty} \frac{\sin 2mx}{x^{2m+1}} \left(\sum_{n=0}^{\infty} \frac{c''_{2n,m}}{x^{2n}} \right) \right]. \tag{1.6}$$

This expansion enables us to study the singularity structure of the one particle momentum density function $n(k)$. $n(k)$ is defined by

$$n(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dr e^{-ikr} \rho(r). \tag{1.7}$$

The $|x|^{-1/2}$ falloff of $\rho(x)$ for large x leads to a $|k|^{-1/2}$ singularity in $n(k)$ at the origin. The terms with $\sin 2mx$ and $\cos 2mx$ in Eq. (1.6) lead to additional points of nonanalyticity for $n(k)$ at $k = \pm 2mk_F (m = 1,2,...)$. At these points some higher derivative of $n(k)$ diverges. For example, at $k = \pm 2k_F$, $d^2n(k)/dk^2$ is divergent. Note that a system of free fermions has a sharp Fermi surface at zero temperature, while for the system of impenetrable bosons only the second derivative of the momentum distribution function diverges at $k = \pm 2k_F$.

In Fig. 2, we show schematically the behavior of $n(k)$ as a function of (k/k_F). The arrows mark the points of nonanalyticity of $n(k)$. Figure 3 shows the branch cut structure of $n(k)$ in the k plane. All the branch points in Fig. 3 are square root branch points.

The full answer for $\rho(r)$ is written out explicitly in Sec. 7. The behavior of $\rho(r)$ for nonzero temperatures and for the case when c is finite has been discussed in Ref. 4.

II. FORMULATION OF THE PROBLEM

A. Relation between the boson problem and the X–Y model

Schultz[2] approached the boson problem by replacing the continuum by a lattice of evenly spaced lattice points, with lattice spacing ϵ. He expressed the density matrix on this lattice as a determinant. Lenard[3] showed that one can

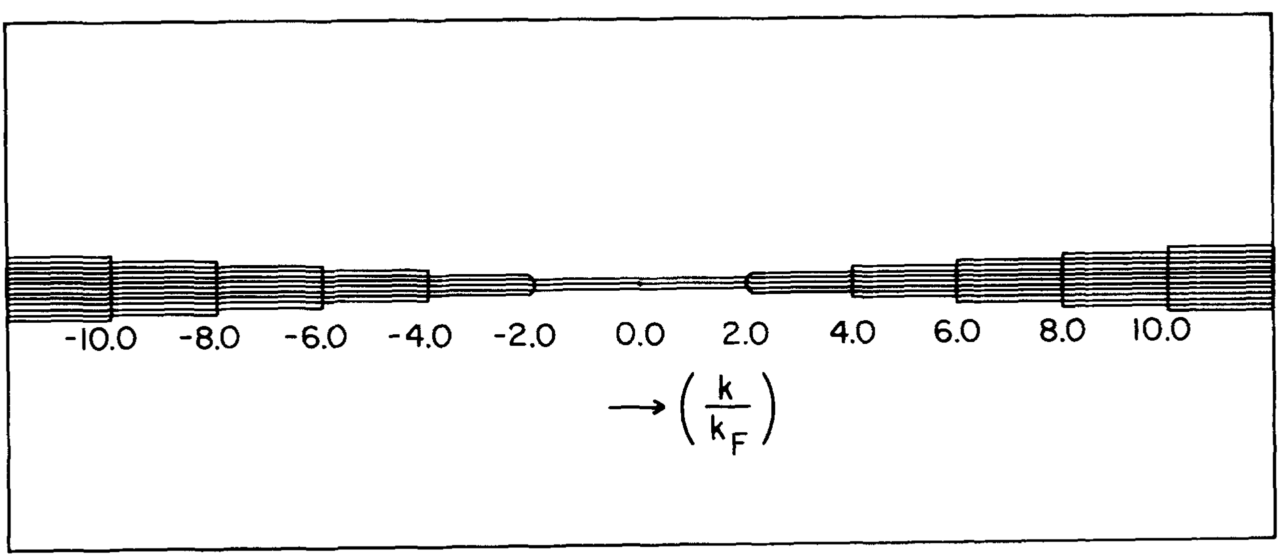

FIG. 3. Analytic structure of $n(k)$ in the k plane showing square root branch points at $k = \pm 2mk_F$, $m = 0,1,2,...$.

take the thermodynamic limit before the continuum limit to obtain $\rho(r)$. Therefore, from Eq. 80 and 87 of Ref. 3 we have

$$\rho(r) = \lim_{\substack{\epsilon\to 0, s\to\infty \\ s\epsilon = r \text{ fixed}}} (\rho\epsilon)^{-1}\rho_s\,, \tag{2.1}$$

where

$$\rho_s = \frac{1}{2}\begin{vmatrix} G_1 & G_2 & \cdots & G_s \\ G_0 & G_1 & \cdots & G_{s-1} \\ \vdots & & & \\ G_{-(s-2)} & G_{-(s-1)} & \cdots & G_1 \end{vmatrix}, \tag{2.2}$$

with

$$G_m = \begin{cases} \dfrac{2}{\pi m}\sin\pi\rho m\epsilon, & \text{for } m\neq 0, \\ -1 + 2\epsilon\rho, & \text{for } m = 0. \end{cases} \tag{2.3}$$

We follow Schultz and compute $\rho(r,\tau)$ by relating it to a certain double scaling limit of the XX correlation function of the spin $\frac{1}{2}XY$ model in one dimension. This model is defined by the Hamiltonian

$$H_N = -\sum_{i=1}^{N}[(1+\gamma)S_i^x S_{i+1}^x + (1-\gamma)S_i^y S_{i+1}^y + hS_i^z], \tag{2.4}$$

where $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$, $\alpha = x,y,z$, and σ_i^α are the usual Pauli matrices. In Eq. (2.4), $\gamma(0\leqslant\gamma\leqslant 1)$, is the anisotropy parameter and h is the magnetic field in the z direction. We impose cyclic boundary conditions $S_{N+1}\equiv S_1$. We denote the ground state transverse correlation function in the thermodynamic limit $(N\to\infty)$ by $\rho_{xx}(R,t;\gamma)$ defined by

$$\rho_{xx}(R,t;\gamma) \equiv \langle S_1^x(0) S_{R+1}^x(t)\rangle\,. \tag{2.5}$$

In Eq. (2.5) the brackets denote the gound state expectation value.

For the anisotropic X–Y model, $\rho_{xx}(R,t=0;\gamma)\equiv\rho_{xx}(R;\gamma)$ can be written as[5]

$$\rho_{xx}(R;\gamma) = \frac{1}{4}\begin{vmatrix} \overline{G}_1 & \overline{G}_2 & \cdots & \overline{G}_R \\ \overline{G}_0 & \overline{G}_1 & \cdots & \overline{G}_{R-1} \\ \vdots & & & \\ \overline{G}_{-(R-2)} & \overline{G}_{-(R-1)} & \cdots & \overline{G}_1 \end{vmatrix}, \tag{2.6}$$

where

$$\overline{G}_m = -(2\pi)^{-1}\int_{-\pi}^{\pi} d\varphi\,\frac{e^{i\varphi m}(h - \cos\varphi + i\gamma\sin\varphi)}{[(h-\cos\varphi)^2 + \gamma^2\sin^2\varphi]^{1/2}}\,. \tag{2.7}$$

for $\gamma = 0$ this reduces to

$$\overline{G}_m = \begin{cases} \dfrac{2}{\pi m}\sin m\varphi_0, & \text{for } m\neq 0, \\ -1 + \dfrac{2}{\pi}\varphi_0, & \text{for } m = 0, \end{cases} \tag{2.8}$$

where

$$\cos\phi_0 = h. \tag{2.9}$$

Comparing Eqs. (2.3) and (2.8), we see that the determinants in Eqs. (2.2) and (2.6) are identical when

$$\rho = \frac{1}{\pi} \text{ and } \epsilon = \arccos(h). \tag{2.10}$$

Under this identification

$$\rho_s = 2\rho_{xx}(s;0). \tag{2.11}$$

As $\epsilon\to 0$, $h\to 1^-$ and we can write $\epsilon = (1-h^2)^{-1/2}$. The equivalence can be established in a similar manner for the time dependent case.

To use Eq. (2.6) we need to evaluate determinants of very large dimension. This in general is very difficult to do. However, when $\gamma\neq 0$, the determinant (2.6) has been studied in great detail in Ref. 6. There it was shown (for the general case $t\neq 0$) that for $h<1$

$$\rho_{xx}(R,t;\gamma) = \rho_{xx}(\infty)\exp\Big[-\sum_{n=1}^{\infty}F^{(2n)}(R,t;\gamma)\Big], \tag{2.12}$$

where

$$\rho_{xx}(\infty) = [2(1+\gamma)]^{-1}[\gamma^2(1-h^2)]^{1/4} \tag{2.13}$$

and

$$F^{(2n)}(R,t;\gamma) = (2n)^{-1}2^{-2n}(2\pi)^{-2n}\int_{-\pi}^{\pi}d\varphi_1\cdots\int_{-\pi}^{\pi}d\varphi_{2n} \times\prod_{j=1}^{2n}\Big[\frac{e^{-iR\varphi_j - itA_j}(A_j - A_{j+1})}{A_j\sin(1/2)(\varphi_j+\varphi_{j+1})}\Big], \tag{2.14}$$

with $\phi_{2n+1}\equiv\phi_1$; $\mathrm{Im}\phi_j<0, j = 1,2,\cdots,2n$; and

$$A_j \equiv A(\varphi_j) = [(\cos\varphi_j - h)^2 + \gamma^2\sin^2\varphi_j]^{1/2}. \tag{2.15}$$

All reference to the original determinant has been eliminated in this formula.

We define the double scaling limit of $\rho_{xx}(R,t;\gamma)$ (denoted by $\lim_C$ following Ref. 7) by $h\to 1^-$, $\gamma\to 0$, $R\to\infty$, $t\to\infty$ such that

$$r = (1-h^2)^{1/2}R, \tag{2.16a}$$

$$\tau = \tfrac{1}{2}(1-h^2)t, \tag{2.16b}$$

and

$$g = \gamma(1-h^2)^{-1/2} \tag{2.16c}$$

are held fixed.

We now use Eqs. (2.1) and (2.11) to recover $\rho(r,\tau)$ as

$$\rho(r,\tau) = \lim_{C}\lim_{g\to 0} 2\pi(1-h^2)^{-1/2}\rho_{xx}(R,t;\gamma). \tag{2.17}$$

The calculation of this paper is based on the assumption that the two limits in Eq. (2.17) can be interchanged. We thus have

$$\rho(r,\tau) = \lim_{g\to 0}\lim_{C} 2\pi(1-h^2)^{-1/2}\rho_{xx}(R,t;\gamma). \tag{2.18}$$

B. Double scaling limit of $\rho_{xx}(R,t;\gamma)$

In $\lim_C$ we may expand $A(\phi_j)$ around $\phi_j = 0$ and rescale ϕ_j by

$$\varphi_j = (1-h^2)^{1/2}k_j \tag{2.19}$$

to obtain

$$A(\varphi_j) \simeq \tfrac{1}{2}(1-h^2)\epsilon(k_j), \tag{2.20}$$

where

$$\epsilon(k_j)\equiv\epsilon_j = [(k_j^2+\mu^2)(k_j^2+\mu^{*2})]^{1/2}, \tag{2.21}$$

with

$$\mu = g + i(1-g^2)^{1/2} \tag{2.22}$$

and μ^* is the complex conjugate of μ, and g is given by Eq. (2.16c). Similarly,

$$\sin\tfrac{1}{2}(\varphi_j+\varphi_{j+1})\simeq\tfrac{1}{2}(1-h^2)^{1/2}(k_j+k_{j+1}). \tag{2.23}$$

We scale the integrals in Eq. (2.14) using the formulas (2.19) – (2.23) and denote the scaled functions by $h^{(2n)}(r,\tau,g)$. Substituting in Eq. (2.18), we get

$$\rho(r,\tau) = \lim_{g\to 0}\pi g^{1/2}\exp[-H(\pi^2;r,\tau,g)], \tag{2.24}$$

where

$$H(z;r,\tau,g) = \sum_{n=1}^{\infty} n^{-1}z^{2n}h^{(2n)}(r,\tau,g) \tag{2.25}$$

and

$$h^{(2n)}(r,\tau,g) = \frac{1}{2}\int_{-\infty}^{\infty}dk_1\cdots\int_{-\infty}^{\infty}dk_{2n} \times\prod_{j=1}^{2n}\left[\frac{e^{-irk_j-i\tau\epsilon_j}(\epsilon_j-\epsilon_{j+1})}{\epsilon_j(k_j+k_{j+1})}\right],$$

$$k_{2n+1}\equiv k_1 . \tag{2.26}$$

We will now restrict our attention to the case $\tau=0$ and denote the above functions by $H(z,r,g)$ and $h^{(2n)}(r,g)$, respectively.

III. ANALYSIS OF $h^{(2n)}(r,g)$

The integrands in Eq. (2.26) have branch points at $k_j = \pm i\mu, \pm i\mu^*$. Their analytic structure is shown in Fig. 4. In the limit $g\to 0$ the branch points pinch the real axis in pairs leading to logarithmic divergences in g. The general structure can be seen to be the following for $g\ll 1$:

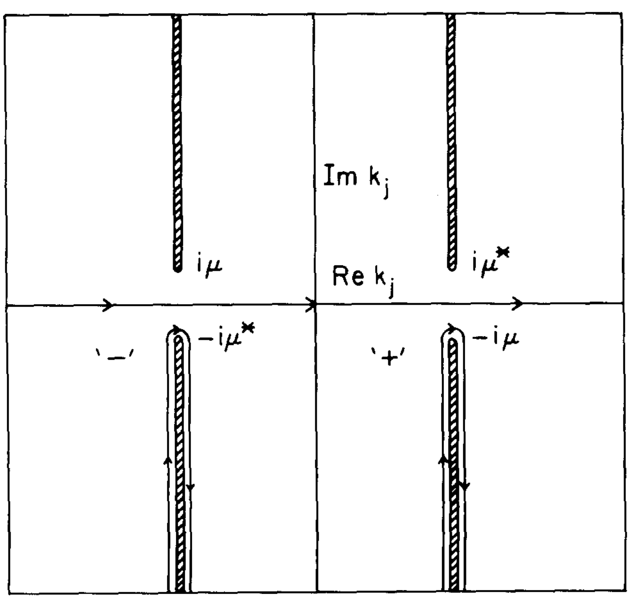

FIG. 4. Analytic structure of the integrand in Eq. (2.26) in the k_j plane. We show the branch cuts + and − around which the contour of integration over k_j is bent.

$$h^{(2n)}(r,g) = C_{2n}^{(2n)}\ln^{2n}(gr) + \sum_{m=1}^{2n} C_{2n-m}^{(2n)}(r)\ln^{2n-m}(gr) + o(1), \tag{3.1}$$

where $C_{2n}^{(2n)}$ are constants and $C_m^{(2n)}(r)$ $(m\neq 2n)$ are functions of r independent g. It is clear that we need to sum these logarithmic divergences systematically to take the limit $g\to 0$.

We can show that the constant $C_{2n}^{(2n)}$ (with the factor n^{-1} of 2.25 included) in Eq. (3.1) is twice the constant C_{2n} appearing in the small variable expansion of the functions $f_{2n}(t)$ of the two dimensional Ising model [see Eq. (3.146) of Ref. 8]. In the Ising model the coefficients multiplying the divergent terms satisfy certain relations so that on summing up the divergences to all orders one gets a simple '$\ln t$' divergence. Something similar has to happen in Eq. (3.1) if we are to get a finite result in the $g\to 0$ limit. The relation between the constants multiplying the leading divergent terms suggests the possibility of expressing the divergent parts of $h^{(2n)}(r,g)$ in terms of the Ising model functions.

