Painlevé functions of the third kind*

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We explicitly construct the one-parameter family of solutions, $\eta(\theta;\nu,\lambda)$, that remain bounded as $\theta\to\infty$ along the positive real θ axis for the Painlevé equation of third kind

$$ww''=(w')^2-\theta^{-1}ww'+2\nu\theta^{-1}(w^3-w)+w^4-1,$$

where, as $\theta\to\infty$, $\eta\sim 1-\lambda\Gamma(\nu+1/2)2^{-2\nu}\theta^{-\nu-1/2}e^{-2\theta}$. We further construct a representation for $\psi(t;\nu,\lambda)=-\ln[\eta(t/2;\nu,\lambda)]$, where $\psi(t;\nu,\lambda)$ satisfies the differential equation

$$\psi''+t^{-1}\psi'=(1/2)\sinh(2\psi)+2\nu t^{-1}\sinh(\psi).$$

The small-θ behavior of $\eta(\theta;\nu,\lambda)$ is described for $|\lambda|<\pi^{-1}$ by

$$\eta(\theta;\nu,\lambda)\sim 2^\sigma B\theta^\sigma.$$

The parameters σ and B are given as explicit functions of λ and ν. Finally an identity involving the Painlevé transcendent $\eta(\theta;\nu,\lambda)$ is proved. These results for the special case $\nu=0$ and $\lambda=\pi^{-1}$ make rigorous the analysis of the scaling limit of the spin–spin correlation function of the two-dimensional Ising model.

I. INTRODUCTION

The Painlevé equation of the third kind is

$$w''=\frac{1}{w}(w')^2-\frac{1}{\theta}w'+\frac{1}{\theta}(\alpha w^2+\beta)+\gamma w^3+\frac{\delta}{w}, \tag{1.1}$$

where prime denotes differentiation with respect to the variable θ and α, β, γ, and δ are constants. The importance of (1.1) in the theory of ordinary differential equations was first discussed by Painlevé[1] and later by Gambier.[2]

In this paper we develop the theory for the one-parameter, bounded (as $\theta\to\infty$ along the positive real axis), solutions of (1.1) when the constants α, β, γ, and δ satisfy

$$\alpha(-\delta)^{1/2}+\beta(\gamma)^{1/2}=0. \tag{1.2}$$

Under the assumption (1.2) there is no loss in generality if we consider in place of (1.1) the equation

$$w''=\frac{1}{w}(w')^2-\frac{1}{\theta}w'+\frac{2\nu}{\theta}(w^2-1)+w^3-\frac{1}{w}, \tag{1.3}$$

where ν is a constant.

If we denote by $\eta(\theta;\nu,\lambda)$ the one-parameter family of solutions of (1.3) that remain bounded as θ approaches infinity along the positive real axis, we shall prove

Theorem 1: The function $\eta(\theta;\nu,\lambda)$ satisfies (1.3) and for sufficiently large, positive θ and $\mathrm{Re}\,\nu>-\frac{1}{2}$, $\eta(\theta;\nu,\lambda)$ has the representation

$$\frac{1-\eta(\theta;\nu,\lambda)}{1+\eta(\theta;\nu,\lambda)}=G(t;\nu,\lambda), \tag{1.4a}$$

$$t=2\theta, \tag{1.4b}$$

where

$$G(t;\nu,\lambda)=\sum_{n=0}^{\infty}\lambda^{2n+1}g_{2n+1}(t;\nu), \tag{1.5}$$

$$g_1(t;\nu)=\int_1^\infty dy\,\frac{\exp(-ty)}{(y^2-1)^{1/2}}\left(\frac{y-1}{y+1}\right)^\nu, \tag{1.6a}$$

and for $n\geq 1$

$$g_{2n+1}(t;\nu)=(-1)^n\int_1^\infty dy_1\cdots\times\int_1^\infty dy_{2n+1}\left[\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right]\times\left[\prod_{j=1}^{2n}(y_j+y_{j+1})^{-1}\right]\left[\prod_{j=1}^{n}(y_{2j}^2-1)\right]. \tag{1.6b}$$

The parameter λ is subject only to the condition $|\lambda|<R(t)$ where $R(t)$ is the radius of convergence of (1.5) viewed in the complex λ plane. Simple bounds on $R(t)$ follow from the inequalities of Ref. 3, Eqs. (3.156)–(3.159). The restriction $\mathrm{Re}\,\nu>-\frac{1}{2}$ can be lifted in (1.6) by first changing the contour of integration to the contour $\mathcal{C}$ which is the contour beginning at infinity and looping around the branch point at $y=1$. The additional factor $\sin\pi(\nu-\frac{1}{2})$ can be incorporated into λ.

It is an important feature concerning the theory of the function $\eta(\theta;\nu,\lambda)$ that if we define $\psi(t;\nu,\lambda)$ by

$$\text{(i)}\quad \eta(\theta;\nu,\lambda)=\exp[-\psi(t;\nu,\lambda)],\quad t=2\theta,$$
$$\text{(ii)}\quad \psi(t;\nu,\lambda)\to 0\quad\text{as}\quad t\to+\infty, \tag{1.7}$$

then we have

Theorem 2: For t sufficiently large and $\mathrm{Re}\,\nu>-\frac{1}{2}$

$$\psi(t;\nu,\lambda)=\sum_{n=0}^{\infty}\lambda^{2n+1}\psi_{2n+1}(t;\nu), \tag{1.8}$$

where we have

$$\psi_1(t;\nu)=2g_1(t;\nu) \tag{1.9a}$$

and for $n\geq 1$

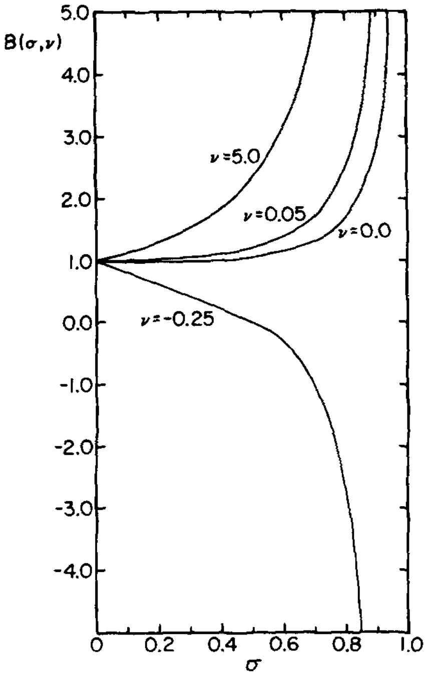

FIG. 1. Quantity $B(\sigma,\nu)$ as a function of σ for various values of ν. The slope of $B(\sigma,\nu)$ at $\sigma=0$ is $2\gamma+\ln 2+\psi(\frac{1}{2}+\nu)$. For $\nu=\nu^*\simeq 0.0245$ the slope is zero. For $\nu>\nu^*$ ($<\nu^*$) the slope at the origin is positive (negative). For $\nu=0$ the minimum of $B(\sigma,\nu)$ occurs at $\sigma\simeq 0.23$. The scale of the figure is too large to see the $B(\sigma,\nu)<1$ behavior for $\nu\geq 0$. For $\nu<0$ $B(\sigma,\nu)$ vanishes at $\sigma=1+2\nu$.

$$\psi_{2n+1}(t;\nu)=\frac{2}{2n+1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n+1}\left[\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{y_j+y_{j+1}}\right]\times\left[\prod_{j=1}^{2n+1}\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}+\prod_{j=1}^{2n+1}\left(\frac{y_j-1}{y_j+1}\right)^{\nu+1/2}\right] \tag{1.9b}$$

with $y_{2n+2}\equiv y_1$ in (1.9b). Again the restriction $\mathrm{Re}(\nu)>-\frac{1}{2}$ can be lifted by using the contour $\mathcal{C}$. To examine the analytic properties of $\eta(\theta;\nu,\lambda)$ and $\psi(t;\nu,\lambda)$ in the complex λ plane, representation (3.38) is useful.

As emphasized by Painlevé[1] the point $\theta=0$ plays a unique role in the theory of the third Painlevé transcendent. It is the only point in the finite θ plane for which a branch point or an essential singular point of a solution of (1.1) can occur. Furthermore it has been shown[1,2] that if $\theta=0$ is an analytic point, then the solution is a meromorphic function. Thus it is important to examine the behavior of a solution of (1.3) in the neighborhood of $\theta=0$. It is easy to demonstrate that for $t\to 0$ ($t=2\theta$) a formal solution of (1.3) is

$$w(t/2)=Bt^{\sigma}\{1-\nu B^{-1}(1-\sigma)^{-2}t^{1-\sigma}+B\nu(1+\sigma)^{-2}t^{1+\sigma}+[\tfrac{1}{4}\nu^2B^{-2}(1-\sigma)^{-4}-\tfrac{1}{16}B^{-2}(1-\sigma)^{-2}]t^{2-2\sigma}+O(t^2)\}, \tag{1.10}$$

where $-1<\mathrm{Re}\,\sigma<1$ but otherwise σ and B are arbitrary.

In general a solution that behaves as (1.10) for $t\to 0$ will not remain bounded as $t\to+\infty$. When $0\leq\lambda<\pi^{-1}$ the bounded solution $\eta(t/2;\nu,\lambda)$ behaves as (1.10) for $t\to 0$ but the coefficients σ and B are now functions of λ and ν. Using Theorem 2 we shall prove

Theorem 3: The solution $\eta(t/2;\nu,\lambda)$ has the small-t expansion (1.10) for $0\leq\lambda<1/\pi$ where

$$\sigma=\sigma(\lambda)=(2/\pi)\arcsin(\pi\lambda) \tag{1.11}$$

and

$$B=B(\sigma,\nu)=2^{-3\sigma}\frac{\Gamma^2((1-\sigma)/2)}{\Gamma^2((1+\sigma)/2)}\,\frac{\Gamma((1+\sigma)/2+\nu)}{\Gamma((1-\sigma)/2+\nu)}, \tag{1.12}$$

where $\Gamma(x)$ is the gamma function.

In Fig. 1 the function $B(\sigma,\nu)$ is graphed. Using Theorem 3 we can determine the small t behavior of $\eta(t/2;\nu,\lambda)$ for $\lambda\geq\pi^{-1}$ (see Sec. IV.I, also the case $\lambda<0$ is discussed).

We conclude our presentation of the theory of the Painlevé transcendent $\eta(\theta;\nu,\lambda)$ by proving a useful identity ($0\leq\lambda\leq\pi^{-1}$).

Theorem 4: If we define the functions

$$f_{2n}(t;\nu)=\frac{(-1)^n}{n}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n}\left[\prod_{j=1}^{2n}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\times\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\frac{1}{y_j+y_{j+1}}\right]\prod_{j=1}^{n}(y_{2j}^2-1) \tag{1.13}$$

with $y_{2n+1}\equiv y_1$, then we have

$$\tfrac{1}{2}[1+\eta(\theta;\nu,\lambda)]\eta^{-1/2}(\theta;\nu,\lambda)\exp\left[\int_\theta^\infty dx\left\{\tfrac{1}{4}x\eta^{-2}(x;\nu,\lambda)\times[(1-\eta^2(x;\nu,\lambda))^2-(\eta'(x;\nu,\lambda))^2]+\frac{\nu}{2\eta(x;\nu,\lambda)}(1-\eta(x;\nu,\lambda))^2\right\}\right]=\exp\left[-\sum_{n=1}^\infty\lambda^{2n}f_{2n}(2\theta;\nu)\right], \tag{1.14a}$$

where prime denotes differentiation with respect to x. Using definition (1.7) of the function $\psi(t;\nu,\lambda)$ the above identity becomes

$$\cosh\tfrac{1}{2}\psi(t;\nu,\lambda)\exp\left\{\tfrac{1}{4}\int_t^\infty ds\,s\left[-\left(\frac{d\psi}{ds}\right)^2+\sinh^2\psi+\frac{4\nu}{s}\sinh^2\tfrac{1}{2}\psi\right]\right\}=\exp\left[-\sum_{n=1}^\infty\lambda^{2n}f_{2n}(t;\nu)\right], \tag{1.14b}$$

where all ψ functions appearing under the integral sign are functions of s, ν, and λ.

Theorems 1, 2, 3, and 4 are proved and discussed in Secs. II, III, IV, and V, respectively.

For the special case $\nu=0$ and $\lambda=\pi^{-1}$ these four theorems make rigorous the analysis of the scaling limit of the spin–spin correlation function of the two-dimensional Ising model carried out by Wu, McCoy, Tracy, and Barouch.[3] It is perhaps not inappropriate to describe in some detail how the above theorems fit into the work of Ref. 3. However it should be stressed that the remainder of this section is irrelevant for the mathematical discussion that follows in Secs. II–V.

If we denote by ξ the correlation length [$\xi = \xi(T)$, T = temperature, and $\xi \to \infty$ as $T \to T_c^{\pm}$ where T_c is the critical temperature] and by $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ the spin–spin correlation function for the two-dimensional Ising model on a square lattice, and if we further assume for simplicity of presentation that the vertical and horizontal interaction energies are equal, then by *scaling limit* we mean that limit

$$\xi \to \infty, \quad R \equiv (M^2+N^2)^{1/2} \to \infty \tag{1.15a}$$

such that

$$t = R/\xi \quad \text{is fixed.} \tag{1.15b}$$

In this limit the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ becomes[3]

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = R^{-1/4}F_{\pm}(t) + R^{-5/4}F_{1\pm}(t) + o(R^{-5/4}), \tag{1.16}$$

where $F_{\pm}(t)$ and $F_{1\pm}(t)$ (these are commonly called *scaling functions*) are functions of the single variable t defined by (1.15b).

In Secs. III and IV of Ref. 3 an expansion valid for large t was developed [these results are summarized by Eqs. (2.26)–(2.30) of Ref. 3]. The expansion for $F_-(t)$ is the right-hand side of (1.14a) of Theorem 4 (for $\lambda = \pi^{-1}$ and $\nu = 0$) times the factor $(2t)^{1/4}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}$. The expansion for $F_+(t)/F_-(t)$ is the right-hand side of (1.5) of Theorem 1 (for $\nu = 0$ and $\lambda = \pi^{-1}$). These infinite series expansions are only useful for large t. For small t the functions $g_{2n+1}(t;\nu)$ of Theorem 1 behave as

$$g_{2n+1}(t;\nu) = c_{2n+1}(\ln t)^{2n+1} + c_{2n}(\ln t)^{2n} + \cdots + c_1(\ln t) + c_0 + o(1) \quad (t \to 0) \tag{1.17}$$

and similarly for the functions $f_{2n}(t;\nu)$ of Theorem 4.

Therefore, to study the small-t behavior of $F_{\pm}(t)$ the representation of $F_{\pm}(t)$ as an infinite series of multiple integrals is not directly the most convenient representation. This representation of $F_{\pm}(t)$ as an infinite series of multiple integrals can be thought of as the coordinate space analog of the dispersion integral representation of the two-point function. What is needed is a way to sum up this dispersion integral representation.

In Ref. 3 this was accomplished in two ways. One way (that of Sec. V) was to develop a separate perturbation scheme valid for small t. The other method (that of Sec. VI of Ref. 3) was to introduce an integral equation that could be solved in terms of Painlevé functions. This approach led to the representation of $F_{\pm}(t)$ in terms of Painlevé functions [these results are summarized by Eq. (2.39) of Ref. 3]. In terms of Theorem 1 of this paper $F_+(t)/F_-(t)$ was shown to be the left-hand side of (1.4a) for $\nu = 0$ and $\lambda = \pi^{-1}$ and in terms of Theorem 4 $F_-(t)$ was shown to be the left-hand side of (1.14a) for $\lambda = \pi^{-1}$ and $\nu = 0$ times the factor $(2t)^{1/4}(\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2)^{1/8}$. The methods used in Sec. VI of Ref. 3 though correct are not rigorous. Theorem 1 of this paper *rigorously* proves that the infinite series representation of $F_+(t)/F_-(t)$ is simply related to Painlevé functions, and likewise Theorem 4 rigorously relates the infinite series representation of $F_-(t)$ to Painlevé functions. Stated somewhat crudely, the Painlevé transcendents $\eta(\theta;\nu,\lambda)$ are the functions that sum the dispersion integral representation of the two-point functions $F_{\pm}(t)$.

In light of Theorems 1 and 4 the small-t behavior of $F_{\pm}(t)$ follows once the small-t behavior of $\eta(t/2;0,\pi^{-1})$ is known. To determine this behavior the analysis of Ref. 3 had to make crucial use of the unpublished thesis of Myers[4] where Painlevé functions of the third kind arose in the study of scattering from a strip. Though Myers' analysis is rigorous it gives only the small-t behavior of $\eta(t/2;\nu,\lambda)$ for the case $\nu = 0$ and $\lambda = \pi^{-1}$. Theorem 3 gives a direct proof (that is, the scattering problem is avoided) of the small-t behavior of $\eta(t/2;\nu,\lambda)$. Theorem 2 is essential to prove Theorem 3.

II. THEOREM 1 AND THE FUNCTION $G(t;\nu,\lambda)$

A. Restricted Painlevé equation of third kind

The most general Painlevé equation of the third kind is given by (1.1) where the constants α, β, γ, and δ are arbitrary. If we assume that the constants α, β, γ, and δ are restricted so that (1.2) is satisfied, then (1.1) can be reduced to (1.3). To demonstrate this we let

$$w(z) = A\eta(\theta), \quad \theta = Bz, \tag{2.1}$$

where z denotes the independent variable in (1.1), and A and B are constants that are to be determined. Substituting (2.1) into (1.1) it follows that

$$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta}\left(\frac{d\eta}{d\theta}\right)^2 - \frac{1}{\theta}\frac{d\eta}{d\theta} + \frac{\alpha A}{B}\frac{1}{\theta}\eta^2 + \frac{\beta}{AB}\frac{1}{\theta} + \frac{\gamma A^2}{B^2}\eta^3 + \frac{\delta}{A^2B^2}\frac{1}{\eta}. \tag{2.2}$$

This equation is of the form (1.3) if we have

$$\frac{\alpha A}{B} = -\frac{\beta}{AB} = 2\nu \tag{2.3a}$$

and

$$\frac{\gamma A^2}{B^2} = -\frac{\delta}{A^2B^2} = 1. \tag{2.3b}$$

From (2.3b) we see A and B are given by

$$A^2 = (-\delta/\gamma)^{1/2}, \quad B^2 = (-\delta\gamma)^{1/2}. \tag{2.4}$$

In order that (2.3a) is satisfied we demand

$$2\nu = \alpha/(\gamma)^{1/2} = -\beta/(-\delta)^{1/2} \tag{2.5}$$

which is just (1.2).

The condition (1.2) arises naturally in the following context. In general if $w(\theta)$ is a solution to (1.1), then $[Aw(\theta)]^{-1}$ is a solution to (1.1) with *different* α, β, γ, and δ (here A is a constant). If we demand that $[Aw(\theta)]^{-1}$ is a solution for the *same* α, β, γ, and δ, then A is fixed and the parameters α, β, γ, and δ must satisfy (1.2). From now on we discuss only (1.3).

B. Function $G(t;\nu,\lambda)$

As stated in the Introduction we denote by $\eta(\theta;\nu,\lambda)$ the one-parameter family of bounded (as $\theta\to\infty$ along the positive real θ axis) solutions to (1.3). We associate with $\eta(\theta;\nu,\lambda)$ the function $G(t;\nu,\lambda)$ where

$$G(t;\nu,\lambda)=\frac{1-\eta(\theta;\nu,\lambda)}{1+\eta(\theta;\nu,\lambda)} \tag{2.6a}$$

and

$$t=2\theta. \tag{2.6b}$$

From (1.3) and (2.6) it follows that $G(t;\nu,\lambda)$ satisfies the differential equation

$$G''+\frac{1}{t}G'-\left(1+\frac{2\nu}{t}\right)G=G''G^2-2(G')^2G+\frac{1}{t}G'G^2+G^3-\frac{2\nu}{t}G^3, \tag{2.7}$$

where the prime denotes differentiation with respect to the variable t.

Theorem 1 states that the one-parameter bounded solutions to (2.7) are given by (1.5) and (1.6). It is the goal of this section to prove Theorem 1. The method of proof is to substitute (1.5)–(1.6) into (2.7) and explicitly demonstrate that this is indeed a solution.

We begin the proof of Theorem 1 by establishing some useful identities which we state as lemmas.

C. Preliminary lemmas

Lemma 2.1: A necessary and sufficient condition that $G(t;\nu,\lambda)$ as defined by (1.5)–(1.6) satisfy (2.7) is for $k=0,1,2,\cdots$,

$$\begin{aligned}g''_{2k+1}+\frac{1}{t}g'_{2k+1}-\left(1+\frac{2\nu}{t}\right)g_{2k+1}
=\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}\Bigg\{&g_{2l+1}\Big[g''_{2(k-l-m)-1}+\frac{1}{t}g'_{2(k-l-m)-1}\\
&-\left(1+\frac{2\nu}{t}\right)g_{2(k-l-m)-1}\Big]g_{2m+1}+2g_{2(k-l-m)-1}[g_{2l+1}g_{2m+1}\\
&-g'_{2l+1}g'_{2m+1}]\Bigg\}\end{aligned} \tag{2.8}$$

where $g_{2n+1}(t;\nu)$ are defined by (1.6) and for $k=0$ the right-hand side of (2.8) is defined to be zero.

Proof: Since for $t>0$ $G(t;\nu,\lambda)$ has a finite radius of convergence in the λ plane we are allowed to equate equal powers of λ when (1.5) is substituted into (2.7). The precise form of the right-hand side of (2.8) follows by simple manipulations of power series. Clearly if (2.8) is true, then multiplication of this equation by λ^{2k+1} and summing over k reproduces (2.7).

If we define $g_{2n+1}(t;\nu)$, $n=0,1,2,\cdots$ by (1.6), then an alternate representation of these functions for $n=1,2,\cdots$ is

Lemma 2.2:

$$\begin{aligned}g_{2n+1}(t;\nu)=(-1)^n\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n+1}
&\times\left[\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\right]\\
&\times\left[\prod_{j=1}^{2n}(y_j+y_{j+1})^{-1}\right](y_1y_{2n+1}-1)\prod_{j=2}^{n}(y_{2j-1}^2-1),\end{aligned} \tag{2.9}$$

where for $n=1$ the last product is replaced by unity.

Proof: (i) $n=1$ case

From (1.6) we have

$$g_3(t;\nu)=(-1)\int_1^\infty dy_1\int_1^\infty dy_2\int_1^\infty dy_3\left[\prod_{j=1}^{3}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\times\left(\frac{y_j-1}{y_j-1}\right)^{\nu}\right]\frac{y_2^2-1}{(y_1+y_2)(y_2+y_3)}. \tag{2.10}$$

If we cyclically permute the integration variable labels in (2.10), then we can write $g_3(t;\nu)$ as

$$\begin{aligned}g_3(t;\nu)=-\tfrac{1}{3}\int_1^\infty dy_1\int_1^\infty dy_2\int_1^\infty dy_3\ &\left[\prod_{j=1}^{3}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\times\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\frac{1}{y_j+y_{j+1}}\right][(y_2^2-1)(y_3+y_1)\\
&+(y_3^2-1)(y_1+y_2)+(y_1^2-1)(y_2+y_3)]\end{aligned} \tag{2.11}$$

with $y_4\equiv y_1$.

The quantity in the second square brackets in (2.11) can be written as

$$\begin{aligned}&(y_2^2-1)(y_3+y_1)+(y_3^2-1)(y_1+y_2)+(y_1^2-1)(y_2+y_3)\\
&\quad=(y_1+y_2)(y_1y_2-1)+(y_1+y_3)(y_1y_3-1)\\
&\qquad+(y_2+y_3)(y_2y_3-1).\end{aligned} \tag{2.12}$$

Using this in (2.11) and writing the three resulting terms as one term (again by cyclically permuting the labels of the integration variables) we obtain

$$g_3(t;\nu)=-\int_1^\infty dy_1\int_1^\infty dy_2\int_1^\infty dy_3\left[\prod_{j=1}^{3}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\times\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\right]\frac{y_1y_3-1}{(y_1+y_2)(y_2+y_3)} \tag{2.13}$$

which is (2.9) for $n=1$.

(ii) $n=2$ case

For $n=2$ the part of the integrand in (1.6b) that is not invariant under cyclic permutations of the integration variable labels is

$$(y_2^2-1)(y_4^2-1)(y_5+y_1). \tag{2.14}$$

Under the five cyclic permutations of the labels (1, 2, 3, 4, 5) the quantity (2.14) becomes the sum of five terms, viz.

$$(y_2^2-1)(y_4^2-1)(y_5+y_1)+(y_3^2-1)(y_5^2-1)(y_1+y_2)$$
$$+(y_4^2-1)(y_1^2-1)(y_2+y_3)+(y_5^2-1)(y_2^2-1)(y_3+y_4)$$
$$+(y_1^2-1)(y_3^2-1)(y_4+y_5)$$
$$=(y_3^2-1)(y_5y_1-1)(y_5+y_1)+(y_4^2-1)(y_1y_2-1)(y_1+y_2)$$
$$+(y_5^2-1)(y_2y_3-1)(y_2+y_3)+(y_1^2-1)(y_3y_4-1)(y_3+y_4)$$
$$+(y_2^2-1)(y_4y_5-1)(y_4+y_5). \tag{2.15}$$

This can be written more compactly as

$$(y_2^2-1)(y_4^2-1)(y_5+y_1)+\text{cyclic permutations}$$
$$=(y_3^2-1)(y_5y_1-1)(y_5+y_1)+\text{cyclic permutations}. \tag{2.16}$$

If (2.16) is used in (1.6b) for $n=2$ we obtain (2.9) for $n=2$.