The functions $f_{2n}(t)$ and the related functions $g_{2n+1}(t)$ are given by[8]

$$f_{2n}(t) = (-1)^n n^{-1}\int_1^{\infty}dy_1\cdots\int_1^{\infty}dy_n \times\prod_{j=1}^{2n}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}(y_j+y_{j+1})}\prod_{j=1}^{n}(y_{2j}^2-1), \tag{3.2}$$

where $y_{2n+1}\equiv y_1$ and

$$g_{2n+1}(t) = (-1)^n\int_1^{\infty}dy_1\cdots\int_1^{\infty}dy_{2n+1} \times\prod_{j=1}^{2n+1}\frac{e^{-ty_j}}{(y_j^2-1)^{1/2}}\prod_{j=1}^{2n}(y_j+y_{j+1})^{-1} \times\prod_{j=1}^{n}(y_{2j}^2-1). \tag{3.3}$$

The result which we derive in Sec. 4–6 is, for $g\ll 1$,

$$H(z;r,g) = 2F(z;gr) + \bar{H}[z;r,G(z;gr)] + o(1), \tag{3.4}$$

where

$$F(z;gr) = \sum_{n=1}^{\infty} z^{2n}f_{2n}(gr) \tag{3.5a}$$

$$G(z;gr) = \sum_{n=0}^{\infty} z^{2n+1}g_{2n+1}(gr) \tag{3.5b}$$

and $\bar{H}[z;r,G(z;gr)]$ is given by Eq. (6.43). We assume that the error estimate remains $o(1)$ after carrying out the sum over n in Eq. (2.25).

Note that in the right hand side of Eq. (3.4), the g dependence is only through the functions $F(z;gr)$ and $G(z;gr)$. These functions can be expressed in terms of Painlevé functions of the third kind[9] and their behavior as $g\to 0$ has been studied in detail.[9] In this limit $F(\pi^{-1};gr)$ has the correct divergence to cancel out the factor $g^{1/2}$ in Eq. (2.24) and $\lim_{g\to 0}G(\pi^{-1};gr)=1$.

Our strategy for deriving Eq. (3.4) is the following: We bend the contours of integration in Eq. (2.26) into the lower half plane as illustrated in Fig. 4. Each integral now separates into a sum of two parts, one each around the '+' and '−' branch cuts. Now, $h^{(2n)}(r,g)$ is a sum of 2^{2n} integrals.

We then separate each integral such that the divergent parts are expressed in terms of $f_{2n}(gr)$ and $g_{2n+1}(gr)$ with coefficients that are functions of r alone. Then summing up $h^{(2n)}(r,g)$ to all orders leads to Eq. (3.4).

In the next section, we will systematize the evaluation of the 2^{2n} integrals in $h^{(2n)}(r,g)$, using a transfer matrix formulation. In Appendix A we give a detailed evaluation of $h^{(2)}(r,g)$ as $g\to 0$. A study of Appendix A is instructive in understanding the basic strategy involved in the solution of the problem. The transfer matrix formulation enables us to analyze the 2^{2n} integrals in $h^{(2n)}(r,g)$ in a systematic manner.

IV. TRANSFER MATRIX FORMULATION

As illustrated in Fig. 4 and Appendix A, each integral separates into a sum of two pieces; one coming from the $+$ branch cut and the other coming from the $-$ branch cut. We denote this by using a state vector $\sigma_j \equiv \pm$ for each "y_j" variable of integration. [$\sigma_j \equiv +(-)$ implies that the y_j integration is carried out around the $+(-)$ branch cut.] All the 2^{2n} integrals in $h^{(2n)}(r,g)$ can be written out systematically as follows:

$$h^{(2n)}(r,g) = \sum_P h_P^{(2n)}(r,g), \tag{4.1}$$

where P is a permulation of (l) $+$ signs and $(2n-l)$ $-$ signs $l=0,1,2,\cdots,2n$, representing the branch cut around which each y_j integral $(j=1,2,\cdots,2n)$ is evaluated.

We introduce the symbol $\doteq$ with the following meaning: The two sides of $\doteq$ are equal only after carrying out integrations satisfying the conditions (i) each y_j variable is integrated on $\int_g^\infty dy_j e^{-ry_j}$ and (ii) the integrand has an overall factor

$$\prod_{j\,\mathrm{odd}} (y_j^2-g^2)^{1/2} \prod_{j\,\mathrm{even}} (y_j^2-g^2)^{-1/2},$$

for an integral over $y_1, y_2, \ldots$.

We introduce the notation $\int dy$ with the following meaning:

$$u(r,g) = \int dy f(y;r) \tag{4.2}$$

means that the equality in Eq. (4.2) holds on integrating over all the y_j variables in $f(y,r)(y\equiv y_1,y_2,\cdots,y_m)$ with the conditions (i) and (ii) above.

On changing the integration variables as in Appendix A we can write

$$h^{(2n)}(r,g) = \int dy \sum_{\sigma_j=\pm} \langle\sigma_1|M_0(1,2)|\sigma_2\rangle \times\langle\sigma_2|M_e(2,3)|\sigma_3\rangle\cdots\langle\sigma_{2n}|M_e(2n,1)|\sigma_1\rangle \tag{4.3}$$

$$= \int dy\, \mathrm{Tr}\Big[\prod_{j=1}^{n} M_0(2j-1,2j)M_e(2j,2j+1)\Big], \tag{4.4}$$

where $2n+1\equiv 1$ and "Tr" denotes the trace of the 2×2 matrix on the right. Here,

$$M_0(2j-1,2j) = \begin{array}{c|cc} & + & - \\ \hline + & \dfrac{e^{-ir}(y_{2j-1}+2i)}{y_{2j-1}+y_{2j}+2i} & \dfrac{y_{2j-1}+2i}{y_{2j-1}+y_{2j}} \\ - & \dfrac{y_{2j-1}-2i}{y_{2j-1}+y_{2j}} & \dfrac{e^{+ir}(y_{2j-1}-2i)}{y_{2j-1}+y_{2j}-2i} \end{array} \tag{4.5}$$

and

$$M_e(2j,2j+1) = \begin{array}{c|cc} & + & - \\ \hline + & \dfrac{e^{-ir}}{(y_{2j}+2i)(y_{2j}+y_{2j+1}+2i)} & \dfrac{1}{(y_{2j}+2i)(y_{2j}+y_{2j+1})} \\ - & \dfrac{1}{(y_{2j}-2i)(y_{2j}+y_{2j+1})} & \dfrac{e^{+ir}}{(y_{2j}-2i)(y_{2j}+y_{2j+1}-2i)} \end{array}. \tag{4.6}$$

It is clear that in the odd variable integrals we can set $g=0$. The even variable integrals diverge in the limit $g\to 0$. The aim is to get enough y_{2j}'s in the numerator to make these integrals converge and at the same time factor out integrals identical to $f_{2n}(gr)$ or $g_{2n+1}(gr)$ (after rescaling as in Appendix A).

The matrices in Eqs. (4.5) and (4.6) have the structure

$$\begin{bmatrix} (11) & (12) \\ (12)^* & (11)^* \end{bmatrix}$$

and will be denoted by [(11),(12)] for the sake of compactness. In this notation

$$M_0(2j-1,2j) = \left[\frac{e^{-ir}(y_{2j-1}+2i)}{y_{2j-1}+y_{2j}+2i},\ \frac{y_{2j-1}+2i}{y_{2j-1}+y_{2j}}\right]. \tag{4.5'}$$

We now use the identity

$$(y_{2j-1}+y_{2j}\pm 2i)^{-1} = (y_{2j-1}\pm 2i)^{-1}[1-y_{2j}(y_{2j-1}+y_{2j}\pm 2i)^{-1}] \tag{4.7}$$

to write

$$M_0(2j-1,2j) = M_0^{(1)}(2j-1,2j) + M_0^{(2)}(2j-1,2j) \tag{4.8a}$$

and

$$M_e(2j,2j+1) = M_e^{(1)}(2j,2j+1) + M_e^{(2)}(2j,2j+1), \tag{4.8b}$$

where

$$M_0^{(1)}(2j-1,2j) = [e^{-ir}, (y_{2j-1}+2i)(y_{2j-1}+y_{2j})^{-1}], \tag{4.9a}$$

$$M_0^{(2)}(2j-1,2j) = -y_{2j}\,[e^{-ir}(y_{2j-1}+y_{2j}+2i)^{-1},0]\,, \tag{4.9b}$$

$$M_e^{(1)}(2j,2j+1) = [e^{-ir}[(y_{2j}+2i)(y_{2j+1}+2i)]^{-1}, [(y_{2j}+2i)(y_{2j}+y_{2j+1})]^{-1}], \tag{4.10a}$$

and

$$M_e^{(2)}(2j,2j+1) = -y_{2j}\,[e^{-ir}[(y_{2j}+2i)(y_{2j+1}+2i) \times(y_{2j}+y_{2j+1}+2i)]^{-1},0]. \tag{4.10b}$$

Hence,

$$H(z;r,g) = \sum_{n=1}^{\infty} n^{-1}z^{2n}h^{(2n)}(r,g) \tag{4.11a}$$

$$= \sum_{n=1}^{\infty} n^{-1}z^{2n}\sum_{l=0}^{2n}\int dy\,\mathrm{Tr}G_l(1\to 2n)\,, \tag{4.11b}$$

where $G_l(1\to 2n)\equiv G_l(y_1,y_2,\cdots,y_{2n})$ enumerates all possible products of $2n$ matrices with l $M_{(y)}^{(2)}$'s and $(2n-l)$ $M_{(y)}^{(1)}$'s. In particular,

$$G_0(1\to 2n) \doteqdot \prod_{j=1}^{2n} M_0^{(1)}(2j-1,2j)M_e^{(1)}(2j,2j+1) \tag{4.12a}$$

and

$$G_{2n}(1\to 2n) \doteqdot \prod_{j=1}^{2n} M_0^{(2)}(2j-1,2j)M_e^{(2)}(2j,2j+1). \tag{4.12b}$$

In Eqs. (4.12), $y_{2n+1}\equiv y_1$.

Separating out $G_0(2n)$ and changing the order of summation in the remaining sum, we have

$$H(z;r,g) = \int dy\Big[\sum_{n=1}^{\infty} n^{-1}z^{2n}\,\mathrm{Tr}G_0(1\to 2n) + \sum_{l=1}^{\infty}\sum_{n=[(l+1)/2]}^{\infty} n^{-1}z^{2n}\,\mathrm{Tr}G_l(1\to 2n)\Big]\,, \tag{4.13}$$

where $[x]$ is the largest integer $\leqslant x$.

Now let us look at the structure of $G_l(2n)$, $l\neq 0$. Each term has $l\,M_{(y)}^{(2)}$'s seperated by a string of $M_{(y)}^{(1)}$'s. There are four such distinct strings given by

$$F_{2k}^{(1)}(2\to 2k+1) \doteqdot \Big[\prod_{j=1}^{k-1} M_e^{(1)}(2j,2j+1)M_0^{(1)} \times(2j+1,2j+2)\Big]M_e^{(1)}(2k,2k+1)\,, \tag{4.14}$$

$$F_{2k+1}^{(2)}(2\to 2k+2) \doteqdot \prod_{j=1}^{k} M_e^{(1)}(2j,2j+1)M_0^{(1)}(2j+1,2j+2), \tag{4.15}$$

$$F_{2k+1}^{(3)}(1\to 2k+1) \doteqdot \prod_{j=1}^{k} M_0^{(1)}(2j-1,2j)M_e^{(1)}(2j,2j+1)\,, \tag{4.16}$$

and

$$F_{2k}^{(4)}(1\to 2k) \doteqdot \Big[\prod_{j=1}^{k-1} M_0^{(1)}(2j-1,2j)M_e^{(1)}(2j,2j+1)\Big] \times M_0^{(1)}(2k-1,2k)\,. \tag{4.17}$$

In Eqs. (4.14) and (4.17) the product is replaced by 1 for $k=1$. In Eq. (4.14) to (4.17), $k=1,2,3,\ldots$ and the "r" and "g" dependence is understood. In the above equations the starting variables is either y_1 or y_2, but it is understood that in the actual product the starting and ending variables are chosen so as to match the variables of $M^{(2)}$'s on either side. For examples, a typical string involving $F_{2k}^{(1)}(y)$ might be

$$M_0^{(2)}(7,8)F_{2k}^{(1)}(8\to 2k+7)M_0^{(2)}(2k+7,2k+8).$$

It is also clear that the value of k in $F_k^{(j)}(y)$ ($j=1,2,3$, or 4) is chosen to fit the missing variables between two successive $M_{(y)}^{(2)}$'s. The case where the two $M_{(y)}^{(2)}$'s are adjacent [either $M_0^{(2)}(2j-1,2j)M_e^{(2)}(2j,2j+1)$ or $M_e^{(2)}(2j,2j+1)\times M_0^{(2)}(2j+1,2j+2)$], is accounted for by choosing

$$F_1^{(2)}=1, \qquad F_1^{(3)}=1. \tag{4.18}$$

We define the generating functions

$$F^{(j)}(y;z)\equiv F^{(j)}(y,z;r,g)\equiv \sum_{n=1}^{\infty} z^{2n}F_{2n}^{(j)}(y), \quad \text{for } j=1,4, \tag{4.19a}$$

and

$$F^{(j)}(y;z)\equiv F^{(j)}(y,z;r,g) \equiv \sum_{n=0}^{\infty} z^{2n+1}F_{2n+1}^{(j)}(y), \quad \text{for } j=2,3. \tag{4.19b}$$

In Eq. (4.19), y symbolically represents all the y_j variables appearing in $F^{(j)}(y,z;r,g)$, $F_{2n}^{(j)}(y)$, and $F_{2n+1}^{(j)}(y)$. The choice of the y_j's is determined by factors on either side, as illustrated in the discussion following Eq. (4.17).

Before introducing a transfer matrix notation for the second term in Eq. (4.13) let us write down $G_1(2n)$ and $G_2(2n)$ explicitly:

$$G_1(1\to 2n) \doteqdot \sum_{j=1}^{n}[M_0^{(2)}(2j-1,2j)F_{2n}^{(1)}(2j\to 2n,1\to 2j-1) + M_e^{(2)}(2j,2j+1)F_{2n}^{(4)}(2j+1\to 2n,1\to 2j)] \tag{4.20a}$$

$$\doteqdot n\,[M_0^{(2)}(1,2)F_{2n}^{(1)}(2\to 2n,1) + M_e^{(2)}(2n,1)F_{2n}^{(4)}(1\to 2n)] \tag{4.20b}$$

since in Eq. (4.19a) all the n terms are identical on integration (they are equivalent to a cyclic relabeling of the variables which does not change the value of the integral). Similarly,

$$\begin{aligned} G_2(1\to 2n) \doteqdot \frac{n}{2}\Big\{&\sum_{k=1}^{n-1}[M_0^{(2)}(1,2)F_{2k}^{(1)}(2\to 2k+1)M_0^{(2)}(2k+1,2k+2)F_{2n-2k}^{(1)}(2k+2\to 2n,1)\\ &+M_e^{(2)}(2n,1)F_{2k}^{(4)}(1\to 2k)M_e^{(2)}(2k,2k+1)F_{2n-2k}^{(4)}(2k+1\to 2n)]\\ &+\sum_{k=1}^{n}[M_0^{(2)}(1,2)F_{2k-1}^{(2)}(2\to 2k)M_e^{(2)}(2k,2k+1)F_{2n-2k+1}^{(3)}(2k+1\to 2n,1)\\ &+M_e^{(2)}(2n,1)F_{2k-1}^{(3)}(1\to 2k-1)M_0^{(2)}(2k-1,2k)F_{2n-2k+1}^{(2)}(2k\to 2n)]\Big\}. \end{aligned} \tag{4.21}$$

We now use a transfer matrix notation to enumerate the matrix products in $G_l(2n)$. $G_l(2n)$ has $l\,M^{(2)}$'s each of which is either even ($M_e^{(2)}$) or odd ($M_0^{(2)}$). We construct the compound transfer matrix

$$\mathscr{N}(y;z)\equiv\mathscr{N}(y,z;r,g)$$
$$\begin{array}{c c c} & o & e \\ \begin{matrix} o \\ e \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} M_0^{(2)}(y)F^{(1)}(y;z) & M_0^{(2)}(y)F^{(2)}(y;z) \\ M_e^{(2)}(y)F^{(3)}(y;z) & M_e^{(2)}(y)F^{(4)}(y;z) \end{bmatrix}} \end{array}. \tag{4.22}$$

Then symbolically

$$G_l(1\rightarrow 2n)\doteq\frac{n}{l}\,[\mathrm{Tr}^{(2)}\mathscr{N}(y;z)^l]_{2n}\,. \tag{4.23}$$

The subscript "$2n$" denotes the power of z^{2n} in the expansion in power of z in the right hand side of Eq. (4.23). $\mathrm{Tr}^{(2)}$ denotes the trace over $\mathscr{N}^l(y;z)$ which is treated as a 2×2 matrix as in Eq. (4.22). For example,

$$\mathrm{Tr}^{(2)}\mathscr{N}(y;z)=M_0^{(2)}(y)F^{(1)}(y;z)+M_e^{(2)}(y)F^{(4)}(y;z), \tag{4.24}$$

$$\begin{aligned}&=M_0^{(2)}(y)F^{(1)}(y;z)M_0^{(2)}(y)F^{(1)}(y;z)\\&\quad+M_e^{(2)}(y)F^{(4)}(y;z)M_e^{(2)}(y)F^{(4)}(y;z)\\&\quad+M_0^{(2)}(y)F^{(2)}(y;z)M_e^{(2)}(y)F^{(3)}(y;z)\\&\quad+M_e^{(2)}(y)F^{(3)}(y;z)M_0^{(2)}(y)F^{(2)}(y;z).\end{aligned} \tag{4.25}$$

It can be readily seen that, together with Eqs. (4.19) and (4.23), Eq. (4.24) leads to (4.20b) and Eq. (4.25) leads to (4.21) on introducing the appropriate integration variables. The notation of Eqs. (4.22) and (4.23) is introduced for bookkeeping purposes and it is to be intepreted only in terms of sums of the form (4.21). The reader is advised to start with Eq. (4.4), substitute Eq. (4.8), and write out some terms explicitly to see that Eqs. (4.12a), (4.13), and (4.23) generate all the contributions to $h^{(2n)}(r,g)$.