(iii) General case

We write integrand of (1.6b) as

$$\left[\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^{\nu}(y_j+y_{j+1})^{-1}\right]$$
$$\times(y_{2n+1}+y_1)\prod_{j=1}^{n}(y_{2j}^2-1), \tag{2.17}$$

where $y_{2n+2}\equiv y_1$. The quantity in square brackets in (2.17) is invariant under cyclic permutations of the integration variable labels. We claim that

$$\prod_{j=1}^{n}(y_{2j}^2-1)(y_{2n+1}+y_1)+\text{cyclic perm.}$$
$$=\prod_{j=2}^{n}(y_{2j-1}^2-1)(y_1y_{2n+1}-1)(y_{2n+1}+y_1)+\text{cyclic perm.} \tag{2.18}$$

From (2.18) the result (2.9) follows. To demonstrate (2.18) we first examine that piece of the left-hand side of (2.18) which is of degree $(2n+1)$. There are $2(2n+1)$ such terms and they are of the form

$$y_2^2y_4^2\cdots y_{2n}^2y_{2n+1}+y_2^2y_4^2\cdots y_{2n}^2y_1+\text{cyclic perm.} \tag{2.19}$$

This can be rewritten as

$$y_3^2y_5^2\cdots y_{2n+1}^2y_1+y_1^2y_3^2\cdots y_{2n-1}^2y_{2n+1}+\text{cyclic perm.}$$
$$=y_3^2y_5^2\cdots y_{2n-1}^2y_1y_{2n+1}(y_1+y_{2n+1})+\text{cyclic perm.} \tag{2.20}$$

Now consider the terms of (2.18) that are of degree $2n-1$. These terms arise by replacing some y_{2j}^2 in (2.19) by -1 or y_1y_{2n+1} by -1. This can be done at n places. Thus the second term of the left-hand side of (2.18) is

$$-[y_3^2y_5^2\cdots y_{2n-3}^2y_{2n+1}y_1+y_3^2y_5^2\cdots y_{2n-5}^2y_{2n-1}^2y_{2n+1}y_1+\cdots$$
$$+y_5^2y_7^2\cdots y_{2n-1}^2+y_3^2y_5^2\cdots y_{2n-1}^2](y_{2n+1}+y_1)$$
$$+\text{cyclic perm.} \tag{2.21}$$

If one compares (2.20) and (2.21) with the right-hand side of (2.18), then one sees both of these terms are present. The third term comes from leaving out an additional y_{2j-1}^2 or y_1y_{2n+1}, a term which is again clearly present on the right-hand side of (2.18). Continuing so, we see that the lemma is proved.

Our final lemma is

Lemma 2.3:

$$g''_{2k+1}(t;\nu)+\frac{1}{t}g'_{2k+1}(t;\nu)-\left(1+\frac{2\nu}{t}\right)g_{2k+1}(t;\nu)$$
$$=2(-1)^k\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}$$
$$\times\left[\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\right]$$
$$\times\left[\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}\right](y_1y_{2k+1}-1)\prod_{j=2}^{k}(y_{2j-1}^2-1)$$
$$\times\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k+1-2m}) \tag{2.22}$$

with $k=1,2,3,\cdots$ and $\Pi_{j=2}^{k}(y_{2j-1}^2-1)$ is defined to be unity for $k=1$.

Proof: For notational convenience we denote by L_ν the differential operator

$$L_\nu=\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\left(1+\frac{2\nu}{t}\right). \tag{2.23}$$

From Lemma 2.2 we have

$$L_\nu g_{2k+1}(t;\nu)$$
$$=(-1)^k\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}$$
$$\times\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^{\nu}\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}(y_1y_{2k+1}-1)$$
$$\times\prod_{j=2}^{k}(y_{2j-1}^2-1)\Big[(y_1+y_2+\cdots+y_{2k+1})^2$$
$$-\frac{1}{t}(y_1+y_2+\cdots+y_{2k+1})-\left(1+\frac{2\nu}{t}\right)\Big]. \tag{2.24}$$

We now proceed to integrate by parts the $1/t$ terms in (2.24). We first note the following identities:

$$\frac{y_{2j-1}+2\nu}{(y_{2j-1}^2-1)^{1/2}}\left(\frac{y_{2j-1}-1}{y_{2j-1}+1}\right)^{\nu}dy_{2j-1}$$
$$=d\left[\frac{1}{(y_{2j-1}^2-1)^{1/2}}\left(\frac{y_{2j-1}-1}{y_{2j-1}+1}\right)^{\nu}(y_{2j-1}^2-1)\right] \tag{2.25a}$$

and

$$\frac{y_{2j}-2\nu}{(y_{2j}^2-1)^{3/2}}\left(\frac{y_{2j}-1}{y_{2j}+1}\right)^{\nu}dy_{2j}$$
$$=-d\left[\frac{1}{(y_{2j}^2-1)^{1/2}}\left(\frac{y_{2j}-1}{y_{2j}+1}\right)^{\nu}\right]. \tag{2.25b}$$

We write the $1/t$ terms in the integrand of (2.24) as

$$-\frac{1}{t}(y_1+2\nu)-\frac{1}{t}(y_2-2\nu)-\frac{1}{t}(y_3+2\nu)-\cdots-\frac{1}{t}(y_{2k+1}+2\nu). \tag{2.26}$$

The $1/t$ part of (2.24) in view of (2.26) is a sum of $2k+1$ terms. Each term is a $(2k+1)$-dimensional integral. We integrate by parts a single integral of each of these multidimensional integrals. The term we choose to integrate by parts is the term with the structure of (2.25). We integrate the factors according to (2.25) and differentiate the remaining multiplicative factors. The differentiation creates terms of two classes. One class of terms will not contain a $1/t$ factor (these terms come from differentiating the exponential factor which brings down a t factor canceling the $1/t$ factor in front) and the other class will contain an overall $1/t$ factor. We denote by $[L_\nu g_{2k+1}(t;\nu)]_1$ that part of (2.24) which upon integration by parts in the above described manner contains no $1/t$ factors, and by $[L_\nu g_{2k+1}(t;\nu)]_2$ the part that contains the $1/t$ factor. Thus we have

$$L_\nu g_{2k+1}(t;\nu) = [L_\nu g_{2k+1}(t;\nu)]_1 + [L_\nu g_{2k+1}(t;\nu)]_2. \tag{2.27}$$

We have, carrying out this integration by parts (all boundary terms vanish),

$$\begin{aligned}
&[L_\nu g_{2k+1}(t;\nu)]_1 \\
&= (-1)^k \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2k+1} \\
&\times \prod_{j=1}^{2k+1} \frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}} \left(\frac{y_j-1}{y_j+1}\right)^\nu \prod_{j=1}^{2k} (y_j+y_{j+1})^{-1} (y_1 y_{2k+1}-1) \\
&\times \{(y_1+y_2+\cdots+y_{2k+1})^2 - (y_1^2-1) - (y_3^2-1) - \cdots \\
&- (y_{2k+1}^2-1) + (y_2^2-1) + (y_4^2-1) + \cdots + (y_{2k}^2-1) - 1\}.
\end{aligned} \tag{2.28}$$

The last factor in (2.28) can be combined to obtain

$$\begin{aligned}
&[L_\nu g_{2k+1}(t;\nu)]_1 \\
&= 2(-1)^k \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2k+1} \\
&\times \prod_{j=1}^{2k+1} \frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}} \left(\frac{y_j-1}{y_j+1}\right)^\nu \prod_{j=1}^{2k} (y_j+y_{j+1})^{-1} (y_1 y_{2k+1}-1) \\
&\times \sum_{l=0}^{k-1} \sum_{m=0}^{k-l-1} (y_{2k-2m}+y_{2k-2m+1})(y_{2l+1}+y_{2l+2}).
\end{aligned} \tag{2.29}$$

Comparing (2.27) and (2.29) with (2.22) we see that to prove this lemma we must establish

$$[L_\nu g_{2k+1}(t;\nu)]_2 = 0. \tag{2.30}$$

We have from the integration by parts

$$\begin{aligned}
&[L_\nu g_{2k+1}(t;\nu)]_2 \\
&= \frac{1}{t}(-1)^k \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2k+1} \\
&\times \prod_{j=1}^{2k+1} \frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}} \left(\frac{y_j-1}{y_j+1}\right)^\nu \left[\sum_{j=0}^{k} (y_{2j+1}^2-1) \frac{d}{dy_{2j+1}} \right. \\
&\left. - \sum_{j=1}^{k} \frac{d}{dy_{2j}} (y_{2j}^2-1)\right] \left[\prod_{j=1}^{2k} (y_j+y_{j+1})^{-1} \right. \\
&\left. \times (y_{2k+1}y_1-1) \prod_{j=2}^{k} (y_{2j-1}^2-1)\right].
\end{aligned} \tag{2.31}$$

Performing the indicated differentiations $[L_\nu g_{2k+1}(t;\nu)]_2$ becomes

$$\begin{aligned}
&[L_\nu g_{2k+1}(t;\nu)]_2 \\
&= \frac{1}{t}(-1)^k \int_1^\infty dy_1 \cdots \int_1^\infty dy_{2k+1} \\
&\times \prod_{j=1}^{2k+1} \frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}} \left(\frac{y_j-1}{y_j+1}\right)^\nu \prod_{j=1}^{2k} (y_j+y_{j+1})^{-1} \\
&\times \prod_{j=2}^{k} (y_{2j-1}^2-1)(y_{2k+1}y_1-1) \left\{ \sum_{j=1}^{k-1} (y_{2j+1}^2-1) \right. \\
&\times \left[-\frac{1}{y_{2j+1}+y_{2j+2}} - \frac{1}{y_{2j}+y_{2j+1}} + \frac{2y_{2j+1}}{y_{2j+1}^2-1}\right] \\
&+ (y_1^2-1) \left[-\frac{1}{y_1+y_2} + \frac{y_{2k+1}}{y_{2k+1}y_1-1}\right] + (y_{2k+1}^2-1) \\
&\times \left[-\frac{1}{y_{2k}+y_{2k+1}} + \frac{y_1}{y_{2k+1}y_1-1}\right] - \sum_{j=1}^{k} \left[2y_{2j} - (y_{2j}^2-1) \right. \\
&\left.\left. \times \left(\frac{1}{y_{2j-1}+y_{2j}} + \frac{1}{y_{2j}+y_{2j+1}}\right)\right]\right\}.
\end{aligned} \tag{2.32}$$

We now claim that the term inside the curly brackets in (2.32) is zero. To see this we group the terms in (2.32) with common denominators. Thus the sum of terms that have the denominator (y_1+y_2) is

$$\frac{1}{y_1+y_2}\left[-(y_1^2-1)+(y_2^2-1)\right] = y_2 - y_1 \tag{2.33a}$$

and similarly for the other denominator factors:

$$\frac{1}{y_{2j}+y_{2j+1}}\left[-(y_{2j+1}^2-1)+y_{2j}^2-1\right] = y_{2j} - y_{2j+1}, \tag{2.33b}$$

$$\frac{1}{y_{2j+1}+y_{2j+2}}\left[-(y_{2j+1}^2-1)+y_{2j+2}^2-1\right] = y_{2j+2} - y_{2j+1}, \tag{2.33c}$$

and

$$\frac{1}{y_{2k}+y_{2k+1}}\left[-(y_{2k+1}^2-1)+y_{2k}^2-1\right] = y_{2k} - y_{2k+1}. \tag{2.33d}$$

As a result of this combination we see that the term in curly brackets in (2.32) becomes

$$\begin{aligned}
&y_2 - y_1 + \sum_{j=1}^{k-1}(y_{2j}-y_{2j+1}) + \sum_{j=1}^{k-1}(y_{2j+2}-y_{2j+1}) + y_{2k} - y_{2k+1} \\
&+ 2\sum_{j=1}^{k-1} y_{2j+1} - 2\sum_{j=1}^{k} y_{2j} + \left[\frac{y_{2k+1}(y_1^2-1)+y_1(y_{2k+1}^2-1)}{y_1 y_{2k+1}-1}\right],
\end{aligned}$$

a quantity which is identically zero. Hence (2.30) follows and thus the lemma is proved.

D. Cases $k=0$, $k=1$, $k=2$

The problem is to show that (2.8) holds for all k. For $k=0$ (2.8) reduces to showing

$$L_\nu g_1(t;\nu) = 0, \tag{2.34}$$

where L_ν is given by (2.23). That is we want to demonstrate

$$\int_1^\infty dy\,\frac{\exp(-ty)}{(y^2-1)^{1/2}}\left(\frac{y-1}{y+1}\right)^\nu\left[y^2-\frac{1}{t}y-\left(1+\frac{2\nu}{t}\right)\right]=0. \tag{2.35}$$

This clearly follows by using (2.25a) in the integration by parts of the $1/t$ term. This result is well known.