Thus, we have

$$\begin{aligned}H(z;r,g)=\int dy\Big[&\sum_{n=1}^{\infty}n^{-1}z^{2n}\,\mathrm{Tr}G_0(1\rightarrow 2n)\\&+\sum_{l=1}^{\infty}\sum_{n=[(l+1)/2]}^{\infty}l^{-1}z^{2n}\,\mathrm{Tr}[\mathscr{N}(y;z)^l]_{2n}\Big].\end{aligned} \tag{4.26}$$

We can do the sum over n in the second term to write

$$\begin{aligned}H(z;r,g)=\int dy\Big[&\sum_{n=1}^{\infty}n^{-1}z^{2n}\,\mathrm{Tr}G_0(1\rightarrow 2n)\\&+\sum_{l=1}^{\infty}l^{-1}\,\mathrm{Tr}\mathscr{N}(y;z)^l\Big].\end{aligned} \tag{4.27}$$

We write

$$\lim_{g\rightarrow 0}\mathscr{N}(y,\pi^{-1};r,g)\equiv\mathscr{N}(y;r). \tag{4.28}$$

It should be kept in mind that $\mathrm{Tr}\mathscr{N}(y;z)^l$ has an overall cyclic structure, namely, the last variable in the trace = the first variable [see for example Eqs. (4.20) and (4.21)].

V. ANALYSIS OF $\mathscr{N}^l(y;z)$

A. Factorization of $F^{(j)}(y;z)$, $j=1,2,3,4$

We first introduce the notion of connectedness. Two matrices are connected if they have at least one variable in common; otherwise they are disconnected. Thus, if two factors are disconnected in the matrix form, the corresponding integral is a product of two integrals. We denote the fact that the two matrices are disconnected by introducing a bar (|) between them. We will aim to reduce $F^{(1)}(y;z)$ to $F^{(4)}(y;z)$ in a form such that the divergent part (as $g\rightarrow 0$) is disconnected from the matrices on both sides. We can then integrate over the variables of this factor independently of the other variables in the matrix product.

We now use the identity

$$M_0^{(1)}(2j-1,2j)=M_0^{(1)'}(2j-1)+y_{2j}M_0^{(1)''}(2j-1,2j), \tag{5.1}$$

where

$$M_0^{(1)'}(2j-1)=[e^{-ir},y_{2j-1}^{-1}(y_{2j-1}+2i)] \tag{5.2a}$$

and

$$M_0^{(1)''}(2j-1,2j)=-y_{2j-1}^{-1}[0,(y_{2j-1}+2i)(y_{2j-1}+y_{2j})^{-1}], \tag{5.2b}$$

to write

$$\begin{aligned}&F_{2k}^{(1)}(2\rightarrow 2k+1)\\&\doteq M_e^{(1)}(2,3)M_0^{(1)'}(3)\Big[\prod_{j=2}^{k-1}M_e^{(1)}(2j,2j+1)\\&\quad\times M_0^{(1)}(2j+1,2j+2)\Big]M_e^{(1)}(2k,2k+1)\\&\quad+y_4M_e^{(1)}(2,3)M_0^{(1)''}(3,4)\\&\quad\times\Big[\prod_{j=2}^{k-1}M_e^{(1)}(2j,2j+1)M_0^{(1)}(2j+1,2j+2)\Big]\\&\quad\times M_e^{(1)}(2k,2k+1).\end{aligned} \tag{5.3}$$

The product is replaced by 1 for $k=1,2$ and $k=1,2,\ldots$.

Carrying out the same procedure repeatedly and defining

$$\begin{aligned}&E_{2n}^{(1)}(2\rightarrow 2n+1)\\&\quad=\Big[\prod_{j=1}^{n-1}y_{2j+2}M_e^{(1)}(2j,2j+1)M_0^{(1)''}(2j+1,2j+2)\Big]\\&\qquad\times M_e^{(1)}(2n,2n+1),\\&\qquad\qquad(\text{product}\equiv 1\text{ for }n=1),\quad n=1,2,3,\ldots,\end{aligned} \tag{5.4}$$

we have

$$F_2^{(1)}(2,3)\doteq E_2^{(1)}(2,3) \tag{5.5a}$$

and

$$\begin{aligned}&F_{2k}^{(1)}(2\rightarrow 2k+1)\\&\quad\doteq\sum_{l=1}^{k-1}E_{2l}^{(1)}(2\rightarrow 2l+1)M_0^{(1)}(2l+1)\\&\qquad\times|F_{2k-2l}^{(1)}(2l+2\rightarrow 2k+1)+E_{2k}^{(1)}(2\rightarrow 2k+1),\\&\qquad\qquad\text{for }k=2,3,\ldots .\end{aligned} \tag{5.5b}$$

In the integrals involving the y_j variables of $E_{2n}^{(1)}(y)$, we can set $g=0$. We adopt the notation that we can set $g=0$ in all the integrals with the $E^{(j)}(y)'$ as integrands.

Defining the generating function

$$E^{(1)}(y;z)\equiv E^{(1)}(y,z;r)\equiv\sum_{n=1}^{\infty}z^{2n}E_{2n}^{(1)}(y), \tag{5.6}$$

we can write the recursion relations (5.5) in a compact form

$$F^{(1)}(y;z) \doteqdot E^{(1)}(y;z) M_0^{(1)'}(y) \mid F^{(1)}(y;z) + E^{(1)}(y;z). \tag{5.7}$$

Matching the power of z on the two sides of Eq. (5.7) reproduces Eq. (5.5) for $k = 1,2,...$, after introducing the appropriate integration variables. Note that we cannot algebraically solve Eq. (5.7) for $F^{(1)}(y;z)$ since it is an equality only under the integral sign.

Next we use the identity

$$M_e^{(1)}(2j,2j+1) = [(y_{2j}+2i)^{-1},0] M_e^{(1)'}(2j+1) + y_{2j} M_e^{(1)''}(2j,2j+1), \tag{5.8}$$

where

$$M_e^{(1)'}(2j+1) = [e^{-ir}(y_{2j+1}+2i)^{-1}, y_{2j+1}^{-1}] \tag{5.9a}$$

and

$$M_e^{(1)''}(2j,2j+1) = -y_{2j+1}^{-1}(y_{2j}+y_{2j+1})^{-1}[0,(y_{2j}+2i)^{-1}]. \tag{5.9b}$$

Using this identity repeatedly, we can write

$$\begin{aligned} &|F_{2k}^{(1)}(2\to 2k+1) \\ &\doteqdot \sum_{l=0}^{k-1} |F_{2k-2l-1}^{(5)}(2\to 2k-2l)| \\ &\quad \times E_{2l+1}^{(2)}(2k-2l+1\to 2k+1) \\ &\quad + |E_{2k}^{(3)}(2\to 2k+1), \quad k=1,2,\ldots. \end{aligned} \tag{5.10}$$

The bars represent the fact that these factors are not connected to the matrices on the side of the bar when introduced in the integral. In Eq. (5.10)

$$\begin{aligned} &F_{2k-1}^{(5)}(2\to 2k) \\ &= \left[\prod_{j=1}^{k-1} M_e^{(1)}(2j,2j+1) M_0^{(1)}(2j+1,2j+2)\right] \\ &\quad \times [(y_{2k}+2i)^{-1},0], \quad (\text{product} \equiv 1 \text{ for } \\ &k=1), \quad k=1,2,\ldots, \end{aligned} \tag{5.11}$$

$$\begin{aligned} &E_{2l+1}^{(2)}(1\to 2l+1) \\ &= M_e^{(1)'}(1)\left[\prod_{j=1}^{l} y_{2j} M_0^{(1)}(2j-1,2j) M_e^{(1)''}(2j,2j+1)\right], \\ &\quad (\text{product} \equiv 1 \text{ for } l=0), \quad l=0,1,2,\ldots, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} &E_{2k}^{(3)}(2\to 2k+1) \\ &= y_2 M_e^{(1)''}(2,3)\left[\prod_{j=2}^{k} y_{2j} M_0^{(1)}(2j-1,2j) M_e^{(1)''}(2j,2j+1)\right], \\ &\quad (\text{product} \equiv 1 \text{ for } k=1), \quad k=1,2,\ldots. \end{aligned} \tag{5.13}$$

Defining the generating functions

$$F^{(5)}(y;z) \equiv F^{(5)}(y,z;r,g) \equiv \sum_{n=0}^{\infty} z^{2n+1} F_{2n+1}^{(5)}(y), \tag{5.14a}$$

$$E^{(2)}(y;z) \equiv E^{(2)}(y,z;r) \equiv \sum_{n=0}^{\infty} z^{2n+1} E_{2n+1}^{(2)}(y), \tag{5.14b}$$

and

$$E^{(3)}(y;z) \equiv E^{(3)}(y,z;r) \equiv \sum_{n=1}^{\infty} z^{2n} E_{2n}^{(3)}(y), \tag{5.14c}$$

and using the recursion relations (5.10) and Eq. (5.7), we can write

$$\begin{aligned} F^{(1)}(y;z) &\doteqdot E^{(1)}(y;z) M_0^{(1)'}(y) | F^{(5)}(y;z) | E^{(2)}(y;z) \\ &\quad + E^{(1)}(y;z) M_0^{(1)'}(y) | E^{(3)}(y;z) + E^{(1)}(y;z). \end{aligned} \tag{5.15}$$

A similar analysis of $F^{(2)}(y;z)$, $F^{(3)}(y;z)$, and $F^{(4)}(y;z)$ results in

$$\begin{aligned} F^{(2)}(y;z) &\doteqdot zE^{(1)}(y;z) M_0^{(1)'}(y) | F^{(5)}(y;z) | E^{(2)}(y;z) M_0^{(1)}(y) \\ &\quad + zE^{(1)}(y;z) M_0^{(1)'}(y) | E^{(3)}(y;z) M_0^{(1)}(y) \\ &\quad + E^{(4)}(y;z), \end{aligned} \tag{5.16}$$

$$\begin{aligned} F^{(3)}(y;z) &\doteqdot E^{(5)}(y;z) | F^{(5)}(y;z) | E^{(2)}(y;z) + E^{(5)}(y;z) | \\ &\quad \times E^{(3)}(y;z) + E^{(6)}(y;z), \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} F^{(4)}(y;z) &\doteqdot zE^{(5)}(y;z) | F^{(5)}(y;z) | E^{(2)}(y;z) M_0^{(1)}(y) \\ &\quad + zE^{(5)}(y;z) | E^{(3)}(y;z) M_0^{(1)}(y) \\ &\quad + zE^{(6)}(y;z) M_0^{(1)}(y), \end{aligned} \tag{5.18}$$

respectively.

In Eqs. (5.16)–(5.18)

$$E^{(j)}(y;z) \equiv E^{(j)}(y,z;r) \equiv \sum_{n=0}^{\infty} z^{2n+1} E_{2n+1}^{(j)}(y), \quad j=4,5,6, \tag{5.19}$$

with

$$\begin{aligned} &E_1^{(4)} = 1, \\ &E_{2n+1}^{(4)}(2\to 2n+2) \\ &\quad = E_{2n}^{(1)}(2\to 2n+1) M_0^{(1)}(2n+1,2n+2), \\ &\qquad \text{for} \quad n \geqslant 1, \end{aligned} \tag{5.20}$$

$$\begin{aligned} &E_1^{(5)} = M_0^{(1)'}(1), \\ &E_{2n+1}^{(5)}(1\to 2n+1) \\ &\quad = y_2 M_0^{(1)''}(1,2) E^{(1)}(2\to 2n+1) M_0^{(1)'}(2n+1), \\ &\qquad \text{for} \quad n \geqslant 1, \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} &E_1^{(6)} = 1, \\ &E_{2n+1}^{(6)}(1\to 2n+1) \\ &\quad = y_2 M_0^{(1)''}(1,2) E_{2n}^{(1)}(2\to 2n+1), \quad \text{for } n \geqslant 1. \end{aligned} \tag{5.22}$$

On the right hand side of Eqs. (5.15)–(5.18) all the divergences (as $g\to 0$) are in the $F^{(5)}(y;z)$ term. $F_{(y;z)}^{(5)}$ is disconnected from the factors (which are convergent) on either side. The integrals over the y_j variables in $F_{2n+1}^{(5)}(y)$ factor out, and these can be done independently of the other integrations. We will analyze $F^{(5)}(y;z)$ in the next subsection.

We will summarize our notation here for reference: (1) In $E_n^{(j)}(y)$ or $F_n^{(j)}(y)$, y stands for the n y_j variables appearing in $E_n^{(j)}(y)$ or $F_n^{(j)}(y)$. (2) In the generating functions $E^{(j)}(y;z)$ or $F^{(j)}(y;z)$, y stands for an arbitrary number of y_j variables. (3) The choice of the y_j variables in a factor is determined by the y_j variables on either side. For examples,

$$\begin{aligned} &E_n^{(j)}(y) F_m^{(k)}(y) E_p^{(l)}(y) \\ &\quad \equiv E_n^{(j)}(y_1\to y_n) F_m^{(k)}(y_n\to y_{n+m-1}) \\ &\qquad \times E_p^{(l)}(y_{n+m-1}\to y_{n+m+p-2}). \end{aligned} \tag{5.23}$$

DIAGRAM	TERM IN INTEGRAND
$\underset{j}{\times}$	dy_j
$\underset{j}{\overset{+}{\times}}$	(y_j+2i) for j odd $(y_j+2i)^{-1}$ for j even
$\underset{j}{\overset{-}{\times}}$	(y_j-2i) for j odd $(y_j-2i)^{-1}$ for j even
$\underset{j}{\times}\text{----}\underset{j+1}{\times}$	$(y_j+y_{j+1})^{-1}$
Example	
$\underset{4}{\overset{+}{\times}}\text{----}\underset{5}{\overset{-}{\times}}$	$\dfrac{(y_5-2i)}{(y_4+2i)(y_4+y_5)}$

FIG. 5. Diagrammatic representation for the different functions of y_j occurring in the integrand.

(4) A bar separating two factors indicates that these factors do not have any y_j variables in common. For example,

$$E_n^{(j)}(y)\,|\,F_m^{(k)}(y) \equiv E_n^{(j)}(y_1\to y_n)F_m^{(k)}(y_{n+1}\to y_{n+m}). \tag{5.24}$$

(5) A product of two generating functions is understood in the following sense: Let

$$F^{(k)}(y;z) \equiv \sum_{n=1}^{\infty} z^n F_n^{(k)}(y) \tag{5.25a}$$

and

$$F^{(m)}(y;z) \equiv \sum_{n=1}^{\infty} z^n F_n^{(m)}(y). \tag{5.25b}$$

Then,

$$F^{(k)}(y;z)F^{(m)}(y;z) \equiv \sum_{n=2}^{\infty} z^n \sum_{l=1}^{n-1} F_l^{(k)}(y_1\to y_l)F_{n-l}^{(m)}(y_l\to y_{n-1}) \tag{5.26}$$

and

$$F^{(k)}(y;z)\,|\,F^{(m)}(y;z) \equiv \sum_{n=2}^{\infty} z^n \sum_{l=1}^{n-1} F_l^{(k)}(y_1\to y_l)F_{n-l}^{(m)}(y_{l+1}\to y_n). \tag{5.27}$$

B. Analysis of $F^{(5)}(y;z)$

At this stage we will intorduce the diagrammatic notation of Fig. 5 to represent the different terms in the integrand. We can write $F^{(5)}_{2k+1}(2\to 2k+2)$ in the following form by multiplying the pairs of matrices $M_e^{(1)}(2j,2j+1)M_0^{(1)}(2j+1,2j+2)$ in Eq. (5.11):

$$F^{(5)}_{2k+1}(2\to 2k+2) \doteq \prod_{j=1}^{k}[a(2j\to 2j+2),b(2j\to 2j+2)] \times[(y_{2k+2}+2i)^{-1},0] \quad (\text{product}\equiv 1 \text{ for } k=0), \tag{5.28}$$

where

$$a(2j\to 2j+2) = \frac{(y_{2j+1}-2i)}{(y_{2j}+2i)(y_{2j}+y_{2j+1})(y_{2j+1}+y_{2j+2})} + \frac{e^{-2ir}}{(y_{2j}+2i)(y_{2j+1}+2i)} \tag{5.29a}$$

$$\equiv \underset{2j}{\overset{+}{\times}}\text{----}\underset{2j+1}{\overset{-}{\times}}\text{----}\underset{2j+2}{\times} + \frac{e^{-2ir}}{(y_{2j+1}+2i)}\underset{2j}{\overset{+}{\times}}\;\underset{2j+1}{\times}\;\underset{2j+2}{\times} \tag{5.29b}$$

and

$$b(2j\to 2j+2) = e^{-ir}\underset{2j}{\overset{+}{\times}}\;\underset{2j+1}{\times}\text{----}\underset{2j+2}{\times} + e^{+ir}\underset{2j}{\overset{+}{\times}}\text{----}\underset{2j+1}{\times}\;\underset{2j+2}{\times}. \tag{5.30}$$

In the second term of Eq. (5.29b), we can integrate over y_{2j+1} (and set $g=0$). Thus,

$$a(2j\to 2j+2) = \underset{2j}{\overset{+}{\times}}\text{----}\underset{2j+1}{\overset{-}{\times}}\text{----}\underset{2j+2}{\times} + e(r)\underset{2j}{\overset{+}{\times}}\;\underset{2j+2}{\times}, \tag{5.29c}$$

where

$$e(r) = e^{-2ir}\int_0^{\infty} dx\, e^{-rx}x(x+2i)^{-1}. \tag{5.31}$$

We can write Eq. (5.28) in the following form:

$$F^{(5)}_{2k+1}(2\to 2k+2) \doteq F^{(6)}_{2k+1}(2\to 2k+2) + \sum_{l=1}^{k} F^{(7)}_{2k+1,2l}(2\to 2k+2), \quad (\text{sum}\equiv 0 \text{ for } k=0),\quad k=0,1,2,\dots, \tag{5.32}$$

where

$$F^{(6)}_{2k+1}(2\to 2k+2) \doteq \left[\left(\prod_{j=1}^{k} a(2j\to 2j+2)\right)\underset{2k+2}{\overset{+}{\times}},0\right], \quad (\text{product}\equiv 1 \text{ for } k=0),\quad k=0,1,2,\dots \tag{5.33}$$

is a diagonal matrix and

$$F^{(7)}_{2k+1,2l}(2\to 2k+2) \doteq \left[\prod_{j=1}^{l-1} a(2j\to 2j+2),0\right][0,b(2l\to 2l+2)] \times\left\{\prod_{j=l+1}^{k}[a(2j\to 2j+2),b(2j\to 2j+2)]\right\} \times\left[\underset{2k+2}{\overset{+}{\times}},0\right],$$

(first product≡1 for $l=1$, second product≡1 for $l=k$),

$$l\leqslant k,\quad k=1,2,\dots. \tag{5.34}$$

Using the identity

$$b(2j\to 2j+2) = \overset{+}{\underset{2j}{\times}} b_1(2j+1,2j+2) + b_2(2j,2j+1)\underset{2j+2}{\times}, \tag{5.35}$$

where

$$b_1(2j+1,2j+2) = e^{-ir}\underset{2j+1}{\times}\text{----}\underset{2j+2}{\times} \tag{5.36a}$$

and

$$b_2(2j,2j+1) = e^{+ir}\overset{+}{\underset{2j}{\times}}\text{----}\underset{2j+1}{\times} \tag{5.36b}$$

we can factorize $F^{(7)}_{2k+1,2l}(2\to 2k+2)$ as

$$\begin{aligned}&F^{(7)}_{2k+1,2l}(2\to 2k+2)\\ &\doteq F^{(6)}_{2l-1}(2\to 2l)\,|\,S^{(1)}_{2k-2l+2}(2l+1\to 2k+2)\\ &\quad + F^{(8)}_{2l}(2\to 2l+1)\,|\,F^{(5)}_{2k-2l+1}(2l+2\to 2k+2),\\ &\qquad l\leqslant k,\quad k=1,2,3,\ldots,\end{aligned} \tag{5.37}$$

where

$$\begin{aligned}&S^{(1)}_{2n}(1\to 2n)\\ &\quad\doteq [0,b_1(1,2)]\prod_{j=1}^{n-1}[a(2j\to 2j+2),b(2j\to 2j+2)]\\ &\quad\times\left[\overset{+}{\underset{2n}{\times}},0\right],\\ &\qquad(\text{product}\equiv 1\text{ for }n=1),\quad n=1,2,\ldots,\end{aligned} \tag{5.38}$$

$$F^{(8)}_{2n}(1\to 2n+1)\doteq\prod_{j=1}^{n-1}[a(2j\to 2j+2),0]\,[0,b_2(2n,2n+1)]$$
$$(\text{product}\equiv 1\text{ for }n=1),\quad n=1,2,\ldots. \tag{5.39}$$

Carrying out a similar factorization for $S^{(1)}_{2n}(1\to 2n)$, we get the relations

$$S^{(1)}_2(1,2)\doteq F^{(8)^T}_2(1,2) \tag{5.40a}$$

and

$$\begin{aligned}&S^{(1)}_{2n}(1\to 2n)\\ &\quad\doteq\sum_{j=1}^{n-1}F^{(8)^T}_{2j}(1\to 2j)\,|\,S^{(1)}_{2n-2j}(2j+1\to 2n)\\ &\qquad+\sum_{j=1}^{n-1}F^{(9)}_{2j+1}(1\to 2j+1)\,|\,F^{(5)}_{2n-2j-1}(2j+2\to 2n)\\ &\qquad+F^{(8)^T}_{2n}(1\to 2n),\\ &\qquad\qquad n=2,3,\ldots,\end{aligned} \tag{5.40b}$$

where

$$F^{(8)^T}_{2j}(1\to 2j)\doteq[0,b_1(1,2)]\prod_{l=1}^{j-1}[a(2l\to 2l+2),0]\left[\overset{+}{\underset{2j}{\times}},0\right],$$
$$(\text{product}\equiv 1\text{ for }j=1),\quad j=1,2,\ldots \tag{5.41}$$

is the transpose of $F^{(8)}_{2j}(y)$ (under the integral) and

$$\begin{aligned}F^{(9)}_{2j+1}(1\to 2j+1)&\doteq[0,b_1(1,2)]\prod_{l=1}^{j-1}[a(2l\to 2l+2),0]\\ &\quad\times[0,b_2(2j,2j+1)],\\ &(\text{product}\equiv 1\text{ for }j=1),\quad j=1,2,\ldots.\end{aligned} \tag{5.42}$$

We define the generating functions

$$F^{(6)}(y;z)\equiv F^{(6)}(y,z;r,g)\equiv\sum_{n=0}^{\infty}z^{2n+1}F^{(6)}_{2n+1}(y), \tag{5.43a}$$

$$S^{(1)}(y;z)\equiv S^{(1)}(y,z;r,g)\equiv\sum_{n=1}^{\infty}z^{2n}S^{(1)}_{2n}(y), \tag{5.43b}$$

$$F^{(8)}(y;z)\equiv F^{(8)}(y,z;r,g)\equiv\sum_{n=1}^{\infty}z^{2n}F^{(8)}_{2n}(y), \tag{5.43c}$$

and

$$F^{(9)}(y;z)\equiv F^{(9)}(y,z;r,g)\equiv\sum_{n=1}^{\infty}z^{2n+1}F^{(9)}_{2n+1}(y). \tag{5.43d}$$

Then Eqs. (5.40) imply that

$$S^{(1)}(y;z)\doteq F^{(8)^T}(y;z)\,|\,S^{(1)}(y;z)+F^{(9)}(y;z)\,|\,F^{(5)}(y;z)+F^{(8)^T}(y;z). \tag{5.44}$$

Solving for $S^{(1)}(y;z)$, we have

$$S^{(1)}(y;z)\doteq[1-F^{(8)^T}(y;z)]^{-1}\{F^{(8)^T}(y;z)+F^{(9)}(y;z)\,|\,F^{(5)}(y;z)\}. \tag{5.45}$$

Using the relations (5.32) and (5.37), we get

$$F^{(5)}(y;z)\doteq[1-F^{(8)}(y;z)-F^{(6)}(y;z)[1-F^{(8)^T}(y;z)]^{-1}F^{(9)}(y;z)]^{-1}F^{(6)}(y;z)[1-F^{(8)^T}(y;z)]^{-1}. \tag{5.46}$$

What Eq. (5.46) says is the following: $F^{(j)}(y;z)$ ($j=5,6,8$, and 9) are 2×2 matrices each term of which is a power series in z. The coefficient of z^n in the power series is an integrand with n variables. If we evaluate all the integrals and sum up the power series, we will get a function of z, r, and g. On substituting these functions in the elements of $F^{(j)}(y;z)$ and carrying out the operation on the right hand side of Eq. (5.46) we get an equality. At various stages of the above procedure we have factored out integrals so that they can be done for the different $F^{(j)}_n(y)$ s independently of one another.

Carrying out the matrix multiplications in Eq. (5.33), (5.39), and (5.42), we can write

$$\int dyF^{(6)}(y;z)=[q^{(1)}(z),0], \tag{5.47a}$$

$$\int dyF^{(8)}(y;z)=[0,q^{(2)}(z)], \tag{5.47b}$$

and

$$\int dyF^{(9)}(y;z)=[q^{(3)*}(z),0], \tag{5.47c}$$

where

$$q^{(1)}(z)\equiv q^{(1)}(z;r,g) = \int dy \sum_{n=0}^{\infty} z^{2n+1} q^{(1)}_{2n+1}(y), \tag{5.48}$$

$$q^{(2)}(z)\equiv q^{(2)}(z;r,g) = \int dy \sum_{n=1}^{\infty} z^{2n} q^{(2)}_{2n}(y), \tag{5.49}$$

and

$$q^{(3)}(z)\equiv q^{(3)}(z;r,g) = \int dy \sum_{n=1}^{\infty} z^{2n+1} q^{(3)}_{2n+1}(y)\,. \tag{5.50}$$

In Eqs. (5.48)–(5.50)

$$q^{(1)}_{2n+1}(2\rightarrow 2n+2) \doteq \left[\prod_{j=1}^{n} a(2j\rightarrow 2j+2)\right] \overset{+}{\underset{2n+2}{\times}}, \quad n=0,1,2,\ldots, \tag{5.51a}$$

$$q^{(2)}_{2n}(1\rightarrow 2n) \doteq b^{*}_{1}(1,2)\left[\prod_{j=1}^{n-1} a(2j\rightarrow 2j+2)\right] \overset{+}{\underset{2n}{\times}}, \quad n=1,2,3,\ldots, \tag{5.51b}$$

and

$$q^{(3)}_{2n+1}(1\rightarrow 2n+1) \doteq b^{*}_{1}(1,2)\left[\prod_{j=1}^{n-1} a(2j\rightarrow 2j+2)\right] b_2(2n,2n+1), \quad n=1,2,\ldots\,. \tag{5.51c}$$

In Eq. (5.51a) the product$\equiv 1$ for $n=0$ and in Eqs. (5.51b) and (5.51c) for $n=1$. Substituting Eq. (5.51) in (5.46) and carrying out the matrix multiplications leads to Eq. (7.15), where

$$\lim_{g\rightarrow 0} q^{(j)}(\pi^{-1};r,g) \equiv q^{(j)}(r). \tag{5.52}$$

C. Analysis of $q^{(j)}(z)$, $j=1,2,3$

We first show that

$$q^{(1)}(z) = \frac{A(z)}{1 - ze(r)A(z)}, \tag{5.53}$$

where

$$A(z;r,g) \equiv A(z) = \int dy \sum_{k=0}^{\infty} z^{2k+1} \alpha_{2k+1}(y), \tag{5.54a}$$

with

$$\alpha_{2k+1}(2\rightarrow 2k+2) \doteq \overset{+}{\underset{2}{\times}} - - - - \overset{-}{\underset{3}{\times}} - - - - \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2k+1}{\times}} - - - \overset{+}{\underset{2k+2}{\times}}, \quad k=0,1,\ldots\,. \tag{5.54b}$$

Clearly,

$$q^{(1)}_{1}(2) \doteq \alpha_1(2), \tag{5.55}$$

$$q^{(1)}_{3}(2\rightarrow 4) \doteq \overset{+}{\underset{2}{\times}} - - - - \overset{-}{\underset{3}{\times}} - - - - \overset{+}{\underset{4}{\times}} + e(r)\, \overset{+}{\underset{2}{\times}}\ \overset{+}{\underset{4}{\times}} \doteq \alpha_3(2\rightarrow 4) + e(r)\alpha_1(2)\alpha_1(4) \tag{5.56a,b}$$

and satisfies Eq. (5.53).