For $k=1$ (2.8) reduces to showing

$$L_\nu g_3(t;\nu)=2g_1(g_1-g_1'^2). \tag{2.36}$$

From Lemma 2.3 we see that

$$L_\nu g_3(t;\nu)=-2\int_1^\infty dy_1\int_1^\infty dy_2\int_1^\infty dy_3 \times\left[\prod_{j=1}^3\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right](y_1y_3-1). \tag{2.37}$$

Using the definition (1.6a) of $g_1(t;\nu)$ we see that the right-hand side of (2.36) is precisely (2.37). Hence (2.8) is true for $k=1$.

The case $k=2$ is somewhat more involved. This case along with $k=3$ must have separate proofs from the case of arbitrary k ($\geqslant 4$) as for $k\leqslant 3$ the structure of (2.8) is lacking certain complexities that are present in the general case. This will become apparent as we proceed into the proof.

However certain general comments concerning (2.8) can be made at this point. To prove (2.8) we have found it necessary to put the integrands of the integral representations of the terms appearing in (2.8) into such a form that the integrands contain the same number of denominator factors. By use of Lemma 2.3 we see that $L_\nu g_{2k+1}(t;\nu)$ has $2k-2$ denominator factors in the integrand of its integral representation. This same number of denominator factors occurs in the term

$$2\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}g_{2(k-m-l)-1}[g_{2l+1}g_{2m+1}-g'_{2l+1}g'_{2m+1}]$$

which appears in (2.8). However the term

$$\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}g_{2l+1}g_{2m+1}L_\nu g_{2(k-m-l)-1}$$

which also appears in (2.8) has only $2k-4$ denominator factors in its integral representation (apply Lemma 2.3 to $L_\nu g_{2(k-m-l)-1}$ and use the definitions of g_{2l+1} and g_{2m+1}). Thus instead of (2.8) we will prove the equivalent identity

$$L_\nu g_{2k+1}(t;\nu)-2\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}g_{2(k-m-l)-1}[g_{2l+1}g_{2m+1}-g'_{2l+1}g'_{2m+1}] =\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}g_{2l+1}g_{2m+1}L_\nu g_{2(k-m-l)-1}. \tag{2.38}$$

The key to proving (2.38) will be to write the left-hand side of (2.38) in a form that contains only $2k-4$ denominator factors of the type (y_j+y_{j+1}). Once this is done the two sides of (2.38) can be successfully compared. The remainder of this section is the proof of (2.38) for $k=2$.

Using Lemma 2.2 for $g_{2l+1}, g_{2m+1}, g'_{2l+1}$, and g'_{2m+1}, Lemma 2.3 for $L_\nu g_{2k+1}$, and the definition (1.6) for $g_{2(k-m-l)-1}$ we can write the left-hand side of (2.38) for $k=2$ as

$$L_\nu g_5-2\sum_{l=0}^{1}\sum_{m=0}^{1-l}g_{2(2-m-l)-1}[g_{2l+1}g_{2m+1}-g'_{2l+1}g'_{2m+1}] =2\int_1^\infty dy_1\cdots\int_1^\infty dy_5\prod_{j=1}^5\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu \times\prod_{j=1}^4(y_j+y_{j+1})^{-1}\mathcal{L}_5(y_1,\ldots,y_5), \tag{2.39}$$

where

$$\begin{aligned}\mathcal{L}_5(y_1,\ldots,y_5) &=(y_1+y_2)(y_2+y_3)[(y_1y_5-1)(y_3^2-1)-(y_5y_3-1)\\ &\quad\times[y_1(y_3+y_4+y_5)-1]]+(y_3+y_4)(y_4+y_5)\\ &\quad\times[(y_1y_5-1)(y_3^2-1)-(y_3y_1-1)[(y_1+y_2+y_3)y_5-1]],\end{aligned} \tag{2.40}$$

where we used the labeling $1,2,\ldots,2l+1$ for g_{2l+1}; $2l+2,\ 2l+3,\ldots,2(k-m)$ for $g_{2(k-l-m)-1}$; and $2(k-m)+1,\ldots,2k+1$ for g_{2m+1}. We note that the $l=0$, $m=0$ term is zero. In this expression for $\mathcal{L}_5$ we use the identities

$$\begin{aligned}&(y_1y_5-1)(y_3^2-1)-(y_1y_3-1)[(y_1+y_2+y_3)y_5-1]\\ &\quad=(y_1y_5-1)y_3[(y_2+y_3)-(y_1+y_2)]\\ &\qquad-(y_1y_3-1)y_5(y_2+y_3)\end{aligned} \tag{2.41a}$$

and

$$\begin{aligned}&(y_1y_5-1)(y_3^2-1)-(y_5y_3-1)[(y_3+y_4+y_5)y_1-1]\\ &\quad=(y_1y_5-1)y_3[(y_3+y_4)-(y_4+y_5)]\\ &\qquad-(y_3y_5-1)y_1(y_3+y_4)\end{aligned} \tag{2.41b}$$

to rewrite $\mathcal{L}_5$ so that (2.39) becomes [note that by (2.41) we have factored out one denominator term]

$$\begin{aligned}&L_\nu g_5-2\sum_{l=0}^{1}\sum_{m=0}^{1-l}g_{2(2-m-l)-1}[g_{2l+1}g_{2m+1}-g'_{2l+1}g'_{2m+1}]\\ &=2\int_1^\infty dy_1\cdots\int_1^\infty dy_5\prod_{j=1}^5\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\\ &\quad\times\left\{(y_1y_5-1)\left[\frac{y_3}{y_4+y_5}-\frac{y_3}{y_3+y_4}+\frac{y_3}{y_1+y_2}-\frac{y_3}{y_2+y_3}\right]\right.\\ &\quad\left.-\frac{y_1(y_3y_5-1)}{y_4+y_5}-\frac{y_5(y_1y_3-1)}{y_1+y_2}\right\}.\end{aligned} \tag{2.42}$$

In the last two terms we make the change of variables $y_1\leftrightarrow y_3$ and $y_3\leftrightarrow y_5$, respectively. Then (2.42) becomes

$$L_\nu g_5-2\sum_{l=0}^{1}\sum_{m=0}^{1-l}g_{2(2-m-l)-1}[g_{2l+1}g_{2m+1}-g'_{2l+1}g'_{2m+1}] =2\int_1^\infty dy_1\cdots\int_1^\infty dy_5\prod_{j=1}^5\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu$$

$$\times(y_1y_5-1)\left[-\frac{y_3}{y_3+y_4}-\frac{y_3}{y_2+y_3}\right]$$
$$=-2\int_1^\infty dy_1\cdots\int_1^\infty dy_5\prod_{j=1}^{5}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}$$
$$\times\left(\frac{y_j-1}{y_j+1}\right)^\nu(y_1y_5-1). \tag{2.43}$$

A few words are in order to explain this last step. Suppose we have

$$h=\int_1^\infty dy_1\int_1^\infty dy_2\prod_{j=1}^{2}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\left(\frac{y_2}{y_1+y_2}\right). \tag{2.44}$$

Making the change of variables $y_1\leftrightarrow y_2$ in (2.44), adding this to (2.44) and dividing by two we find

$$h=\tfrac{1}{2}\int_1^\infty dy_1\int_1^\infty dy_2\prod_{j=1}^{2}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu. \tag{2.45}$$

The result (2.45) was used in the last step of (2.43). We now compare (2.43) with the right-hand side of (2.38). We have [recall (2.34)]

$$\sum_{l=0}^{1}\sum_{m=0}^{1-l}g_{2l+1}g_{2m+1}L_\nu g_{3-2m-2l}=g_1^2L_\nu g_3. \tag{2.46}$$

Using (2.37) for $L_\nu g_3$ and (1.6a) for g_1 we conclude that (2.46) is exactly (2.43). Thus (2.8) is true for $k=2$.

E. Integral representation of (2.38) for general k

Before we proceed to the case $k=3$, we derive an integral representation for the left-hand side of (2.38) for general k. If we use Lemma 2.2 for g_{2l+1}, g_{2m+1}, g'_{2l+1}, and g'_{2m+1}, Lemma 2.3 for $L_\nu g_{2k+1}$, and definition (1.6b) for $g_{2(k-m-l)-1}$ and use the labeling $1,2,\ldots,2l+1$ for g_{2l+1}; $2l+2, 2l+3,\ldots,2(k-m)$ for $g_{2(k-l-m)-1}$, and $2(k-m)+1,\ldots,2k+1$ for g_{2m+1}, we find that the left-hand side of (2.38) can be written as

$$L_\nu g_{2k+1}-2\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1}g_{2(k-m-l)-1}[g_{2m+1}g_{2l+1}-g'_{2m+1}g'_{2l+1}]$$
$$=2(-1)^k\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}$$
$$\times\left(\frac{y_j-1}{y_j+1}\right)^\nu\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}\mathcal{L}_{2k+1}(y_1,\ldots,y_{2k+1}) \tag{2.47}$$

with $\mathcal{L}_{2k+1}$ given by

$$\mathcal{L}_{2k+1}=\sum_{m=1}^{k-1}(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)$$
$$\times\Big\{(y_1y_{2k+1}-1)(y_{2k-2m+1}^2-1)-(y_{2k+1}y_{2k-2m+1}-1)$$
$$\times\Big(y_1\sum_{n_2=0}^{2m}y_{2k+1-n_2}-1\Big)\Big\}+\sum_{l=1}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2k}+y_{2k+1})$$
$$\times\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\Big\{(y_1y_{2k+1}-1)(y_{2l+1}^2-1)$$
$$-(y_1y_{2l+1}-1)\Big(y_{2k+1}\sum_{n_3=1}^{2l+1}y_{n_3}-1\Big)\Big\}$$
$$+\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})$$
$$\prod_{\substack{n_1=2\\ n_1\neq l+1,\,k-m+1}}^{k}(y_{2n_1-1}^2-1)\Big\{(y_1y_{2k+1}-1)(y_{2l+1}^2-1)$$
$$\times(y_{2k-2m+1}^2-1)-(y_1y_{2l+1}-1)(y_{2k+1}y_{2k+1-2m}-1)$$
$$\times\Big(\sum_{n_3=1}^{2l+1}y_{n_3}\sum_{n_2=0}^{2m}y_{2k+1-n_2}-1\Big)\Big\}. \tag{2.48}$$

As in the $k=2$ case, the $l=0$, $m=0$ term canceled. The first term in (2.48) (the term involving the sum $\sum_{m=1}^{k-1}$) is the $l=0$, $m\neq0$ terms of (2.38); the second term in (2.48) (the term involving the sum $\sum_{l=1}^{k-1}$) is the $l\neq0$, $m=0$ terms of (2.38); and the third term which involves the double sum is the $l\neq0$, $m\neq0$ terms of (2.38). We write the first term in curly brackets in (2.48) as

$$(y_1y_{2k+1}-1)(y_{2k-2m+1}^2-1)-(y_{2k+1}y_{2k-2m+1}-1)$$
$$\times\Big(y_1\sum_{n_2=0}^{2m}y_{2k+1-n_2}-1\Big)$$
$$=(y_1y_{2k+1}-1)y_{2k-2m+1}(y_{2k-2m+1}-y_{2k+1})$$
$$-y_1\sum_{n_2=1}^{2m}y_{2k+1-n_2}(y_{2k-2m+1}y_{2k+1}-1)$$
$$=(y_1y_{2k+1}-1)y_{2k-2m+1}\Big(\sum_{n_2=1}^{2m}y_{2k+1-n_2}-\sum_{n_2=0}^{2m-1}y_{2k+1-n_2}\Big)$$
$$-y_1(y_{2k-2m+1}y_{2k+1}-1)\sum_{n_2=1}^{2m}y_{2k+1-n_2}$$
$$=-y_{2k-2m+1}(y_1y_{2k+1}-1)\sum_{n_2=0}^{2m-1}y_{2k+1-n_2}$$
$$+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m}y_{2k+1-n_2}, \tag{2.49}$$

the second term in curly brackets as

$$(y_1y_{2k+1}-1)(y_{2l+1}^2-1)-(y_1y_{2l+1}-1)\Big(y_{2k+1}\sum_{n_3=1}^{2l+1}y_{n_3}-1\Big)$$
$$=y_{2l+1}(y_1y_{2k+1}-1)(y_{2l+1}-y_1)-y_{2k+1}\Big(\sum_{n_3=2}^{2l+1}y_{n_3}\Big)(y_1y_{2l+1}-1)$$
$$=-y_{2l+1}(y_1y_{2k+1}-1)\sum_{n_3=1}^{2l}y_{n_3}+(y_{2k+1}-y_{2l+1})\sum_{n_3=2}^{2l+1}y_{n_3}, \tag{2.50}$$