We will now prove Eq. (5.53) by induction. Let

$$q^{(1)}_{2k-1}(4\rightarrow 2k+2) \doteq \left[\frac{A(z)}{1 - ze(r)A(z)}\right]_{2k-1}, \tag{5.57}$$

where the subscript $2k-1$ denotes the coefficients of z^{2k-1} in the expansion on the right hand side (without the integration over y). Then,

$$q^{(1)}_{2k+1}(2\rightarrow 2k+1) \doteq a(2\rightarrow 4) q^{(1)}_{2k-1}(4\rightarrow 2k+1) \tag{5.58}$$

$$\doteq \left[\overset{+}{\underset{2}{\times}} - - - - \overset{-}{\underset{3}{\times}} - - - - \underset{4}{\times} + e(r)\, \overset{+}{\underset{2}{\times}}\ \underset{4}{\times}\right] \times \{\alpha_{2k-1} + e(r)[A^2(z)]_{2k-2} + \cdots + e^{k-1}(r)\alpha_1^{k}\}, \tag{5.59}$$

where inside the curly bracket the variables of integration start from y_4. Therefore,

$$q^{(1)}_{2k+1}(2\rightarrow 2k+2) \doteq \alpha_{(2k+1}(2\rightarrow 2k+2) + e^{k}(r)\alpha_1^{k+1} + \sum_{l=1}^{k-1} e^{l}(r) \times \left\{\alpha_1(2)[A^{l}(z)]_{2k-l} + \overset{+}{\underset{2}{\times}} - - - - \overset{-}{\underset{3}{\times}} - - - - \underset{4}{\times}\ [A^{l+1}(z)]_{2k-l+1}\right\}. \tag{5.60}$$

Now

$$\overset{+}{\underset{2}{\times}}\text{- - - -}\overset{-}{\underset{3}{\times}}\text{- - - -}\underset{4}{\times}\;[A^2(z)]_{2k-2}\doteqdot\overset{+}{\underset{2}{\times}}\text{- - - -}\overset{-}{\underset{3}{\times}}\text{- - - -}\underset{4}{\times}\sum_{l=1}^{k-1}\alpha_{2l-1}(4\longrightarrow 2l+2)\alpha_{2k-2l-1}(2l+4\longrightarrow 2k+2) \tag{5.61}$$

$$\doteqdot\sum_{l=1}^{k-1}\alpha_{2l+1}(2\longrightarrow 2l+2)\alpha_{2k-2l-1}(2l+4\longrightarrow 2k+2) \tag{5.62}$$

$$\doteqdot[A^2(z)]_{2k}-\alpha_1(2)\alpha_{2k-1}(4\longrightarrow 2k+2)\,. \tag{5.63}$$

Similarly, it can be shown that

$$\overset{+}{\underset{2}{\times}}\text{- - - -}\overset{-}{\underset{3}{\times}}\text{- - - -}\underset{4}{\times}\;[A^{l+1}(z)]_{2k-l-1}\doteqdot(A^{l+1}(z))_{2k-l+1}-\alpha_1(2)[A^l(z)]_{2k-l}\,,\quad\text{for } l=2,3,\cdots,k-1. \tag{5.64}$$

Using the relations (5.63) and (5.64) in Eq. (5.60), we get

$$q^{(1)}_{2k+1}(2\longrightarrow 2k+2)\doteqdot\left[\frac{A(z)}{1-ze(r)A(z)}\right]_{2k+1} \tag{5.65}$$

Hence, by induction, Eq. (5.53) holds.

In $\alpha_{2k+1}(2\longrightarrow 2k+2)$ we use the identity

$$\overset{-}{\underset{2j-1}{\times}}\text{- - - -}\overset{+}{\underset{2j}{\times}}=-\underset{2j-1}{\times}\text{- - - -}\underset{2j}{\times}+\underset{2j-1}{\times}\;\overset{+}{\underset{2j}{\times}} \tag{5.66}$$

to write

$$\alpha_{2k+1}(2\longrightarrow 2k+2)\doteqdot\Big(\overset{+}{\underset{2}{\times}}\text{- - - -}\underset{3}{\times}\Big)\alpha_{2k-1}(4\longrightarrow 2k+2)$$
$$-\overset{+}{\underset{2}{\times}}\text{- - - -}\underset{3}{\times}\text{- - - -}\underset{4}{\times}\text{- - - -}\overset{-}{\underset{5}{\times}}\text{- - - -}\overset{+}{\underset{6}{\times}}\cdots\overset{-}{\underset{2k+1}{\times}}\text{- - - -}\overset{+}{\underset{2k+2}{\times}}\,. \tag{5.67}$$

Using the identity (5.66) for $j=3,4,\cdots,k+1$ and defining

$$\gamma_k(2\rightarrow k+1)\doteqdot\overset{+}{\underset{2}{\times}}\text{- - - -}\underset{3}{\times}\text{- - - -}\underset{4}{\times}\cdots\underset{k}{\times}\text{- - - -}\underset{k+1}{\times}\,,\quad k=1,2,\ldots, \tag{5.68}$$

we obtain the recursion relations

$$\alpha_1(2)\doteqdot\gamma_1(2) \tag{5.69a}$$

and

$$\alpha_{2k+1}(2\longrightarrow 2k+2)\doteqdot(-1)^k\gamma_{2k+1}(2\longrightarrow 2k+2)$$
$$+\sum_{l=1}^{k}(-1)^{l+1}\gamma_{2l}(2\longrightarrow 2l+1)\alpha_{2k-2l+1}(2l+2\longrightarrow 2k+2)\,,\quad k=1,2,\ldots\,. \tag{5.69b}$$

We define the generating functions

$$\Gamma^{(1)}(z)\equiv\Gamma^{(1)}(z;r,g)=\int dy\sum_{k=0}^{\infty}(-1)^k z^{2k+1}\gamma_{2k+1}(y), \tag{5.70a}$$

$$\Gamma^{(2)}(z)\equiv\Gamma^{(2)}(z;r,g)=\int dy\sum_{k=1}^{\infty}(-1)^{k+1}z^{2k}\gamma_{2k}(y)\,. \tag{5.70b}$$

Then Eq. (5.69) implies that

$$A(z)=\frac{\Gamma^{(1)}(z)}{1-\Gamma^{(2)}(z)}\,. \tag{5.71}$$

We can rewrite $g_{2k+1}(gr)$ (see Eq. (3.3)] as

$$g_{2k+1}(gr)=(-1)^k\int_g^\infty dy_2\cdots\int_g^\infty dy_{2k+2}\prod_{j=2}^{2k+2}e^{-ry_i}\prod_{j=2}^{2k+1}(y_j+y_{j+1})^{-1}(y^2_{2k+2}-g^2)^{-1/2}\prod_{j=1}^{k}\frac{(y^2_{2j+1}-g^2)^{1/2}}{(y^2_{2j}-g^2)^{1/2}}, \tag{5.72}$$

$$=(-1)^k\int dy\Big[\underset{2}{\times}\text{- - - -}\underset{3}{\times}\text{- - - -}\underset{4}{\times}\cdots\underset{2k+1}{\times}\text{- - - -}\underset{2k+2}{\times}\Big]\,. \tag{5.73}$$

Now in $\gamma_{2k+1}(2\longrightarrow 2k+2)$ we use the identity

$$\overset{+}{\underset{2}{\times}}=(2i)^{-1}\Big[\underset{2}{\times}-y_2\overset{+}{\underset{2}{\times}}\Big] \tag{5.74}$$

to write

$$\int dy\gamma_{2k+1}(2\longrightarrow 2k+2)=(2i)^{-1}\Big[(-1)^k g_{2k+1}(gr)-\int dyy_2\overset{+}{\underset{2}{\times}}\text{- - - -}\underset{3}{\times}\cdots\underset{2k+1}{\times}\text{- - - -}\underset{2k+2}{\times}\Big]\,. \tag{5.75}$$

Next we use the identity

$$\underset{2j+1}{\times}\text{----}\underset{2j+2}{\times} = y_{2j+1}^{-1}\left[\underset{2j+1}{\times}\underset{2j+2}{\times} - y_{2j+2}\underset{2j+1}{\times}\text{----}\underset{2j+2}{\times}\right] \tag{5.76}$$

for $j = 1,2,\cdots,k$ to write

$$\int dy\gamma_1(2) = (2i)^{-1}\left[g_1(gr) - \int dy y_2 \overset{+}{\underset{2}{\times}}\right] \tag{5.77a}$$

and

$$\int dy\gamma_{2k+1}(2\to 2k+2)$$

$$= (-1)^k(2i)^{-1}\Bigg\{g_{2k+1}(gr) + \sum_{l=1}^{k} g_{2k-2l+1}(gr)\int dy\left[\prod_{j=1}^{l}\frac{y_{2j}}{y_{2j+1}}\overset{+}{\underset{2}{\times}}\text{----}\underset{3}{\times}\text{----}\underset{4}{\times}\cdots\underset{2l}{\times}\text{----}\underset{2l+1}{\times}\right]$$

$$-\int dy\left(\frac{\Pi_{j=1}^{k+1}y_{2j}}{\Pi_{j=1}^{k}y_{2j+1}}\right)\overset{+}{\underset{2}{\times}}\text{----}\underset{3}{\times}\text{----}\underset{4}{\times}\cdots\underset{2k+1}{\times}\text{----}\underset{2k+1}{\times}\Bigg\},\quad k=1,2,\ldots. \tag{5.77b}$$

It is clear that in the last term and in the integrals multiplying the $g_{2m+1}(gr)$'s in the second term we can set $g = 0$. We do this and define

$$e^{(1)}(z;r) = \sum_{l=0}^{\infty} z^{2l}e_{2l}^{(1)}(r) \tag{5.78a}$$

and

$$e^{(2)}(z;r) = \sum_{l=0}^{\infty} z^{2l+1}e_{2l+1}^{(1)}(r), \tag{5.78b}$$

where

$$e_0^{(1)}(r) = 1 \tag{5.79a}$$

and

$$e_l^{(1)}(r) = \int_0^\infty dy_1\cdots\int_0^\infty dy_l(y_1+2i)^{-1}\prod_{j=1}^{l}e^{-ry_j}\prod_{j=1}^{l-1}(y_j+y_{j+1})^{-1},\quad l\geqslant 1. \tag{5.79b}$$

Then the recursion relation (5.77) implies that, as $g\to 0$,

$$\Gamma^{(1)}(z) = (2i)^{-1}[G(z;gr)e^{(1)}(z;r) - e^{(2)}(z;r)] + O(g^2). \tag{5.80}$$

Similarly, we can show that

$$\Gamma^{(2)}(z) = 1 - e^{(1)}(z;r) + \Gamma^{(2)}(z)e^{(3)}(z;r) + O(g^2), \tag{5.81}$$

where

$$e^{(3)}(z;r) = \sum_{l=0}^{\infty} z^{2l+1}e_{2l+1}^{(3)}(r), \tag{5.82}$$

with

$$e_l^{(3)}(r) = \int_0^\infty dy_1\cdots\int_0^\infty dy_l\prod_{j=1}^{l}e^{-ry_j}\prod_{j=1}^{l-1}(y_j+y_{j+1})^{-1}. \tag{5.83}$$

This completes the factorization of $q^{(1)}(z)$. $q^{(2)}(z)$ and $q^{(3)}(z)$ can be analysed in a similar manner by relating them to $q^{(1)}(z)$ and functions of r. These functions are in turn related to $e^{(j)}(z;r)$ $(j = 1,2,3)$. We will not write down the details of this calculation.

Note that the g dependence of the functions $q^{(j)}(z)$ $(j = 1,2,3)$ is only through $G(gr;z)$. We use the fact

$$\lim_{g\to 0} G(gr,\pi^{-1}) = 1$$

to take this limit in $q^{(j)}(z)$. This leads to Eq. (7.7)–(7.9) on using $e^{(3)}(r) = (2r)^{-1}$ (which is shown later in this section).

We will next reduce $e^{(j)}(\pi^{-1};r)$ $(j = 1,2,3)$ to their final form [Eqs. (7.3), (7.4) and (5.96)]. The procedure is the same in all three cases and will be illustrated for $e^{(1)}(\pi^{-1};r)$.

Scaling out r in Eq. (5.79b) $(ry_j\to y_j)$, we can write $e_l^{(1)}(r)$ in the form of an integral over iterated kernels

$$e_l^{(1)}(r) = \int_0^\infty d\sigma_1\cdots\int_0^\infty d\sigma_l(y_1+2ir)^{-1}K(1,2)K(2,3)\cdots K(l-1,l),\quad l\geqslant 1, \tag{5.84}$$

where

$$d\sigma_j = e^{-y_j}dy_j,\quad K(j,j+1) = (y_j+y_{j+1})^{-1}. \tag{5.85}$$

The kernel $(y_j+y_{j+1})^{-1}$ has been studied in Ref. 9, where the eigenvalues and eigenfunctions are explicitly written down. We use the results in Sec. E of this reference (for the special case $\nu=0$)

$$\int_0^\infty d\sigma_2 K(1,2)\chi_p(2)=\lambda_p\chi_p(1), \tag{5.86}$$

where

$$\lambda_p=\pi\,\mathrm{sech}\pi p \tag{5.87}$$

are the eigenvalues and

$$\chi_p(x)=(2\lambda_p)^{-1/2}\int_1^\infty d\xi\exp[-(\xi-1)x/2]\varphi_p(\xi) \tag{5.88}$$

are the eigenfunctions of the integral equation (5.86).

In Eq. (5.86)

$$\varphi_p(\xi)=C_pF(\tfrac12+ip,\tfrac12-ip,1;\tfrac12-\tfrac12\xi) \tag{5.89}$$

where

$$C_p=(p\tanh\pi p)^{1/2} \tag{5.90}$$

and $F(a,b,c;z)$ is the hypergeometric function. $\phi_p(\xi)$ satisfies the integral equation

$$\int_1^\infty d\xi\frac{\varphi_p(\xi)}{\xi+x}=\lambda_p\,\varphi_p(x). \tag{5.91}$$

Writing the eigenfunction expansion for $K(j,j+1)$ as

$$K(j,j+1)=\int_0^\infty dp\,\lambda_p\,\chi_p(j)\chi_p(j+1) \tag{5.92}$$

and doing the integrations over $y_2\to y_l$ in Eq. (5.84), we can write

$$e_l^{(1)}(r)=\int_0^\infty dp\,\lambda_p^{\,l-1}\left[\int_0^\infty dx\,e^{-x}\chi_p(x)\right]\times\left[\int_0^\infty dy\,e^{-y}(y+2ir)^{-1}\chi_p(y)\right]. \tag{5.93}$$

Using Eq. (5.88) and (5.91), we can show that

$$\int_0^\infty dxe^{-x}\chi_p(x)=(2p\lambda_p\tanh\pi p)^{1/2}. \tag{5.94}$$

Using this in Eq. (5.92), doing the sum in Eq. (5.78a) (which in a geometric progression is $z^2\lambda^2\rho$), and setting $z=\pi^{-1}$, we get Eq. (7.3). A similar analysis leads to Eq. (7.4).

We can similarly show that

$$e_l^{(3)}(r)=\frac{2}{r}\int_0^\infty dpp\lambda_p^{\,l}\tanh\pi p. \tag{5.95}$$

Substituting in Eq. (5.82), summing up the series, setting $z=\pi^{-1}$, and doing the final integral over p, we get

$$e^{(3)}(\pi^{-1};r)=(2r)^{-1}. \tag{5.96}$$

This completes our analysis of the second term in Eq. (4.26) in the limit $g\to0$. On taking this limit we get the expression for $H^{(1)}(r)$ in the final result (see Sec. VII).

VI. ANALYSIS OF $G_0(1\to2n)$

A. Factorization of $G_0(1\to2n)$

We can write

$$G_0(1\to2n)\doteq\prod_{j=1}^n[a(2j\to2j+2),b(2j\to2j+2)]. \tag{6.1}$$

The product in Eq. (6.1) has a cyclic structure, i.e., $y_{2n+1}\equiv y_1$ and $y_{2n+2}\equiv y_2$. Separating the a's as in Eq. (5.32), we have

$$G_0(1\to2n)\doteq\left[\prod_{j=1}^n a(2j\to2j+2),0\right]+\sum_{l=1}^n[F_{2n,2l}^{(10)}(1\to2n)+F_{2n,2l}^{(11)}(2\to2n,1)], \tag{6.2}$$

where

$$F_{2n,2l}^{(10)}(1\to2n)\doteq[0,b_1(1,2)]\prod_{j=1}^{n-l}[a(2j\to2j+2),b(2j\to2j+2)]\times\prod_{j=n-l+1}^{n-1}[a(2j\to2j+2),0]\left[\overset{+}{\underset{2n}{\times}},0\right] \tag{6.3}$$

and

$$F_{2n,2l}^{(11)}(2\to2n,1)\doteq\prod_{j=1}^{n-l}[a(2j\to2j+2),b(2j\to2j+2)]\times\prod_{j=n-l+1}^{n-1}[a(2j\to2j+2),0]\times[0,b_2(2n,1)] \tag{6.4}$$

with $l\leqslant n$, $n=1,2,\ldots$. In Eq. (6.2) only the first term has a cyclic structure. In Eqs. (6.3) and (6.4) the first product$\equiv1$ for $l=n$ and the second product$\equiv1$ for $l=1$.

By methods similar to those used in Sec. V we can derive the following recursion relations:

$$F_{2n,2l}^{(10)}(1\to2n)\doteq F_{2n}^{(8)^T}(1\to2n)+\sum_{k=1}^{n-l}\{F_{2k}^{(8)^T}(1\to2k)|F_{2n-2k,2l}^{(10)}(2k+1\to2n)+F_{2k+1}^{(9)}(1\to2k+1)|F_{2n-2k-1,2l-1}^{(12)}(2k+2\to2n)\}, \tag{6.5}$$

where

$$F_{2n-1,2l-1}^{(12)}(2\to2n)\doteq\prod_{j=1}^{n-l}[a(2j\to2j+2),b(2j\to2j+2)]\prod_{j=n-l+1}^{n-1}[a(2j\to2j+2),0]\left[\overset{+}{\underset{2n}{\times}},0\right], \tag{6.6}$$

and

$$F^{(11)}_{2n,2l}(2\to 2n,1)\doteq F^{(8)}_{2n}(2\to 2n,1)+\sum_{k=1}^{n-l}\{F^{(6)}_{2k-1}(2\to 2k)\,|\,F^{(13)}_{2n-2k+1,2l}(2k+1\to 2n,1)$$
$$+F^{(8)}_{2k}(2\to 2k+1)\,|\,F^{(11)}_{2n-2k,2l}(2k+2\to 2n,1)\},\tag{6.7}$$

where

$$F^{(13)}_{2n-1,2l}(1\to 2n-1)\doteq[0,b_1(1,2)]\prod_{j=1}^{n-l-1}[a(2j\to 2j+2),b(2j\to 2j+2)]$$
$$\times\prod_{j=n-l}^{n-2}[a(2j\to 2j+2),0]\,[0,b_2(2n-2,2n-1)],\quad n\geqslant l+1,\ l\geqslant 1.\tag{6.8}$$

In Eqs. (6.5)–(6.7), $l\leqslant n$ and $n=1,2,\ldots$.

Defining the generating functions

$$F^{(k)}(z,\omega;y)\equiv F^{(k)}(z,\omega;y,r,g)\equiv\sum_{n=1}^{\infty}\sum_{l=1}^{n}z^{2n}\omega^{2l}F^{(k)}_{2n,2l}(y),\quad\text{for }k=10\text{ and }11,\tag{6.9a}$$

$$F^{(12)}(z,\omega;y)\equiv F^{(12)}(z,\omega;y,r,g)\equiv\sum_{n=1}^{\infty}\sum_{l=1}^{n}z^{2n-1}\omega^{2l-1}F^{(12)}_{2n-1,2l-1}(y),\tag{6.9b}$$

and

$$F^{(13)}(z,\omega;y)\equiv F^{(13)}(z,\omega;y,r,g)\equiv\sum_{n=2}^{\infty}\sum_{l=1}^{n-1}z^{2n-1}\omega^{2l}F^{(13)}_{2n-1,2l}(y),\tag{6.9c}$$

we have

$$F^{(10)}(z,1;y)\doteq F^{(8)^T}(y;z)\,\Big|\,F^{(10)}(z,1;y)+\frac{1}{2}z\frac{d}{dz}F^{(8)^T}(y;z)+F^{(9)}(y;z)\,\Big|\,F^{(12)}(z,1;y)\tag{6.10}$$

and

$$F^{(11)}(z,1;y)\doteq\frac{1}{2}z\frac{d}{dz}F^{(8)}(y;z)+F^{(6)}(y;z)\,|\,F^{(13)}(z,1;y)+F^{(8)}(y;z)\,|\,F^{(11)}(z,1;y).\tag{6.11}$$

The reason for evaluating these functions at $\omega=1$ will be clear later on. We can now reduce $F^{(12)}(z,1;y)$ and $F^{(13)}(z,1;y)$ by deriving recursion relations to get

$$F^{(12)}(z,1;y)\doteq\frac{1}{2}\frac{d}{dz}[zF^{(6)}(y;z)]+F^{(6)}(y;z)\,|\,F^{(10)}(z,1;y)+F^{(8)}(y;z)\,|\,F^{(12)}(z,1;y)\tag{6.12}$$

and

$$F^{(13)}(z,1;y)\doteq\frac{1}{2}\left(z\frac{d}{dz}-1\right)F^{(9)}(y;z)+F^{(8)^T}(y;z)\,|\,F^{(13)}(z,1;y)+F^{(9)}(y;z)\,|\,F^{(11)}(z,1;y).\tag{6.13}$$

Solving for $F^{(12)}(z,1;y)$ and $F^{(13)}(z,1;y)$ and substituting in Eqs. (6.10) and (6.11), we get

$$F^{(10)}(z,1;y)\doteq\frac{1}{2}[1-F^{(8)^T}(y;z)-F^{(9)}(y;z)[1-F^{(8)}(y;z)]^{-1}F^{(6)}(y;z)]^{-1}$$
$$\times\left\{z\frac{d}{dz}F^{(8)^T}(y;z)+F^{(9)}(y;z)[1-F^{(8)}(y;z)]^{-1}\left(z\frac{d}{dz}+1\right)F^{(6)}(y;z)\right\}\tag{6.14}$$

and

$$F^{(11)}(z,1;y)\doteq\frac{1}{2}[1-F^{(8)}(y;z)-F^{(6)}(y;z)[1-F^{(8)^T}(y;z)]^{-1}F^{(9)}(y;z)]^{-1}$$
$$\times\left\{z\frac{d}{dz}F^{(8)}(y;z)+F^{(6)}(y;z)[1-F^{(8)^T}(y;z)]^{-1}\left(z\frac{d}{dz}-1\right)F^{(9)}(y;z)\right\}.\tag{6.15}$$

$F^{(10)}(z,1;y)$ and $F^{(11)}(z,1;y)$ each have one term without any derivative. On taking the trace the contribution of these nonderivative terms to $\mathrm{Tr}[F^{(10)}(z,1;y)+F^{(11)}(z,1;y)]$ is of the form

$$\mathrm{Tr}[A^{-1}B-(A^T)^{-1}B^T]=\mathrm{Tr}[A^{-1}B-(A^{-1}B)^T]=0,\tag{6.16}$$

where

$$A=1-F^{(8)^T}(y;z)-F^{(9)}(y;z)[1-F^{(8)}(y;z)]^{-1}F^{(6)}(y;z)\tag{6.17a}$$

and

$$B=\tfrac{1}{2}F^{(9)}(y;z)[1-F^{(8)}(y;z)]^{-1}F^{(6)}(y;z)\tag{6.17b}$$

since

$$F^{(6)^T}(y;z)\doteq F^{(6)}(y;z)\quad\text{and}\quad F^{(9)^T}(y;z)=F^{(9)}(y;z).\tag{6.18}$$

Thus, the nonderivative terms cancel out on taking the trace.

Substrituting the expressions (5.47) for $F^{(6)}(y;z)$, $F^{(8)}(y;z)$, and $F^{(9)}(y;z)$ in Eqs. (6.14) and (6.15) and carrying out the matrix multiplications, we can show that

$$\int dy\, \mathrm{Tr}[F^{(10)}(z,1;y) + F^{(11)}(z,1;y)] = -\frac{1}{2} z \frac{d}{dz} \ln q(z)\,, \tag{6.19}$$

where $q(z)$ is given by

$$\begin{aligned} q(z) \equiv q(z;r,g) = {} & [1 - q^{(2)}(z) q^{(2)*}(z)]^2 - q^{(1)}(z) q^{(3)*}(z) - q^{(1)*}(z) q^{(3)}(z) + q^{(1)}(z) q^{(1)*}(z) q^{(3)}(z) q^{(3)*}(z) \\ & - q^{(1)}(z) q^{(2)*^2}(z) q^{(3)}(z) - q^{(1)*}(z) q^{(2)^2}(z) q^{(3)*}(z)\,. \end{aligned} \tag{6.20}$$

We need to evaluate $\Sigma_{n=1}^{\infty} n^{-1} z^{2n}\, \mathrm{Tr} G_0(1\to 2n)$. Using Eq. (6.2), we have

$$\sum_{n=1}^{\infty} n^{-1} z^{2n}\, \mathrm{Tr} G_0(1\to 2n) \doteqdot 2\, \mathrm{Re} \sum_{n=1}^{\infty} n^{-1} z^{2n} \prod_{j=1}^{n} a(2j\to 2j+2) + 2\int_0^z \frac{dz'}{z'}\, \mathrm{Tr}[F^{(10)}(z',1;y) + F^{(11)}(z',1;y)]\,. \tag{6.21}$$

The second term can be evaluated using Eq. (6.19). The first term will be analyzed in the next subsection.

B. Evaluation of $\Pi_{j=1}^{n} a(2j\to 2j+2)$

Because of its cyclic structure this term requires a different factorization than that of $q^{(1)}(z)$. Using the equation (5.29c) for $a(2j\to 2j+2)$, we can write

$$\prod_{j=1}^{n} a(2j\to 2j+2) \doteqdot \bar{a}_{2n}(2\to 2n,1) + e(r) \sum_{\substack{l=1 \\ n=1,2,\ldots}}^{n} d_{2n-1,2l-1}(2\to 2n)\,, \tag{6.22}$$

where

$$\bar{a}_{2n}(2\to 2n,1) \doteqdot (y_1+y_2)^{-1} \overset{+}{\underset{2}{\times}} \text{----} \overset{-}{\underset{3}{\times}} \text{----} \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2n-1}{\times}} \text{----} \overset{+}{\underset{2n}{\times}} \text{----} \overset{-}{\underset{1}{\times}} \tag{6.23}$$

and

$$d_{2n-1,2l-1}(2\to 2n) \doteqdot \Big[\prod_{j=1}^{n-1} a(2j\to 2j+2)\Big] \overset{+}{\underset{2n-2l+2}{\times}} \text{----} \overset{-}{\underset{2n-2l+3}{\times}} \text{----} \overset{+}{\underset{2n-2l+4}{\times}} \cdots \overset{-}{\underset{2n-1}{\times}} \text{----} \overset{+}{\underset{2n}{\times}}\,,$$

$$(\text{product} \equiv 1 \ \text{for}\ l = n)\,. \tag{6.24}$$

We can write down the recursion relations

$$d_{2n-1,2n-1}(2\to 2n) \doteqdot \alpha_{2n-1}(2\to 2n) \tag{6.25a}$$

and

$$d_{2n-1,2l-1}(2\to 2n) \doteqdot \alpha_{2n-1}(2\to 2n) + e(r) \sum_{m=0}^{n-l-1} \alpha_{2m+1}(2\to 2m+2) d_{2n-2m-3,2l-1}(2m+4\to 2n),$$

$$\text{for}\ l < n;\quad n = 2,3,\ldots\,. \tag{6.25b}$$

Define

$$D(z,\omega) \equiv D(z,w;r,g) = \int dy \sum_{n=1}^{\infty} \sum_{l=1}^{n} z^{2n-1} \omega^{2l-1} d_{2n-1,2l-1}(y)\,. \tag{6.26}$$

Then,

$$D(z,\omega) = \int dy \sum_{n=1}^{\infty} \sum_{l=1}^{n} z^{2n-1} \omega^{2l-1} \alpha_{2n-1}(y) + z\, e(r) A(z) D(z,\omega)\,. \tag{6.27}$$

Hence,

$$D(z,1) = \frac{1}{2} \frac{d}{dz} [zA(z)] + z\, e(r) A(z) D(z,1)\,. \tag{6.28}$$

Solving for $D(z,1)$, we get

$$D(z,1) = -[2e(r)]^{-1} \frac{d}{dz} \{\ln[1 - z\, e(r) A(z)]\}\,. \tag{6.29}$$

Now we use the identity (5.66),

$$\overset{-}{\underset{2j+1}{\times}} \text{----} \overset{+}{\underset{2j+2}{\times}} \doteqdot - \underset{2j+1}{\times} \text{----} \underset{2j+2}{\times} + \underset{2j+1}{\times} \overset{+}{\underset{2j+2}{\times}}\,, \tag{5.66}$$

for $j = 0,1,\cdots,n-1$ in Eq. (6.23) to get

$$\bar{a}_{2n}(2\to 2n,1) \doteqdot (-1)^n (y_1+y_2)^{-1} \underset{2}{\times} \text{----} \underset{3}{\times} \text{----} \underset{4}{\times} \cdots \underset{2n}{\times} \text{----} \underset{1}{\times} + \sum_{k=0}^{n-1} (-1)^k p_{2n,2n-2k}(2\to 2n,1)\,, \tag{6.30}$$

where

$$p_{2n,2k}(2\to 2n,1) \doteq \underset{2}{\overset{+}{\times}} \text{----} \underset{3}{\overset{-}{\times}} \text{----} \underset{4}{\overset{+}{\times}} \cdots \underset{2k-1}{\overset{-}{\times}} \text{----} \underset{2k}{\overset{+}{\times}} \text{----} \underset{2k+1}{\times} \cdots \underset{2n}{\times} \text{----} \underset{1}{\times} . \tag{6.31}$$

In Eq. (6.30) the first term is equal to $nf_{2n}(gr)$ (on integrating over y). Define

$$P(z,\omega) \equiv P(z,\omega;r,g) + \int dy \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{n+k} z^{2n} \omega^{2k} p_{2n,2k}(y) . \tag{6.32}$$

Then Eq. (6.30) leads to

$$\int dy \sum_{n=1}^{\infty} n^{-1} z^{2n} \bar{a}_{2n} = F(z;gr) + 2\int_0^z \frac{dz'}{z'} P(z',1) . \tag{6.33}$$

We now use Eq. (5.66) for $j = k - 1$ in Eq. (6.31) to get

$$p_{2n,2k}(2\to 2n,1) \doteq -p_{2n,2k-2}(2\to 2n,1) + p_{2k-2,2k-2}(2\to 2k-1)\gamma_{2n-2k+2}(2k\to 2n,1) , \quad \text{for } 2\leqslant k\leqslant n, \quad n = 2,3,... \tag{6.34}$$

and

$$p_{2n,2}(y) = \gamma_{2n}(y), \quad n = 1,2,... . \tag{6.35}$$

Thus,

$$\sum_{n=2}^{\infty} \sum_{k=2}^{n} (-1)^{n+k} z^{2n} \omega^{2k} p_{2n,2k}(y) \doteq \sum_{n=2}^{\infty} \sum_{k=2}^{n} (-1)^{n+k} z^{2n} \omega^{2k} [-p_{2n,2k-2}(y) + p_{2k-2,2k-2}(y)\gamma_{2n-2k+2}(y)] \tag{6.36}$$

Adding and subtracting the missing terms on the left and right leads to

$$P(z,\omega) - \omega^2 \Gamma^{(2)}(z) = \omega^2 \left\{ P(z,\omega) - [1 - \Gamma^{(2)}(z)] \int dy \sum_{n=1}^{\infty} (z\omega)^{2n} p_{2n,2n}(y) \right\} . \tag{6.37}$$

We can show that

$$\int dy \sum_{n=1}^{\infty} z^{2n} p_{2n,2n}(y) = \Gamma^{(2)}(z)[1 - \Gamma^{(2)}(z)]^{-1}. \tag{6.38}$$

Substituting Eq. (6.38) in (6.37) and solving for $P(z,\omega)$, we get

$$P(z,\omega) = \frac{\omega^2[\Gamma^{(2)}(z) - \Gamma^{(2)}(z\omega)]}{(1-\omega^2)[1 - \Gamma^{(2)}(z\omega)]} . \tag{6.39}$$

For $\omega = 1$, both the numerator and the denominator in Eq. (6.39) are equal to zero. Using L'Hospital's rule, we have

$$P(z,1) = -\frac{z}{2}\frac{d}{dz} \ln[1 - \Gamma^{(2)}(z)] . \tag{6.40}$$

Equations (6.22), (6.29), (6.33), and (6.40) lead to

$$\int dy \sum_{n=1}^{\infty} n^{-1} z^{2n} \sum_{j=1}^{n} a(2j\to 2j+2) = F(z;gr) - \ln[1 - \Gamma^{(2)}(z) - z\, e(r)\Gamma^{(1)}(z)] . \tag{6.41}$$

Sustituting Eq. (6.41) in (6.21) and doing the integral using Eq. (6.19) leads to

$$\int dy \sum_{n=1}^{\infty} n^{-1} z^{2n} \,\mathrm{Tr} G_0(1\to 2n) = 2F(z;gr) - 2\,\mathrm{Re}\{\ln[1 - \Gamma^{(2)}(z) - z\, e(r)\Gamma^{(1)}(z)]\} - \ln q(z) . \tag{6.42}$$

We can now summarize the results of Secs. IV–VI to write down $\bar{H}[z;r,G(z;gr)]$ [see Eq. (3.4)]

$$\bar{H}[z;r,G(z;gr)] = \int dy \sum_{l=1}^{\infty} l^{-1} \,\mathrm{Tr}\mathscr{N}^l(y;z) - 2\,\mathrm{Re}\ln[1 - \Gamma^{(2)}(z) - z\, e(r)\Gamma^{(1)}(z)] - \ln q(z) . \tag{6.43}$$

The lower limit of integration in the first term of Eq. (6.43) is now zero. We have shown that in all the terms on the right hand side of Eq. (6.43) the g dependence is only through $G(z;gr)$.