and the third term in curly brackets as

$$(y_1y_{2k+1}-1)(y_{2l+1}^2-1)(y_{2k-2m+1}^2-1)-(y_1y_{2l+1}-1)$$
$$\times(y_{2k+1}y_{2k+1-2m}-1)\sum_{n_3=1}^{2l+1}y_{n_3}\sum_{n_2=0}^{2m}y_{2k+1-n_2}-1)$$
$$=-(y_1y_{2k+1}-1)[y_{2l+1}(y_{2k-2m+1}^2-1)(y_1-y_{2l+1})$$
$$+y_{2k-2m+1}(y_{2l+1}^2-1)(y_{2k+1}-y_{2k-2m+1})$$
$$+y_{2l+1}y_{2k-2m+1}(y_1-y_{2l+1})(y_{2k+1}-y_{2k-2m+1})]$$

$$-(y_1y_{2l+1}-1)(y_{2k+1}y_{2k-2m+1}-1)\times\left[y_1\sum_{n_2=1}^{2m} y_{2k+1-n_2}+y_{2k+1}\sum_{n_3=2}^{2l+1} y_{n_3}+\sum_{n_3=2}^{2l+1} y_{n_3}\sum_{n_2=1}^{2m} y_{2k+1-n_2}\right],$$

so that $\mathcal{L}_{2k+1}$ becomes

$$\begin{aligned}
\mathcal{L}_{2k+1} &= \sum_{m=1}^{k-1}(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)\\
&\quad\times\left[-y_{2k-2m+1}(y_1y_{2k+1}-1)\times\sum_{n_2=0}^{2m-1} y_{2k+1-n_2}+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m} y_{2k+1-n_2}\right]\\
&\quad+\sum_{l=1}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2k}+y_{2k+1})\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\\
&\quad\times\left[-y_{2l+1}(y_1y_{2k+1}-1)\sum_{n_3=1}^{2l} y_{n_3}+(y_{2k+1}-y_{2l+1})\sum_{n_3=2}^{2l+1} y_{n_3}\right]\\
&\quad-\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})\\
&\quad\times\prod_{\substack{n_1=2\\ n_1\neq k-m+1,\, l+1}}^{k}(y_{2n_1-1}^2-1)\Big\{(y_1y_{2k+1}-1)\\
&\quad\times[y_{2l+1}(y_{2k-2m+1}^2-1)(y_1-y_{2l+1})+y_{2k-2m+1}(y_{2l+1}^2-1)\\
&\quad\times(y_{2k+1}-y_{2k-2m+1})+y_{2l+1}y_{2k-2m+1}(y_1-y_{2l+1})\\
&\quad\times(y_{2k+1}-y_{2k+1-2m})]+(y_1y_{2l+1}-1)(y_{2k+1}y_{2k-2m+1}-1)\\
&\quad\times\left[y_1\sum_{n_2=1}^{2m} y_{2k+1-n_2}+y_{2k+1}\sum_{n_3=2}^{2l+1} y_{n_3}\right.\\
&\quad\left.+\sum_{n_3=2}^{2l+1} y_{n_3}\sum_{n_2=1}^{2m} y_{2k+1-n_2}\right]\Big\}\,.
\end{aligned}\tag{2.51}$$

Frequently when working with the quantity $\mathcal{L}_{2k+1}$ we will perform operations upon $\mathcal{L}_{2k+1}$ (for instance, symmetrizing the integration variable labels) that leave the value of the right-hand side of (2.47) unchanged. Under these circumstances we will use the symbol "=" to mean that $\mathcal{L}_{2k+1}$ as given above and the right-hand side of the equation have identical values when substituted into (2.47). From the context of the equation it will be clear when we are using this meaning of "=".

F. Graphs and $\mathcal{L}_{2k+1}$

It is convenient to develop a graphical representation of the various terms that occur in $\mathcal{L}_{2k+1}$. The basic factor appearing in (2.47) is the quantity

$$\mathcal{L}_{2k+1}(y_1,\ldots,y_{2k+1})\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}. \tag{2.52}$$

We can represent all such terms by the following rules:

(1) $\Pi_{j=1}^{2k}(y_j+y_{j+1})^{-1}$ is represented by a straight line with $2k+1$ points [see Fig. 2(a)].

(2) $(y_j+y_{j+1})\,\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$ is represented by a straight line with $2k+1$ points and one additional line connecting the points j and $j+1$ [see Fig. 2(b)].

(3) $(y_j^2-1)\,\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$ is represented by a straight line with $2k+1$ points and a circle centered about the jth point [see Fig. 2(c)].

(4) $y_j\,\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$ is represented by a straight line with $2k+1$ points and a "×" through the jth point [see Fig. 2(d)].

(5) Suppose we have a term $\mathcal{L}'$ which is a part of $\mathcal{L}_{2k+1}$. The order of $\mathcal{L}'$ is $2k+1$ and by the graph of $\mathcal{L}'$ we mean the graph of the integrand

$$\mathcal{L}'\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}$$

as constructed in accordance with rules (1)–(4).

Sometimes we wish to multiply some integrand factor $\mathcal{L}'$ by the factor $\mathcal{L}''$. If $\mathcal{L}'$ is a single graph, the product will be in general many graphs. To illustrate this multiplication of the graph $\mathcal{L}'$ by some other factor $\mathcal{L}''$ we draw the graph of $\mathcal{L}'$ and merely place $\mathcal{L}''$ to the extreme left. Of course, we may also explicitly draw all the graphs corresponding to $\mathcal{L}'\mathcal{L}''$ in accordance with rules (1)–(4).

G. Case $k=3$

For $k=3$ we write

$$\mathcal{L}_7=\sum_{l=0}^{2}\sum_{m=0}^{2-l}\mathcal{L}_7(l,m), \tag{2.53}$$

where from (2.51) it follows that

$$\begin{aligned}
\mathcal{L}_7(0,0)&=0,\\
\mathcal{L}_7(0,1)&=(y_1+y_2)(y_4+y_5)(y_3^2-1)\\
&\quad\times[-y_5(y_1y_7-1)(y_6+y_7)+(y_1-y_5)(y_6+y_5)],\\
\mathcal{L}_7(0,2)&=(y_1+y_2)(y_2+y_3)(y_5^2-1)[-y_3(y_1y_7-1)\\
&\quad\times(y_7+y_6+y_5+y_4)+(y_1-y_3)(y_6+y_5+y_4+y_3)],\\
\mathcal{L}_7(1,0)&=(y_3+y_4)(y_6+y_7)(y_5^2-1)\\
&\quad\times[-y_3(y_1y_7-1)(y_1+y_2)+(y_7-y_3)(y_2+y_3)],\\
\mathcal{L}_7(2,0)&=(y_5+y_6)(y_6+y_7)(y_3^2-1)[-y_5(y_1y_7-1)\\
&\quad\times(y_1+y_2+y_3+y_4)+(y_7-y_5)(y_2+y_3+y_4+y_5)],
\end{aligned}$$

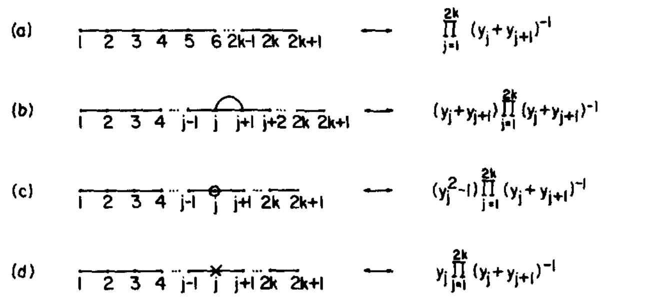

FIG. 2. (a) Graphical representation of $\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$. (b) Graphical representation of $(y_j+y_{j+1})\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$. (c) Graphical representation of $(y_j^2-1)\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$. (d) Graphical representation of $y_j\Pi_{l=1}^{2k}(y_l+y_{l+1})^{-1}$.

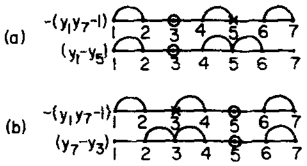

FIG. 3. (a) Graph of $\mathcal{L}_7(0,1)$ as defined by (2.54). (b) Graph of $\mathcal{L}_7(1,0)$ as defined by (2.54).

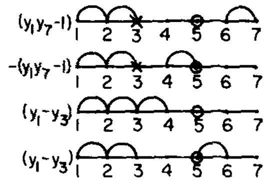

FIG. 5. Graph of $\mathcal{L}_7(0,2)$ as defined by (2.54).

$$\mathcal{L}_7(1,1)=-(y_3+y_4)(y_4+y_5)\{(y_1y_7-1)[y_3(y_5^2-1)\times(y_1-y_3)+y_5(y_3^2-1)(y_7-y_5)+y_3y_5(y_1-y_3)\times(y_7-y_5)]+(y_1y_3-1)(y_7y_5-1)[y_1(y_6+y_5)+y_7(y_2+y_3)+(y_2+y_3)(y_6+y_5)]\}. \tag{2.54}$$

The graphs of $\mathcal{L}_7(0,1)$ and $\mathcal{L}_7(1,0)$ are given in Fig. 3(a) and Fig. 3(b), respectively. From the graphs it is clear that $\mathcal{L}_7(0,1)$ and $\mathcal{L}_7(1,0)$ are equal [in the sense of "=" following (2.51)].

$\mathcal{L}_7(0,1)$ consists of two terms as illustrated in Fig. 3(a). If we let $1\leftrightarrow 5$ in the second term, the integrand is antisymmetric and thus when integrated gives zero. Hence

$$\mathcal{L}_7(0,1)=-(y_1+y_2)(y_4+y_5)(y_3^2-1)y_5(y_1y_7-1)(y_6+y_7) \tag{2.55}$$

and similarly $(1\leftrightarrow 3)$

$$\mathcal{L}_7(1,0)=-(y_3+y_4)(y_6+y_7)(y_5^2-1)y_3(y_1y_7-1)(y_1+y_2). \tag{2.56}$$

Both (2.55) and (2.56) can be reduced further. This reduction is essentially the same as that of (2.44) and (2.45) [in (2.55) symmetrize $5\leftrightarrow 6$ and in (2.56) symmetrize $2\leftrightarrow 3$]. Thus $\mathcal{L}_7(0,1)$ and $\mathcal{L}_7(1,0)$ become

$$\mathcal{L}_7(0,1)=-\tfrac{1}{2}(y_1y_7-1)(y_1+y_2)(y_4+y_5)(y_5+y_6)\times(y_6+y_7)(y_3^2-1) \tag{2.57}$$

and

$$\mathcal{L}_7(1,0)=-\tfrac{1}{2}(y_1y_7-1)(y_1+y_2)(y_2+y_3)(y_3+y_4)\times(y_6+y_7)(y_5^2-1), \tag{2.58}$$

respectively. The graph of $\mathcal{L}_7(0,1)$ is displayed in Fig. 4.

We now examine the term $\mathcal{L}_7(0,2)$. There are four basic terms in $\mathcal{L}_7(0,2)$ and these are displayed in Fig. 5.

The second graph has the reduction

$$-(y_1y_7-1)(y_1+y_2)(y_2+y_3)y_3(y_4+y_5)(y_5^2-1)\to-\tfrac{1}{2}(y_1y_7-1)(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_4+y_5)(y_5^2-1) \tag{2.59}$$

which is obtained by symmetrizing the y_3 variable $(3\leftrightarrow 4)$. This reduction always occurs when the graph is of the type Fig. 5(b). The general structure required for this reduction is shown in Fig. 6. The third graph of $\mathcal{L}_7(0,2)$ [Fig. 5(c)] gives zero weight to the integral (2.47) since the integrand is antisymmetric under the interchange $1\leftrightarrow 3$. Thus $\mathcal{L}_7(0,2)$ becomes

$$\mathcal{L}_7(0,2)=-(y_1y_7-1)(y_1+y_2)(y_2+y_3)y_3(y_5^2-1)(y_6+y_7)-\tfrac{1}{2}(y_1y_7-1)(y_1+y_2)(y_2+y_3)(y_3+y_4)\times(y_4+y_5)(y_5^2-1)+(y_1-y_3)(y_1+y_2)\times(y_2+y_3)(y_5^2-1)(y_5+y_6). \tag{2.60}$$

This reduced form for $\mathcal{L}_7(0,2)$ is shown in Fig. 7. The second term in (2.60) [Fig. 7(b)] has the correct number of factored denominators (in a graph this always corresponds to four loops).

A similar reduction for $\mathcal{L}_7(2,0)$ gives

$$\mathcal{L}_7(2,0)=(y_5+y_6)(y_6+y_7)(y_3^2-1)[-y_5(y_1y_7-1)(y_1+y_2)-\tfrac{1}{2}(y_1y_7-1)(y_4+y_5)(y_3+y_4)+(y_7-y_5)(y_2+y_3)]. \tag{2.61}$$

The graph of (2.61) is shown in Fig. 8 and should be compared with Fig. 7.