We can now use Eq. (2.23), (2.26), and (2.32) of Ref. 7, which say that

$$\lim_{g\to 0} (gr)^{1/2} \exp[-2F(\pi^{-1},gr)] = \pi^{-1}\rho_\infty , \tag{6.44}$$

where we have expressed the constants in Eq. (2.32) in terms of ρ_∞ given by Eq. (1.7). This completes the analysis of the $g\to 0$ limit of $g^{1/2}\exp[-H(\pi^{-1};r,g)]$. Setting

$$\lim_{g\to 0} \Gamma^{(1)}(\pi^{-1};r,g) = \Gamma^{(1)}(r), \tag{6.45}$$

$$\lim_{g\to 0} q(\pi^{-1};r,g) = q(r), \tag{6.46}$$

we get the final answer for $q(r)$ which is written out in detail in the next section.

VII. RESULTS

Using the results of Secs. IV–VI, we can write down the final form of $\rho(r)$:

$$\rho(r) = \frac{\rho_\infty}{r^{1/2}} (q(r)\{e^{(1)}(r) - [(2r)^{-1} + \pi^{-1}e(r)]\Gamma^{(1)}(r)\}$$
$$\times\{e^{(1)*}(r) - [(2r)^{-1} + \pi^{-1}e^*(r)]\Gamma^{(1)*}(r)\})$$
$$\times \exp[-H^{(1)}(r)]) , \tag{7.1}$$

where ρ_∞ is given by Eq. (1.4) and

$$e(r) = e^{-2ir}\int_0^\infty dx\, e^{-rx}x(x+2i)^{-1}, \tag{7.2}$$

$$e^{(1)}(r) = 1 + \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dp\left(\frac{p}{\sinh^3\pi p}\right)^{1/2}$$
$$\times\int_0^\infty dy\, e^{-y}\frac{\chi_p(y)}{y+2ir}, \tag{7.3}$$

$$e^{(2)}(r) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dp\left(\frac{p\,\mathrm{sech}\pi p}{\tanh^3\pi p}\right)^{1/2}$$
$$\times\int_0^\infty dy e^{-y}\frac{\chi_p(y)}{y+2ir}. \tag{7.4}$$

$\chi_p(y)$ satisfies Eq. (5.86) and is explicitly written down in Eqs. (5.88) and (5.89):

$$\Gamma^{(1)}(r) = (2i)^{-1}[e^{(1)}(r) - e^{(2)}(r)] \tag{7.5}$$

and

$$q(r) = [1 - q^{(2)}(r)q^{(2)*}(r)]^2 - q^{(1)}(r)q^{(3)*}(r)$$
$$- q^{(1)*}(r)q^{(3)}(r) + q^{(1)}(r)q^{(1)*}(r)q^{(3)}(r)q^{(3)*}(r)$$
$$- q^{(1)}(r)q^{(2)*^2}(r)q^{(3)}(r) - q^{(1)*}(r)q^{(2)^2}(r)q^{(3)*}(r), \tag{7.6}$$

with

$$q^{(1)}(r) = \frac{\Gamma^{(1)}(r)}{e^{(1)}(r) - [(2r)^{-1} + \pi^{-1}e(r)]\Gamma^{(1)}(r)}, \tag{7.7}$$

$$q^{(2)}(r) = \frac{e^{ir}\{(2r)^{-1} - \pi^{-1}e(r)[e^{(1)}(r) - 1]\}}{e^{(1)}(r)}q^{(1)}(r)$$
$$- \frac{e^{ir}[e^{(1)}(r) - 1]}{e^{(1)}(r)}, \tag{7.8}$$

and

$$q^{(3)}(r) = \frac{e^{ir}\{(2r)^{-1} - \pi^{-1}e(r)[e^{(1)}(r) - 1]\}}{e^{(1)}(r)}q^{(2)}(r)$$
$$+ e^{2ir}[e^{(1)}(r)]^{-1}[(2r)^{-1} + 2ie^{(2)*}(r)$$
$$- \pi^{-1}e^{-2ir}e^*(r)e^{(1)*}(r)]. \tag{7.9}$$

In Eq. (7.1), $H^{(1)}(r)$ is given by

$$H^{(1)}(r) = \sum_{n=2}^\infty H_n^{(1)}(r), \tag{7.10}$$

where

$$H_n^{(1)}(r) = \int_0^\infty dy_1\cdots\int_0^\infty dy_n$$
$$\times\prod_{j=1}^n e^{-ry_j}\prod_{j=0} y_{2j+1}\prod_{j=1} y_{2j}^{-1}f_n(y;r) \tag{7.11}$$

and

$$f_n(y;r) \equiv f_n(y_1,y_2,\cdots,y_n;r)$$
$$= \left[\sum_{l=1}^\infty l^{-1}\,\mathrm{Tr}\mathcal{N}(y;r))^l\right]_n, \tag{7.12}$$

where $\mathcal{N}(y;r)$ is a 4×4 matrix of functions of y and r. The summation inside the bracket on the right hand side of Eq. (7.12) generates terms with $m\, y_j$'s ($m = 2,3,4...$). The subscript n denotes that we keep all the terms with exactly $n\, y_j$'s in $f_n(y;r)$. The discussion following Eqs. (4.21) and (4.22) explains the method of generating these terms in detail.

In Eq. (7.12)

$$\mathcal{N}(y;r) = \begin{bmatrix} M_0^{(2)}(y;r)F^{(1)}(y;r) & M_0^{(2)}(y;r)F^{(2)}(y;r) \\ M_e^{(2)}(y;r)F^{(3)}(y;r) & M_e^{(2)}(y;r)F^{(4)}(y;r)\end{bmatrix}, \tag{7.13}$$

where $M_0^{(2)}(y;r)$ and $M_e^{(2)}(y;r)$ are given by Eqs. (4.8b) and (4.9b), respectively:

$$F^{(j)}(y;r) = E^{(j,1)}(y;r)F^{(5)}(r)E^{(j,2)}(y;r) + E^{(j,3)}(y;r),$$
$$j = 1,2,3, \text{ and } 4, \tag{7.14}$$

where

$$F^{(5)}(r) = \frac{1}{q(r)}\begin{bmatrix} q^{(1)}(r)[1 - q^{(1)*}(r)q^{(3)}(r)] + q^{(1)*}(r)q^{(2)^2}(r) & q^{(1)}(r)q^{(2)*}(r) + q^{(1)*}(r)q^{(2)}(r) \\ q^{(1)}(r)q^{(2)*}(r) + q^{(1)*}(r)q^{(2)}(r) & q^{(1)*}(r)[1 - q^{(1)}(r)q^{(3)*}(r)] + q^{(1)}(r)q^{(2)*^2}(r)\end{bmatrix}, \tag{7.15}$$

$$E^{(1,1)}(y;r) = E^{(2,1)}(y;r) = E^{(1)}(y;r)M_0^{(1)'}(y;r),$$

$$[M_0^{(1)'}(y;r) \text{ is given by Eq. (5.2a) and } E^{(1)}(y;r) \text{ by Eqs. (5.4) and (5.6)}], \tag{7.16}$$

$$E^{(3,1)}(y;r) = E^{(4,1)}(y;r) = E^{(5)}(y;r), \quad [E^{(5)}(y;r) \text{ is given by Eqs. (5.19) and (5.21)}], \tag{7.17}$$

$$E^{(1,2)}(y;r) = E^{(3,2)}(y;r) = E^{(2)}(y;r), \quad [E^{(2)}(y;r) \text{ is given by Eqs. (5.12) and (5.14b)}], \tag{7.18}$$

$$E^{(2,2)}(y;r) = E^{(4,2)}(y;r) = \pi^{-1}E^{(2)}(y;r)M_0^{(1)}(y;r), \quad [M_0^{(1)}(y;r) \text{ is given by Eq. (4.8a)}], \tag{7.19}$$

$$E^{(1,3)}(y;r) = E^{(1)}(y;r)M_0^{(1)'}(y;r)E^{(3)}(y;r) + E^{(1)}(y;r), \quad [E^{(3)}(y;r) \text{ is given by Eqs. (5.13) and (5.14c)}], \tag{7.20}$$

$$E^{(2,3)}(y;r) = \pi^{-1}E^{(1,3)}(y;r)M_0^{(1)}(y), \tag{7.21}$$

$$E^{(3,3)}(y;r) = E^{(5)}(y;r)E^{(3)}(y;r) + E^{(6)}(y;r), \quad [E^{(6)}(y;r) \text{ is given by Eqs. (5.19) and (5.22)}], \tag{7.22}$$

and

$$E^{(4,3)}(y;r) = E^{(3,3)}(y;r)M_0^{(1)}(y;r). \tag{7.23}$$

The 2×2 matrices $E^{(j)}$ have functions of the y_k's. Substituting these in Eq. (7.13), we carry out the trace in Eq. (7.12) and pick up all the terms with n variables to obtain, $f_n(y;r)$. In Eq. (7.16)–(7.29)

$$E^{(j)}(y;r) \equiv E^{(j)}(\pi^{-1},y;r), \quad \text{for } j = 1,2,3,5, \text{ and } 6. \tag{7.24}$$

VIII. ASYMPTOTIC EXPANSION OF $\rho(r)$ FOR LARGE r

All the functions inside the curly bracket in Eq. (7.1) can be expressed in terms of $e^{(1)}(r)$, $e^{(2)}(r)$, and $e(r)$. It may be shown that, for $r \gg 1$,

$$e(r) = \frac{e^{-2ir}}{2ir^2}\left[1 - \frac{1}{ir} - \frac{3}{2r^2} + 0(r^{-3})\right]. \tag{8.1}$$

To expand $e^{(1)}(r)$ [Eq. (7.3)] and $e^{(2)}(r)$ [Eq. (7.4)], we expand the $(y+2ir)^{-1}$ factor in the denominators in powers of r^{-1} to write

$$e^{(1)}(r) = 1 + \sum_{n=1}^{\infty} C_n^{(1)} r^{-n} \tag{8.2a}$$

and

$$e^{(2)}(r) = \sum_{n=1}^{\infty} C_n^{(2)} r^{-n}, \tag{8.2b}$$

where

$$C_n^{(1)} = (-1)^{n-1}(2i)^{-n}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dp\left(\frac{p}{\sinh^3 \pi p}\right)^{1/2} \times \int_0^\infty dy\, e^{-y} y^{n-1}\chi_p(y) \tag{8.3a}$$

and

$$C_n^{(2)} = (-1)^{n-1}(2i)^{-n}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dp\left(\frac{p \operatorname{sech} \pi p}{\tanh^3 \pi p}\right)^{1/2} \times \int_0^\infty dy\, e^{-y} y^{n-1}\chi_p(y). \tag{8.3b}$$

Using Eq. (5.88), we can write

$$\int_0^\infty dy\, e^{-y} y^{n-1}\chi_p(y) = \frac{(n-1)!2^n}{(2\lambda_p)^{1/2}}\int_1^\infty d\xi \frac{\phi_p(\xi)}{(\xi+1)^n}, \tag{8.4}$$

where $\phi_p(\xi)$ is defined by Eq. (5.89). Using Eq. (5.91), we have

$$\int_1^\infty d\xi \frac{\varphi_p(\xi)}{(\xi+1)^n} = \frac{(-1)^{n-1}\lambda_p}{(n-1)!}\frac{d^{n-1}\varphi_p(x)}{dx^{n-1}}\bigg|_{x=1} \tag{8.5a}$$

$$= \frac{2^{1-n}\lambda_p C_p}{[(n-1)!]^2}\frac{\Gamma(n-\frac{1}{2}+ip)\Gamma(n-\frac{1}{2}-ip)}{\Gamma(\frac{1}{2}+ip)\Gamma(\frac{1}{2}-ip)}, \tag{8.5b}$$

where, in Eq. (8.5b), $\Gamma(z)$ is the gamma function. Substituting in Eqs. (8.4) and (8.3) and doing the final integration over p, we can evaluate $C_n^{(1)}$ and $C_n^{(2)}$. We write down the first few terms in Eq. (8.2). For $r \gg 1$,

$$e^{(1)}(r) = 1 + \frac{1}{8ir} + \frac{3}{128r^2} - \frac{15}{1024ir^3} - \frac{525}{2\times 4^7 r^4} + O(r^{-5}) \tag{8.6a}$$

and

$$e^{(2)}(r) = \frac{1}{4ir} + \frac{3}{32r^2} - \frac{45}{512ir^3} - \frac{525}{4^6 r^4} + O(r^{-5}) \tag{8.6b}$$

Using the expansions (8.1) and (8.6), we can derive an asymptic expansion for large r for the curly bracket in Eq. (7.1). The expansion of $H^{(1)}(r)$ is more difficult. We first note the following: An integral of the form

$$\int_0^\infty dy_1 \cdots \int_0^\infty dy_n \prod_{j=1}^{n} e^{-ry_j}\frac{f_n(y)}{\Pi(y_j+y_{j+1})}$$

falls off as r^{-m}, where

$$\begin{aligned} m = n &+ \text{number of } y_j \text{ factors in the numerator of } f_n(y) \\ &- \text{number of } y_j \text{ factors in the denominator of } f_n(y) \\ &- \text{number of } (y_j+y_{j+1}) \text{ factors in the denominator.} \end{aligned} \tag{8.7}$$

For example,

$$\int_0^\infty dy_1\int_0^\infty dy_2\, e^{-r(y_1,y_2)}\frac{y_1 y_2}{(y_1+y_2)} \sim r^{-3}. \tag{8.8}$$

A study of the matrices in $\mathcal{N}(y;r)$ shows that

$$\int dy\, \operatorname{tr}\mathcal{N}(y;r)^l \sim \begin{cases} r^{-2l-1} & \text{for } l \text{ odd,} \\ r^{-2l} & \text{for } l \text{ even.} \end{cases} \tag{8.9}$$

Thus, to obtain the expansion to $0(x^{-4})$ in the Introduction, we only need to study $\mathrm{Tr}\mathcal{N}(y;r)$ and $\mathrm{Tr}\mathcal{N}(y;r)^2$. The first term needs to be evaluated to the leading and next leading order in x^{-1}, while the second term needs to be evaluated only to the leading order. Now,

$$\mathrm{Tr}\mathcal{N}(y;r) = \mathrm{Tr}\left[M_0^{(2)}(y;r)F^{(1)}(y;r) + M_e^{(2)}(y;r)F^{(4)}(y;r)\right]. \tag{8.10}$$

We can use the expressions (7.14) for $F^{(1)}(y;r)$ and $F^{(4)}(y;r)$ to write out the right hand side of Eq. (8.10).