From (2.54) we can write $\mathcal{L}_7(1,1)$ as

$$\mathcal{L}_7(1,1)=-(y_3+y_4)(y_4+y_5)\{(y_1y_7-1)[y_3y_5(y_1-y_3)(y_7-y_5)+(y_5^2-1)y_3(y_1+y_2)+(y_3^2-1)y_5(y_7+y_6)]+(y_2+y_3)[y_7(y_1y_3-1)(y_5y_7-1)-y_3(y_1y_7-1)(y_5^2-1)]+(y_5+y_6)\times[y_1(y_1y_3-1)(y_5y_7-1)-y_5(y_1y_7-1)(y_3^2-1)]+(y_1y_3-1)(y_5y_7-1)(y_2+y_3)(y_5+y_6)\}. \tag{2.62}$$

We now use the identities

$$y_{2k+1}(y_1y_{2l+1}-1)(y_{2k-2m+1}y_{2k+1}-1)-y_{2l+1}(y_{2k-2m+1}^2-1)(y_1y_{2k+1}-1)=(y_{2k+1-2m}^2-1)(y_{2l+1}-y_{2k+1})+y_{2k+1}y_{2k-2m+1}\times(y_1y_{2l+1}-1)(y_{2k+1}-y_{2k-2m+1}) \tag{2.63}$$

and

$$y_1(y_1y_{2l+1}-1)(y_{2k-2m+1}y_{2k+1}-1)-y_{2k-2m+1}(y_{2l+1}^2-1)(y_1y_{2k+1}-1)$$

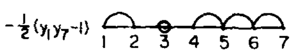

FIG. 4. Quantity (2.57).

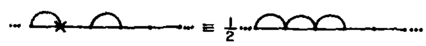

FIG. 6. General reduction formula. See discussion following (2.59).

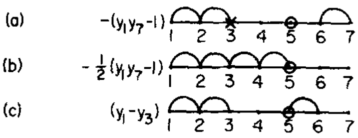

FIG. 7. Quantity (2.60).

$$= (y_{2l+1}^2 - 1)(y_{2k-2m+1} - y_1) + y_1 y_{2l+1}(y_1 - y_{2l+1})(y_{2k-2m+1}y_{2k+1} - 1)$$

for $k=3$, $l=1$, and $m=1$ in (2.62). The second and third terms multiplying the factor (y_1y_7-1) can be further reduced (these terms are of the general structure of Fig. 6). Carrying this out we can write $\mathcal{L}_7(1,1)$ as

$$\begin{aligned}\mathcal{L}_7(1,1) = &-(y_3+y_4)(y_4+y_5)[(y_1y_7-1)[y_3y_5(y_1-y_3)(y_7-y_5)\\ &+\tfrac{1}{2}(y_1+y_2)(y_2+y_3)(y_5^2-1)+\tfrac{1}{2}(y_5+y_6)(y_6+y_7)\\ &\times(y_3^2-1)]+(y_2+y_3)[(y_5^2-1)(y_3-y_7)\\ &+y_5y_7(y_1y_3-1)(y_7-y_5)]+(y_5+y_6)\\ &\times[(y_3^2-1)(y_5-y_1)+y_1y_3(y_1-y_3)(y_5y_7-1)]\\ &+(y_1y_3-1)(y_5y_7-1)(y_2+y_3)(y_5+y_6)].\end{aligned} \tag{2.64}$$

We now examine the term

$$\begin{aligned}&y_1y_7(y_3+y_4)(y_4+y_5)y_3y_5(y_1-y_3)(y_7-y_5)\\ &\quad=(y_3+y_4)(y_4+y_5)y_1y_3y_5y_7[(y_1+y_2)\\ &\quad\quad-(y_2+y_3)][(y_7+y_6)-(y_6+y_5)]\end{aligned} \tag{2.65}$$

occurring in $\mathcal{L}_7(1,1)$. We draw the graph of (2.65) in Fig. 9. The first term cancels the second and third term, the fourth (let $5\to 7$ and $7\to 5$ in the first and third graphs).

We now combine the terms $\mathcal{L}_7(0,2)$, $\mathcal{L}_7(2,0)$, and $\mathcal{L}_7(1,1)$. One way to create denominator factors from a term like (y_j-y_k) is to write this as $(y_j+y_{j+1}) - (y_{j+1}+y_{j+2}) + \cdots - (y_{k-1}+y_k)$. This identity has been extensively used already. However there are terms where this is of no use. For instance in (2.60) for $\mathcal{L}_7(0,2)$ there occurs the term (y_1-y_3). If we were to rewrite this as $(y_1+y_2)-(y_2+y_3)$ we would introduce the factors $(y_1+y_2)^2$ and $(y_2+y_3)^2$. We do not want terms of this form. Such a problem term occurs in (2.61) for $\mathcal{L}_7(2,0)$ [the (y_7-y_5) term] and two such terms in (2.64). We combine these terms:

$$\begin{aligned}J_7 \equiv\ &(y_1-y_3)(y_1+y_2)(y_2+y_3)(y_5+y_6)(y_5^2-1)\\ &+(y_7-y_5)(y_2+y_3)(y_5+y_6)(y_6+y_7)(y_3^2-1)\\ &-(y_3-y_7)(y_2+y_3)(y_3+y_4)(y_4+y_5)(y_5^2-1)\\ &-(y_5-y_1)(y_3+y_4)(y_4+y_5)(y_5+y_6)(y_3^2-1).\end{aligned} \tag{2.66}$$

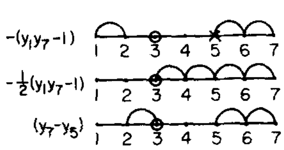

FIG. 8. Quantity (2.61).

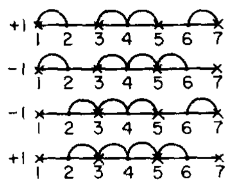

FIG. 9. Quantity (2.65).

The graph of J_7 is shown in Fig. 10. The first term cancels the third term and the second term is canceled by the fourth term. This can be seen by the change of variables $1\to 3$, $2\to 4$, $3\to 7$, $4\to 6$, $5\to 5$, $6\to 2$, and $7\to 1$ (this relabeling is seen most easily by comparing the first graph with the third graph of Fig. 10).

Thus we have

$$\begin{aligned}&\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}[(y_1-y_3)(y_1+y_2)(y_2+y_3)(y_5+y_6)(y_5^2-1)]\\ &=\frac{(y_1-y_3)(y_5^2-1)}{(y_3+y_4)(y_4+y_5)(y_6+y_7)} \to \frac{(y_3-y_7)(y_5^2-1)}{(y_7+y_6)(y_6+y_5)(y_1+y_2)}\\ &=\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}[(y_3-y_7)\\ &\times(y_5^2-1)(y_2+y_3)(y_3+y_4)(y_4+y_5)]\end{aligned}$$

which is the third graph. Hence we have demonstrated

$$J_7 = 0 \tag{2.67}$$

where we use the sense of "=" as discussed after (2.51).

We now examine the term

$$-(y_3+y_4)(y_4+y_5)y_1y_3(y_1-y_3)(y_5y_7-1)(y_5+y_6) \tag{2.68}$$

in $\mathcal{L}_7(1,1)$ [see (2.64)]. This clearly gives zero contribution since the above integrand [multiplied as always by $\Pi_{j=1}^{6}(y_j+y_{j+1})^{-1}$] is antisymmetric under the interchange $1\leftrightarrow 3$. The same is true for the term

$$(y_3+y_4)(y_4+y_5)(y_5+y_6)y_5y_7(y_1y_3-1)(y_7-y_5) \tag{2.69}$$

occurring in $\mathcal{L}_7(1,1)$. Collecting these results we have

$$\begin{aligned}&\mathcal{L}_7(0,2)+\mathcal{L}_7(2,0)+\mathcal{L}_7(1,1)\\ &=-(y_1y_7-1)(y_1+y_2)(y_2+y_3)(y_5^2-1)[y_3(y_6+y_7)\\ &\quad+\tfrac{1}{2}(y_3+y_4)(y_4+y_5)]-(y_1y_7-1)(y_5+y_6)(y_6+y_7)\\ &\quad\times(y_3^2-1)[y_5(y_1+y_2)+\tfrac{1}{2}(y_4+y_5)(y_3+y_4)]\\ &\quad-(y_1y_7-1)(y_3+y_4)(y_4+y_5)[\tfrac{1}{2}(y_2+y_3)(y_1+y_2)\\ &\quad\times(y_5^2-1)+\tfrac{1}{2}(y_5+y_6)(y_6+y_7)(y_3^2-1)]\\ &\quad+(y_3+y_4)(y_4+y_5)y_3y_5(y_1-y_3)(y_7-y_5)\\ &\quad-(y_3+y_4)(y_4+y_5)(y_1y_3-1)(y_5y_7-1)(y_2+y_3)(y_5+y_6).\end{aligned} \tag{2.70}$$

Though the last term in (2.70) contains four denomina-

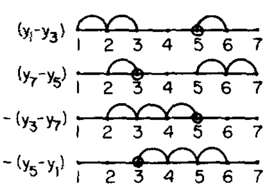

FIG. 10. Quantity (2.66).

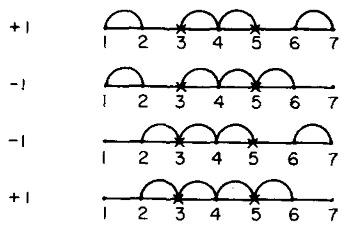

FIG. 11. Quantity $y_3y_5(y_3+y_4)(y_4+y_5)(y_1-y_3)(y_7-y_5)$.

tor type factors, the presence of the two terms $(y_1y_3-1)(y_5y_7-1)$ is not desired. Thus we expect further reductions of this term along with the other terms in (2.70) that do not have four denominator factors.

We examine the combination

$$(y_3+y_4)(y_4+y_5)[y_3y_5(y_1-y_3)(y_7-y_5) - (y_1y_3-1)(y_5y_7-1)(y_2+y_3)(y_5+y_6)]. \tag{2.71}$$

Now

$$\begin{aligned} y_3y_5(y_1-y_3)(y_7-y_5) &= y_3y_5(y_1+y_2)(y_6+y_7) - y_3y_5(y_1+y_2)(y_5+y_6) \\ &\quad - y_3y_5(y_2+y_3)(y_6+y_7) + y_3y_5(y_2+y_3)(y_5+y_6) \end{aligned} \tag{2.72}$$

so that the term $y_3y_5(y_3+y_4)(y_4+y_5)(y_1-y_3)(y_7-y_5)$ can be viewed as a sum of four terms. These terms are displayed in Fig. 11. In the first graph we let $1 \leftrightarrow 3$ and $5 \leftrightarrow 7$, in the second graph $1 \leftrightarrow 3$, and $5 \leftrightarrow 7$ in the third graph to obtain

$$y_3y_5(y_1-y_3)(y_7-y_5) = (y_1-y_3)(y_7-y_5)(y_2+y_3)(y_5+y_6). \tag{2.73}$$

Using (2.73) the expression (2.71) becomes

$$\begin{aligned} &(y_3+y_4)(y_4+y_5)(y_2+y_3)(y_5+y_6) \\ &\quad \times[(y_1-y_3)(y_7-y_5) - (y_1y_3-1)(y_5y_7-1)] \\ &= -(y_2+y_3)(y_3+y_4)(y_4+y_5)(y_5+y_6) \\ &\quad \times[(y_1y_7-1)(y_3y_5-1) + y_1(y_5-y_3) + y_7(y_3-y_5)]. \end{aligned} \tag{2.74}$$

By letting $5 \leftrightarrow 3$ in the second and third terms in (2.74) we see that the integrand obtained from (2.74) [that is, multiply (2.74) by $\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}$] is antisymmetric. Hence (2.74) is equivalent to

$$-(y_2+y_3)(y_3+y_4)(y_4+y_5)(y_5+y_6)(y_1y_7-1)(y_3y_5-1). \tag{2.75}$$

Multiplying (2.75) by $\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}$ we have

$$-\frac{(y_1y_7-1)(y_3y_5-1)}{(y_1+y_2)(y_6+y_7)} \tag{2.76}$$

which will be integrated over in (2.47). We relabel the variables by $1\to 3$, $3\to 7$, $5\to 1$, and $7\to 5$ (keeping the even labels fixed) so that (2.76) is equivalent to

$$-\frac{(y_1y_7-1)(y_3y_5-1)}{(y_2+y_3)(y_5+y_6)} \tag{2.77}$$

which implies (2.75) is equivalent to

$$-(y_1+y_2)(y_6+y_7)(y_3+y_4)(y_4+y_5)(y_1y_7-1)(y_3y_5-1). \tag{2.78}$$

Using these results (2.70) becomes

$$\begin{aligned} &\mathcal{L}_7(0,2)+\mathcal{L}_7(2,0)+\mathcal{L}_7(1,1) \\ &= -(y_1y_7-1)\{(y_1+y_2)(y_2+y_3)(y_5^2-1)[y_3(y_6+y_7) \\ &\quad + \tfrac{1}{2}(y_3+y_4)(y_4+y_5)] + (y_5+y_6)(y_6+y_7)(y_3^2-1) \\ &\quad \times[y_5(y_1+y_2) + \tfrac{1}{2}(y_3+y_4)(y_4+y_5)] \\ &\quad + (y_3+y_4)[\tfrac{1}{2}(y_1+y_2)(y_2+y_3)(y_5^2-1) \\ &\quad + \tfrac{1}{2}(y_5+y_6)(y_6+y_7)(y_3^2-1) \\ &\quad + (y_3y_5-1)(y_1+y_2)(y_6+y_7)]\}. \end{aligned} \tag{2.79}$$

Making use of the identity

$$\begin{aligned} y_3(y_5^2-1) &= (y_3+y_4)(y_5^2-1) - (y_4+y_5)(y_4y_5-1) \\ &\quad + (y_4^2-1)(y_5+y_6) - y_6(y_4^2-1) \end{aligned} \tag{2.80}$$

we see that by a relabeling of the integration variable labels the quantity

$$\frac{y_3(y_5^2-1)}{(y_3+y_4)(y_4+y_5)(y_5+y_6)}$$

can be replaced by

$$\frac{(y_3+y_4)(y_5^2-1)}{(y_3+y_4)(y_4+y_5)(y_5+y_6)} - \frac{1}{2}\,\frac{(y_4+y_5)(y_4y_5-1)}{(y_3+y_4)(y_4+y_5)(y_5+y_6)}$$

in (2.79). A similar transformation on the term $y_5(y_1+y_2)$ in (2.79) results in the equivalent expression for (2.79),

$$\begin{aligned} &\mathcal{L}_7(0,2)+\mathcal{L}_7(2,0)+\mathcal{L}_7(1,1) \\ &= -(y_1y_7-1)\{(y_1+y_2)(y_2+y_3)(y_5^2-1)[(y_3+y_4)(y_6+y_7) \\ &\quad + \tfrac{1}{2}(y_3+y_4)(y_4+y_5)] + (y_5+y_6)(y_6+y_7)(y_3^2-1) \\ &\quad \times[(y_4+y_5)(y_1+y_2) + \tfrac{1}{2}(y_3+y_4)(y_4+y_5)] \\ &\quad + (y_3+y_4)(y_4+y_5)[\tfrac{1}{2}(y_1+y_2)(y_2+y_3)(y_5^2-1) \\ &\quad + \tfrac{1}{2}(y_5+y_6)(y_6+y_7)(y_3^2-1)] - \tfrac{1}{2}(y_4+y_5)(y_4y_5-1) \\ &\quad \times(y_1+y_2)(y_2+y_3)(y_6+y_7) - \tfrac{1}{2}(y_3+y_4)(y_3y_4-1) \\ &\quad \times(y_1+y_2)(y_5+y_6)(y_6+y_7) + (y_1+y_2)(y_3+y_4) \\ &\quad \times(y_4+y_5)(y_6+y_7)(y_3y_5-1)\}. \end{aligned} \tag{2.81}$$

The last three terms of (2.81) cancel. To see this we multiply these terms by the factor $\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}$ to obtain

$$\begin{aligned} &-\frac{1}{2}\,\frac{(y_1y_7-1)(y_4y_5-1)}{(y_3+y_4)(y_5+y_6)} - \frac{1}{2}\,\frac{(y_1y_7-1)(y_3y_4-1)}{(y_2+y_3)(y_4+y_5)} \\ &\quad + \frac{(y_1y_7-1)(y_3y_5-1)}{(y_2+y_3)(y_5+y_6)}. \end{aligned} \tag{2.82}$$

Letting $3\to 2$, $4\to 3$ in the first term and $4\to 5$, $5\to 6$ in the second term we see that (2.82) is zero. Hence using this in (2.81) and adding the result to $\mathcal{L}_7(0,1) + \mathcal{L}_7(1,0)$ [see (2.57) and (2.58)] we find that (2.47) for the case $k=3$ becomes

$$2(-1)^3\int_1^\infty dy_1\cdots\int_1^\infty dy_7\prod_{j=1}^{7}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu$$

$$\times\prod_{j=1}^{6}(y_j+y_{j+1})^{-1}\mathcal{L}_7$$

$$=2\int_1^\infty dy_1\cdots\int_1^\infty dy_7\prod_{j=1}^{7}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu$$

$$\times(y_1y_7-1)\left(\frac{3}{2}\ \frac{(y_3^2-1)}{(y_2+y_3)(y_3+y_4)}+\frac{3}{2}\ \frac{(y_5^2-1)}{(y_4+y_5)(y_5+y_6)}\right.$$

$$\left.+\frac{(y_5^2-1)}{(y_5+y_6)(y_6+y_7)}+\frac{(y_3^2-1)}{(y_1+y_2)(y_2+y_3)}\right).\tag{2.83}$$

We now compare the result (2.83) with the right-hand side of (2.38). From (2.38) and (2.34)

$$\sum_{l=0}^{2}\sum_{m=0}^{2-l}g_{2l+1}g_{2m+1}L_\nu g_{2(3-m-l)-1}$$

$$=g_1^2L_\nu g_5+2g_1g_3L_\nu g_3.\tag{2.84}$$

Using the definition of the function g_1 and g_3 and Lemma 2.3 for $L_\nu g_3$ and $L_\nu g_5$ we can write (2.84) as

$$\sum_{l=0}^{2}\sum_{m=0}^{2-l}g_{2l+1}g_{2m+1}L_\nu g_{2(3-m-l)-1}$$

$$=2\int_1^\infty dy_1\cdots\int_1^\infty dy_7\prod_{j=1}^{7}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu$$

$$\times\left(\frac{(y_1y_5-1)(y_3^2-1)}{(y_2+y_3)(y_3+y_4)}+\frac{(y_1y_5-1)(y_3^2-1)}{(y_3+y_4)(y_4+y_5)}\right.$$

$$\left.+\frac{(y_1y_5-1)(y_3^2-1)}{(y_1+y_2)(y_2+y_3)}+2\ \frac{(y_1y_3-1)(y_5^2-1)}{(y_4+y_5)(y_5+y_6)}\right).\tag{2.85}$$

By relabeling the integration variable subscripts we see that (2.85) and (2.83) are identical. Hence we have proved identity (2.8) for $k=3$.

H. Cases $k\geqslant 4$

We have proved (2.38) for $k=1,\ 2,$ and 3. To prove Theorem 1 we must prove (2.38) [and hence (2.8)] for $k\geqslant 4$. In the preceding section the $k=3$ case of (2.38) was presented. Rather than give the most direct proof possible for $k=3$, we presented a proof that parallels as much as possible the general proof of this section. Even so the general proof is involved and at places special cases are presented to help see the cancellation that is taking place.

1. Alternative form for $\mathcal{L}_{2k+1}$

We start with $\mathcal{L}_{2k+1}$ as given by (2.51) and write

$$\mathcal{L}_{2k+1}=\sum_{l=1}^{k-1}\sum_{m=0}^{k-l-1}\mathcal{L}_{2k+1}(l,m)\tag{2.86}$$

with $\mathcal{L}_{2k+1}(0,0)\equiv 0$. Equation (2.51) can be rewritten (by adding and subtracting terms) as

$$\mathcal{L}_{2k+1}=\sum_{m=1}^{k-1}(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)$$

$$\times\left(-y_{2k-2m+1}(y_1y_{2k+1}-1)\sum_{n_2=0}^{2m-1}y_{2k+1-n_2}\right.$$

$$\left.+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m}y_{2k+1-n_2}\right)+\sum_{l=1}^{k-1}(y_{2l+1}+y_{2l+2})$$

$$\times(y_{2k}+y_{2k+1})\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\left(-y_{2l+1}(y_1y_{2k+1}-1)\right.$$

$$\left.\times\sum_{n_3=1}^{2l}y_{n_3}+(y_{2k+1}-y_{2l+1})\sum_{n_3=2}^{2l+1}y_{n_3}\right)$$

$$-\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})$$

$$\times\prod_{\substack{n_1=2\\ n_1\neq k-m+1,\ l+1}}^{k}(y_{2n_1-1}^2-1)\left((y_1y_{2k+1}-1)\right.$$

$$\times y_{2l+1}y_{2k-2m+1}(y_1-y_{2l+1})(y_{2k+1}-y_{2k-2m+1})$$

$$+(y_1y_{2l+1}-1)(y_{2k+1}y_{2k-2m+1}-1)$$

$$\times\sum_{n_3=2}^{2l+1}y_{n_3}\sum_{n_2=1}^{2m}y_{2k+1-n_2}+(y_1y_{2k+1}-1)(y_{2l+1}^2-1)y_{2k-2m+1}$$

$$\times\sum_{n_2=0}^{2m-1}y_{2k+1-n_2}+(y_1y_{2k+1}-1)(y_{2k-2m+1}^2-1)y_{2l+1}$$

$$\times\sum_{n_3=1}^{2l}y_{n_3}+\sum_{n_2=1}^{2m}y_{2k+1-n_2}[y_1(y_1y_{2l+1}-1)$$

$$\times(y_{2k+1}y_{2k-2m+1}-1)-y_{2k-2m+1}(y_{2l+1}^2-1)(y_1y_{2k+1}-1)]$$

$$+\sum_{n_3=2}^{2l+1}y_{n_3}[y_{2k+1}(y_1y_{2l+1}-1)(y_{2k-2m+1}y_{2k+1}-1)$$

$$\left.-y_{2l+1}(y_{2k-2m+1}^2-1)(y_1y_{2k+1}-1)]\right).\tag{2.87}$$

We first examine the $l=0,\ m=1$ term of (2.87), i.e.,

$$\mathcal{L}_{2k+1}(0,1)=\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)(y_1+y_2)(y_{2k-2}+y_{2k-1})$$

$$\times[-y_{2k-1}(y_1y_{2k+1}-1)(y_{2k}+y_{2k+1})$$

$$+(y_1-y_{2k-1})(y_{2k-1}+y_{2k})].\tag{2.88}$$

The term containing (y_1-y_{2k-1}) in (2.88) gives zero contribution to $\prod_{j=1}^{2k}(y_1+y_{j+1})^{-1}\mathcal{L}_{2k+1}(0,1)$ (let $1\leftrightarrow 2k-1$). Hence we have

$$\mathcal{L}_{2k+1}(0,1)=-\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)(y_1+y_2)(y_{2k-2}+y_{2k-1})y_{2k-1}$$

$$\times(y_1y_{2k+1}-1)(y_{2k}+y_{2k+1}).\tag{2.89}$$

Furthermore, symmetrizing the y_{2k-1} variable (recall argument associated with Fig. 6) we have

$$\mathcal{L}_{2k+1}(0,1)=-\tfrac{1}{2}\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)(y_1+y_2)(y_{2k-2}+y_{2k-1})$$

$$\times(y_{2k-1}+y_{2k})(y_{2k}+y_{2k+1})(y_1y_{2k+1}-1).\tag{2.90}$$

Similar transformations result in

$$\angle_{2k+1}(1,0) = -\tfrac{1}{2}\prod_{n_1=3}^{k}(y_{2n_1-1}^2-1)(y_1+y_2)(y_2+y_3)$$
$$\times(y_3+y_4)(y_{2k}+y_{2k+1})(y_1y_{2k+1}-1). \qquad (2.91)$$

Both $\angle_{2k+1}(0,1)$ and $\angle_{2k+1}(1,0)$ have the required four denominator type factors and a single $(y_1y_{2k+1}-1)$ factor.