We will illustrate the method of deriving the asymptotic expansion by a typical term in Eq. (8.10), namely,

$$\mathrm{Tr}M_0^{(2)}(y;r)E^{(1)}(y;r) = \mathrm{Tr}\sum_{n=1}^{\infty}\pi^{-2n}M_0^{(2)}(y_1,y_2)E_{2n}^{(1)}(y_2,y_3,\cdots,y_{2n},y_1). \tag{8.11}$$

Expanding the $(y_1+y_2+2i)^{-1}$ term in $M_0^{(2)}(y_1,y_2)$ [see Eq. (4.8b)], we have

$$M_0^{(2)}(y_1,y_2) = -y_2 e^{-ir}[(2i)^{-1} + \tfrac{1}{4}(y_1+y_2),0] + \text{ higher order terms,} \tag{8.12}$$

where the higher order terms contribute to $O(x^{-5})$ and higher. Next we multiply out the matrices in $E_{2n}^{(1)}(y_2 \to y_{2n}, y_1)$ [see Eq. (5.4)] keeping the terms only to leading and next leading order. This leads to

$$E^{(2)}(2\to 2n,1) = \left\{e^{-ir}\left[y_1(y_1+2i)^{-1}t_{2n-1}^{(1)}(2\to 2n) - \sum_{m=1}^{n-1} t_{2m-1}^{(1)}(2\to 2m)t_{2n-2m+1}^{(2)*}(2m+1\to 2n,1)\right], t_{2n}^{(3)}(2\to 2n,1)\right\} + \text{ higher order terms,} \tag{8.13}$$

where

$$t^{(1)}_{2k-1}(2\to 2k) = (-1)^{k+1}\Big(\prod_{j=2}^{k} y_{2j}\Big/\prod_{j=1}^{k-1} y_{2j+1}\Big) \overset{+}{\underset{2}{\times}} \text{----} \overset{-}{\underset{3}{\times}} \text{----} \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2k-1}{\times}} \text{----} \overset{+}{\underset{2k}{\times}}, \quad k = 1,2,..., \tag{8.14}$$

$$t^{(2)}_{2k+1}(1\to 2k+1) = (-1)^{k+1} y_{2k+1}\Big(\prod_{j=1}^{k} y_{2j}\Big/\prod_{j=0}^{k} y_{2j+1}\Big) \underset{1}{\times} \text{----} \overset{+}{\underset{2}{\times}} \text{----} \underset{3}{\times} \cdots \overset{-}{\underset{2k-1}{\times}} \text{----} \overset{+}{\underset{2k}{\times}} \text{----} \underset{2k+1}{\times}, \quad k = 1,2,..., \tag{8.15}$$

and

$$t^{(3)}_{2k}(1\to 2k+1) = (-1)^{k+1} y_{2k+1} \prod_{j=1}^{k}\left(\frac{y_{2j}}{y_{2j+1}}\right) \overset{+}{\underset{2}{\times}} \text{----} \overset{-}{\underset{3}{\times}} \text{----} \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2k-1}{\times}} \text{----} \overset{+}{\underset{2k}{\times}} \text{----} \underset{2k+1}{\times}, \quad k = 1,2,\dots . \tag{8.16}$$

Substituting in Eq. (8.11) and making use of Eqs. (7.10) to (7.12), we see that

$$\text{Contribution from } \operatorname{Tr}M^{(2)}_0(y;r)E^{(1)}(y;r) \text{ to } H^{(1)}(r) = \sum_{n=1}^{\infty} 2\pi^{-2n}\operatorname{Re}[I^{(1)}_{2n}(r) + I^{(2)}_{2n}(r)] + O(r^{-5}), \tag{8.17}$$

where

$$I^{(1)}_{2n}(r) = (-1)^n e^{-2ir}\int_0^\infty dy_1 \cdots \int_0^\infty dy_{2n} \prod_{j=1}^{2n} e^{-ry_j} y_1(y_1 + 2i)^{-1}[(2i)^{-1} + \tfrac{1}{4}(y_1 + y_2)]$$
$$\times\left\{ \overset{+}{\underset{2}{\times}} \text{----} \overset{-}{\underset{3}{\times}} \text{----} \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2n-1}{\times}} \text{----} \overset{+}{\underset{2n}{\times}} \right\} \tag{8.18}$$

and

$$I^{(2)}_{2n}(r) = (-1)^n e^{-2ir}\int_0^\infty dy_1 \cdots \int_0^\infty dy_{2n} \prod_{j=1}^{2n} e^{-ry_j} y_1 [(2i)^{-1} + \tfrac{1}{4}(y_1 + y_2)]$$
$$\times \sum_{m=1}^{n-1}\left\{ \overset{+}{\underset{2}{\times}} \text{----} \overset{-}{\underset{3}{\times}} \text{----} \overset{+}{\underset{4}{\times}} \cdots \overset{-}{\underset{2m-1}{\times}} \text{----} \overset{+}{\underset{2m}{\times}} \right\}\left\{ \underset{2n+1}{\times} \text{----} \overset{-}{\underset{2m+2}{\times}} \text{----} \overset{+}{\underset{2m+3}{\times}} \cdots \overset{+}{\underset{2n-1}{\times}} \text{----} \overset{-}{\underset{2n}{\times}} \text{----} \underset{1}{\times} \right\}. \tag{8.19}$$

Expanding the $(y_{2j+1} - 2i)$ and $(y_{2j} + 2i)$ factors in Eq. (8.18) and making use of Eq. (8.7), we see that

$$I^{(1)}_{2n}(r) = \frac{1}{4}e(r)\left(1 + \frac{1}{2i}\frac{d}{dr}\right)\left[\int_0^\infty dy_2 \cdots \int_0^\infty dy_{2n} \prod_{j=2}^{2n} e^{-ry_j}\prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1}\right]$$
$$+ \frac{e^{-2ir}}{16}\int_0^\infty dy_1 \cdots \int_0^\infty dy_{2n} \prod_{j=1}^{2n} e^{-ry_j} y_1(y_1 + y_2)\prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1} + O(r^{-5}). \tag{8.20}$$

Using the expansion (8.1) for $e(r)$ and scaling r in the integrals in Eq. (8.20) $(r y_j \to y_j)$, we can write

$$I^{(1)}_{2n}(r) = \frac{e^{-2ir}}{8r^2}\left[\left(\frac{1}{ir} + \frac{5}{2r^2}\right)\int_0^\infty dy_2 \cdots \int_0^\infty dy_{2n} \prod_{j=2}^{2n} e^{-y_j}\prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1}\right.$$
$$\left.+ \frac{1}{2r^2}\int_0^\infty dy_2 \cdots \int_0^\infty dy_{2n} \prod_{j=2}^{2n} e^{-y_j} y_2 \prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1}\right] + O(r^{-5}). \tag{8.21}$$

The integrals in Eq. (8.21) can be analyzed by techniques similar to those used in the analysis of $e^{(1)}(r)$ and $e^{(2)}(r)$. In particular, we can show that

$$\sum_{n=1}^{\infty} \pi^{-2n}\int_0^\infty dy_2 \cdots \int_0^\infty dy_{2n} \prod_{j=2}^{2n} e^{-y_j}\prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1} = 2i\pi^{-1}C^{(2)}_1 = \frac{1}{2\pi} \tag{8.22}$$

and

$$\sum_{n=1}^{\infty} \pi^{-2n}\int_0^\infty dy_2 \cdots \int_0^\infty dy_{2n} \prod_{j=2}^{2n} e^{-y_j}\prod_{j=2}^{2n-1}(y_j + y_{j+1})^{-1} y_2 = 4\pi^{-1}C^{(2)}_2 = \frac{3}{8\pi}. \tag{8.23}$$

Thus we have

$$\sum_{n=1}^{\infty} \pi^{-2n} I^{(1)}_{2n}(r) = \frac{e^{-2ir}}{16\pi r^3}\left(-i + \frac{23}{8r}\right) + O(r^{-5}). \tag{8.24}$$

The integral in $I^{(2)}_{2n}(r)$ factorizes into two factors, each of which can be analyzed in a similar manner. Adding the two contributions and using Eq. (8.17), we get

$$\text{contribution from } \operatorname{Tr}M^{(2)}_0(y;r)E^{(1)}(y;r) \text{ to } H^{(1)}(r) = -\frac{3\sin 2r}{64r^3} + \frac{63\cos 2r}{512\,r^4} + O(r^{-5}). \tag{8.25}$$

The remaining terms in $\operatorname{Tr}\mathcal{N}(y;r)$ and $\operatorname{Tr}\mathcal{N}(y;r)^2$ can be analyzed similarly. The calculation is tedious but straightforward on using the integral equation techniques developed in the analysis of $e^{(1)}(r)$. The final result is

$$H^{(1)}(r) = -\frac{\sin 2r}{32r^3} - \frac{1}{1024r^4} + \left(\frac{31}{8} - 119\cos 2r\right) + O(r^{-5}). \tag{8.26}$$

Substituting $H^{(1)}(r)$ and the expansion for the curly bracket in Eq. (7.1) and identifying $r \equiv x$ (since $k_f = 1$), we get the large x expansion in Eq. (1.3).

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APPENDIX A: FACTORIZATION OF $h^{(2)}(r,g)$

In this Appendix we will carry out a factorization of $h^{(2)}(r,g)$ as $g \to 0$ in terms of $f_2(gr)$, $g_1(gr)$, and factors that are functions of r alone.

Along the branch cut $-$, we make a change of variables

$$k_j = -\sqrt{1-g^2} - iy_j, \quad g \leqslant y_j \leqslant \infty, \tag{A1a}$$

so that

$$[(k_j^2 + \mu^2)(k_j^2 + \mu^{*2})]^{1/2} \to (y_j - 2i)(y_j^2 - g^2)^{1/2} + O(g^2), \quad \text{as } g \to 0. \tag{A1b}$$

Along the branch cut $+$

$$k_j = \sqrt{1-g^2} - iy_j, \quad g \leqslant y_j \leqslant \infty, \tag{A2a}$$

so that

$$[(k_j^2 + \mu^2)(k_j^2 + \mu^{*2})]^{1/2} \to (y_j + 2i)(y_j^2 - g^2)^{1/2} + O(g^2), \quad \text{as } g \to 0. \tag{A2b}$$

Thus, using the obvious notation, as $g \to 0$

$$h^{(2)}(r,g) = \tfrac{1}{2}[h^{(2)}_{++}(r,g) + h^{(2)}_{+-}(r,g) + h^{(2)}_{-+}(r,g) + h^{(2)}_{--}(r,g)] + o(1), \tag{A3}$$

where

$$h^{(2)}_{++}(r,g) = 2e^{-2ir}\int_g^\infty dy_1 \int_g^\infty dy_2 \frac{e^{-r(y_1+y_2)}(y_1+2i)(y_1^2-g^2)^{1/2}}{(y_2+2i)(y_2^2-g^2)^{1/2}(y_1+y_2+2i)^2}, \tag{A4}$$

$$\begin{aligned} h^{(2)}_{+-}(r,g) = &\int_g^\infty dy_1 \int_g^\infty dy_2 \frac{e^{-r(y_1+y_2)}}{(y_1+2i)(y_1^2-g^2)^{1/2}(y_2-2i)(y_2^2-g^2)^{1/2}(y_1+y_2)^2} \\ &\times [(y_1+2i)^2(y_1^2-g^2) + (y_2-2i)^2(y_2^2-g^2)], \end{aligned} \tag{A5}$$

and

$$h^{(2)}_{-+}(r,g) = h^{(2)}_{+-}(r,g), \quad h^{(2)}_{--}(r,g) = h^{(2)*}_{++}(r,g). \tag{A6}$$

Therefore,

$$h^{(2)}(r,g) = \mathrm{Re}\, h^{(2)}_{++}(r,g) + h^{(2)}_{+-}(r,g), \tag{A7}$$

$$\begin{aligned} h^{(2)}_{++}(r,g) = 2\,e^{-2ir}&\int_g^\infty dy_1 \int_g^\infty dy_2 \frac{e^{-r(y_1+y_2)}(y_1^2-g^2)^{1/2}}{(y_2^2-g^2)^{1/2}} \\ &\times \left[\frac{(y_1+2i)}{(y_2+2i)(y_1+y_2+2i)^2} - \frac{1}{2i(y_1+2i)} + \frac{1}{2i(y_1+2i)}\right] \end{aligned} \tag{A8}$$

$$\begin{aligned} = &\frac{e^{-2ir}}{i}\int_g^\infty \frac{dy_1 e^{-ry_1}(y_1^2-g^2)^{1/2}}{y_1+2i}\int_g^\infty dy_2 \frac{e^{-ry_2}}{(y_2^2-g^2)^{1/2}} - \frac{e^{-2ir}}{i}\int_g^\infty dy_1 \int_g^\infty dy_2 \\ &\times \frac{e^{-r(y_1+y_2)}(y_1^2-g^2)^{1/2}y_2[(y_1+2i)^2 + 4y_2(y_1+2i) + y_2^2 + 4i(y_1+2i) + 2iy_2]}{(y_2^2-g^2)^{1/2}(y_1+2i)(y_1+y_2+2i)^2}. \end{aligned} \tag{A9}$$

In the second integral we can set $g = 0$. In the first integral in the y_1 integration we can set $g = 0$, and in the y_2 integration we rescale $y_2 \to gy_2$ to get, for $g \to 0$,

$$\begin{aligned} h^{(2)}_{++}(r,g) = g_1(gr)&\frac{e^{-2ir}}{i}\int_0^\infty \frac{dy_1 e^{-ry_1}y_1}{y_1+2i} - \frac{e^{-2ir}}{i} \\ &\times \int_0^\infty dy_1 \int_0^\infty dy_2 \frac{e^{-r(y_1+y_2)}}{(y_1+y_2+2i)}\left[\frac{y_1+y_2}{y_1+y_2+2i} + \frac{y_1}{2(y_2+2i)} + \frac{y_2}{2(y_1+2i)}\right]. \end{aligned} \tag{A10}$$

Now

$$h^{(2)}_{+-}(r,g) = \int_g^\infty dy_1 \int_g^\infty dy_2 \, \frac{e^{-r(y_1+y_2)}(y_1^2-g^2)^{1/2}(y_1y_2-4)}{(y_2^2-g^2)^{1/2}(y_1+y_2)^2(y_2^2+4)}, \tag{A11}$$

$$= \int_g^\infty dy_1 \int_g^\infty dy_2 \, \frac{e^{-r(y_1+y_2)}(y_1^2-g^2)^{1/2}}{(y_2^2-g^2)^{1/2}(y_1+y_2)^2}\left(\frac{y_1y_2-4}{y_2^2+4}+1-1\right). \tag{A12}$$

Rescaling and setting $g=0$ wherever allowed, we have

$$h^{(2)}_{+-}(r,g) = -\int_1^\infty dy_1 \int_1^\infty dy_2 \, \frac{e^{-gr(y_1+y_2)}(y_1^2-1)^{1/2}}{(y_2^2-1)^{1/2}(y_1+y_2)^2} + \int_0^\infty dy_1 \int_0^\infty dy_2 \, \frac{e^{-r(y_1+y_2)}y_1}{(y_1+y_2)(y_2^2+4)}. \tag{A13}$$

We identify the first term with $f_2(gr)$, and substituting in Eq. (A7) we get the result, as $g \to 0$,

$$h^{(2)}(r,g) = 2f_2(gr) + g_1(gr)\,\mathrm{Re}\left(\frac{e^{-2ir}}{i}\int_0^\infty \frac{dx\, x\, e^{-rx}}{x+2i}\right) + 2\int_0^\infty dy_1 \int_0^\infty dy_2 \, \frac{e^{-r(y_1+y_2)}y_1}{(y_1+y_2)(y_2^2+4)}$$
$$- \mathrm{Re}\left\{\frac{e^{-2ir}}{i}\int_0^\infty dy_1 \int_0^\infty dy_2 \, \frac{e^{-r(y_1+y_2)}}{(y_1+y_2+2i)}\left[\frac{y_1+y_2}{(y_1+y_2+2i)} + \frac{y_1}{2(y_2+2i)} + \frac{y_2}{2(y_1+2i)}\right]\right\}. \tag{A14}$$

Thus, our objective of expressing the g dependence only through the functions $f_2(gr)$ and $g_1(gr)$ is achieved.

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