We now analyze

$$\sum_{m=2}^{k-1}\angle_{2k+1}(0,m) \text{ and } \sum_{l=2}^{k-1}\angle_{2k+1}(l,0).$$

From (2.87) we have

$$\sum_{m=2}^{k-1}\angle_{2k+1}(0,m)$$
$$=\sum_{m=2}^{k-1}(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)$$
$$\times\Big[-y_{2k-2m+1}(y_1y_{2k+1}-1)\sum_{n_2=0}^{2m-3}y_{2k+1-n_2}-y_{2k-2m+1}$$
$$\times(y_1y_{2k+1}-1)(y_{2k-2m+2}+y_{2k-2m+3})+(y_1-y_{2k-2m+1})$$
$$\times(y_{2k-2m+1}+y_{2k-2m+2})+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m-2}y_{2k+1-n_2}\Big]. \qquad (2.92)$$

Symmetrizing the $y_{2k-2m+1}$ variable in the second term in square brackets in (2.92) $(y_{2k-2m+1}\leftrightarrow y_{2k-2m+2})$ and observing that the term $(y_{2k-2m}+y_{2k-2m+1})(y_1-y_{2k-2m+1})(y_{2k-2m+1}+y_{2k-2m+2})(y_1+y_2)$ is equivalent to zero $(1\leftrightarrow 2k-2m+1)$ the quantity (2.92) becomes

$$\sum_{m=2}^{k-1}\angle_{2k+1}(0,m)$$
$$=\sum_{m=2}^{k-1}(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)$$
$$\times[-y_{2k-2m+1}(y_1y_{2k+1}-1)$$
$$\times\sum_{n_2=0}^{2m-3}y_{2k+1-n_2}-\tfrac{1}{2}(y_{2k-2m+1}+y_{2k-2m+2})(y_{2k-2m+2}$$
$$+y_{2k-2m+3})(y_1y_{2k+1}-1)+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m-2}y_{2k+1-n_2}]. \qquad (2.93)$$

Similarly for $\sum_{l=2}^{k-1}\angle_{2k+1}(l,0)$ we have

$$\sum_{l=2}^{k-1}\angle_{2k+1}(l,0)$$
$$=\sum_{l=2}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2k}+y_{2k+1})\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)$$
$$\times\Big[-y_{2l+1}(y_1y_{2k+1}-1)\sum_{n_3=1}^{2l-2}y_{n_3}-\tfrac{1}{2}(y_{2l-1}+y_{2l})$$
$$\times(y_{2l}+y_{2l+1})(y_1y_{2k+1}-1)+(y_{2k+1}-y_{2l+1})\sum_{n_3=2}^{2l-1}y_{n_3}\Big]. \qquad (2.94)$$

We now claim

$$\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}\prod_{\substack{n_1=2\\ n_1\neq l+1,\,k-m+1}}^{k}(y_{2n_1-1}^2-1)(y_{2l+1}+y_{2l+2})$$
$$\times(y_{2k-2m}+y_{2k-2m+1})y_1y_{2l+1}y_{2k-2m+1}y_{2k+1}$$
$$\times(y_1-y_{2l+1})(y_{2k+1}-y_{2k-2m+1})=0 \qquad (2.95)$$

which is the generalization of (2.65). To demonstrate this we write

$$y_1-y_{2l+1}=(y_1+y_2)-(y_2+y_3)+\cdots$$
$$+(y_{2l-1}+y_{2l})-(y_{2l}+y_{2l+1}) \qquad (2.96)$$
$$y_{2k+1}-y_{2k-2m+1}=(y_{2k+1}+y_{2k})-(y_{2k}+y_{2k-1})$$
$$+\cdots-(y_{2k-2m}+y_{2k-2m+1}).$$

Then for a fixed l and m each term in (2.95) can be written as a sum of $4l(m+1)$ terms. A typical term is of the form

$$(-1)^{p+q}\prod_{\substack{n_1=2\\ n_1\neq l+1,\,k-m+1}}^{k}(y_{2n_1-1}^2-1)(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})$$
$$\times y_1y_{2l+1}y_{2k-2m+1}y_{2k+1}(y_p+y_{p+1})(y_q+y_{q+1}), \qquad (2.97)$$

where $p=1,2,\ldots,2l$ and $q=2k+1,\ 2k,\ldots,2k-2m$.

Keeping q fixed we examine the term with p replaced by $2l+1-p$. It is

$$(-1)^{p+q}\prod_{\substack{n_1=2\\ n_1\neq l+1,\,k-m+1}}^{k}(y_{2n_1-1}^2-1)(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})$$
$$\times y_1y_{2l+1}y_{2k-2m+1}y_{2k+1}(y_{2l+1-p}+y_{2l+2-p})(y_q+y_{q+1}). \qquad (2.98)$$

These two terms [(2.97) and (2.98)] are equivalent as can be seen from their graphs (see Fig. 12). They differ by an overall minus sign and thus add to give zero. Since this is true for fixed l, m, and q, we have pairwise cancellation as the index p runs through $1,2,\ldots,2l$. Hence it follows that (2.95) is true.

The term

$$(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})(y_1y_{2k+1}-1)$$
$$\times(y_{2l+1}^2-1)\,y_{2k-2m+1}\sum_{n_2=0}^{2m-1}y_{2k+1-n_2} \qquad (2.99)$$

occurring in (2.87) is equivalent to

$$(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})(y_1y_{2k+1}-1)(y_{2l+1}^2-1)$$
$$\times\tfrac{1}{2}(y_{2k-2m+1}+y_{2k-2m+2})(y_{2k-2m+2}+y_{2k-2m+3})$$
$$+(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})(y_1y_{2k+1}-1)$$
$$\times y_{2k-2m+1}\sum_{n_2=0}^{2m-3}y_{2k+1-n_2} \qquad (2.100)$$

as can be seen by symmetrizing y_{2k-m+1} when it multiplies

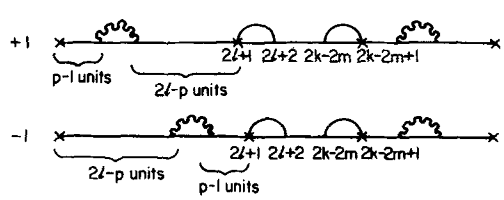

FIG. 12. Quantities (2.97) and (2.98). The loops that are moved are with wavy lines.

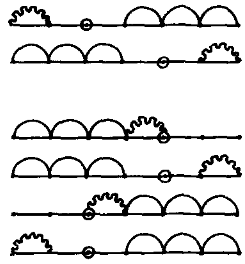

FIG. 13. (a) $l+m=1$ terms for $k=3$. (b) $l+m=2$ terms for $k=3$.

the last two terms of the sum $\sum_{n_2=0}^{2m-1} y_{2k+1-n_2}$. Likewise we can symmetrize the variable y_{2l+1} occurring in (2.87) when it multiplies the last two terms of the sum

$$\sum_{n_3=1}^{2l} y_{n_3}.$$

Collecting all these results and using the identities given by (2.63) we find that $\mathcal{L}_{2k+1}$ as given by (2.87) can be written in the equivalent form

$$
\begin{aligned}
\mathcal{L}_{2k+1} &\\
={}& -\tfrac{1}{2}\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)(y_1+y_2)(y_2+y_3)(y_3+y_4)(y_{2k}+y_{2k+1})\\
&\times(y_1y_{2k+1}-1)-\tfrac{1}{2}\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)(y_1+y_2)(y_{2k-2}+y_{2k-1})\\
&\times(y_{2k-1}+y_{2k})(y_{2k}+y_{2k+1})+\sum_{m=2}^{k-1}\prod_{\substack{n_1=2\\ n_1\neq k-m+1}}^{k}(y_{2n_1-1}^2-1)\\
&\times(y_1+y_2)(y_{2k-2m}+y_{2k-2m+1})\Big[-\tfrac{1}{2}(y_1y_{2k+1}-1)\\
&\times(y_{2k-2m+1}+y_{2k-2m+2})(y_{2k-2m+2}+y_{2k-2m+3})\\
&+(y_1-y_{2k-2m+1})\sum_{n_2=1}^{2m-2}y_{2k+1-n_2}-y_{2k-2m+1}(y_1y_{2k+1}-1)\\
&\times\sum_{n_2=0}^{2m-3}y_{2k+1-n_2}\Big]+\sum_{l=2}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2k}+y_{2k+1})\\
&\times\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\Big[-\tfrac{1}{2}(y_1y_{2k+1}-1)(y_{2l-1}+y_{2l})\\
&\times(y_{2l}+y_{2l+1})+(y_{2k+1}-y_{2l+1})\sum_{n_3=2}^{2l-1}y_{n_3}-y_{2l+1}\\
&\times(y_1y_{2k+1}-1)\sum_{n_3=1}^{2l-2}y_{n_3}\Big]-\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}\prod_{\substack{n_1=2\\ n_1\neq k-m+1,\,l+1}}^{k}(y_{2n_1-1}^2-1)\\
&\times(y_{2l+1}+y_{2l+2})(y_{2k-2m}+y_{2k-2m+1})\Big\{\tfrac{1}{2}(y_1y_{2k+1}-1)\\
&\times(y_{2l+1}^2-1)(y_{2k-2m+1}+y_{2k-2m+2})(y_{2k-2m+2}+y_{2k-2m+3})\\
&+\tfrac{1}{2}(y_1y_{2k+1}-1)(y_{2k-2m+1}^2-1)(y_{2l}+y_{2l-1})(y_{2l}+y_{2l+1})\\
&-y_{2l+1}y_{2k-2m+1}(y_1-y_{2l+1})(y_{2k+1}-y_{2k-2m+1})\\
&+(y_1y_{2k+1}-1)(y_{2l+1}^2-1)\,y_{2k-2m+1}\sum_{n_2=0}^{2m-3}y_{2k+1-n_3}\\
&+(y_1y_{2k+1}-1)(y_{2k-2m+1}^2-1)\,y_{2l+1}\sum_{n_3=1}^{2l-2}y_{n_3}\\
&+\sum_{n_2=1}^{2m}y_{2k+1-n_2}\Big[(y_{2l+1}^2-1)(y_{2k-2m+1}-y_1)+y_1y_{2l+1}\\
&\times(y_1-y_{2l+1})(y_{2k-2m+1}y_{2k+1}-1)\Big]+\sum_{n_3=2}^{2l+1}y_{n_3}\\
&\times\Big[(y_{2k-2m+1}^2-1)(y_{2l+1}-y_{2k+1})+y_{2k+1}y_{2k-2m+1}(y_1y_{2l+1}-1)\\
&\times(y_{2k+1}-y_{2k-2m+1})\Big]+(y_1y_{2l+1}-1)\Big]\\
&\times(y_{2k-2m+1}y_{2k+1}-1)\sum_{n_3=2}^{2l+1}y_{n_3}\sum_{n_2=1}^{2m}y_{2k+1-n_2}\Big\}\,.
\end{aligned}
\tag{2.101}
$$

The advantage of the representation (2.101) for $\mathcal{L}_{2k+1}$ as opposed to the previous representations [as for example (2.51)] is, for one, the separation of the "end effects" and the "bulk effects" of the integrand. Also the splitting (2.63) has been introduced into (2.101).

2. Summing $\mathcal{L}_{2k+1}(l,m)$ for $m+l=k-1$

For the case $k=3$ the graphs that appear in $\mathcal{L}_7(0,1)$ and $\mathcal{L}_7(1,0)$ when all reductions have been completed [see Fig. 13(a)] can be obtained from the set of graphs for $\mathcal{L}_7(2,0)+\mathcal{L}_7(0,2)+\mathcal{L}_7(1,1)$ [see Fig. 13(b)]. We claim that this is a general result. That is to say, if we sum all $\mathcal{L}_{2k+1}(l,m)$ such that $l+m=k-1$, then from the final reduced form for this sum there is a simple prescription to obtain the remaining terms. Therefore, we examine the sum

$$S_{2k+1}=\sum_{\substack{l=0\\ l+m=k-1}}^{k-1}{}'\sum_{m=0}^{k-l-1}{}'\mathcal{L}_{2k+1}(l,m) \tag{2.102}$$

and proceed to reduce this to the desired form [four loops in all graphs and a single $(y_1y_{2k+1}-1)$ factor].

From (2.101) and the definition of S_{2k+1} we have

$$
\begin{aligned}
S_{2k+1}&\\
={}&(y_1+y_2)(y_2+y_3)\prod_{n_1=3}^{k}(y_{2n_1-1}^2-1)\Big\{-\tfrac{1}{2}(y_1y_{2k+1}-1)(y_3+y_4)\\
&\times(y_4+y_5)+(y_1-y_3)\sum_{n_2=1}^{2k-4}y_{2k+1-n_2}-y_3(y_1y_{2k+1}-1)\\
&\times\sum_{n_2=0}^{2k-5}y_{2k+1-n_2}\Big\}+(y_{2k-1}+y_{2k})(y_{2k}+y_{2k+1})\\
&\times\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)\Big\{-\tfrac{1}{2}(y_1y_{2k+1}-1)(y_{2k-2}+y_{2k-1})\\
&\times(y_{2k-3}+y_{2k-2})+(y_{2k+1}-y_{2k-1})\sum_{n_3=2}^{2k-3}y_{n_3}\\
&-y_{2k-1}(y_1y_{2k+1}-1)\sum_{n_3=1}^{2k-4}y_{n_3}\Big\}-\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})\\
&\times(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k}(y_{2n_1-1}^2-1)\Big\{\tfrac{1}{2}(y_1y_{2k+1}-1)\\
&\times(y_{2l+1}^2-1)(y_{2l+3}+y_{2l+4})(y_{2l+4}+y_{2l+5})\\
&+\tfrac{1}{2}(y_1y_{2k+1}-1)(y_{2l+3}^2-1)(y_{2l}+y_{2l+1})(y_{2l-1}+y_{2l})\\
&-y_{2l+1}y_{2l+3}(y_1-y_{2l+1})(y_{2k+1}-y_{2l+3})+(y_1y_{2k+1}-1)\\
&\times(y_{2l+1}^2-1)\,y_{2l+3}\sum_{n_2=0}^{2k-2l-5}y_{2k+1-n_2}+(y_1y_{2k+1}-1)
\end{aligned}
$$

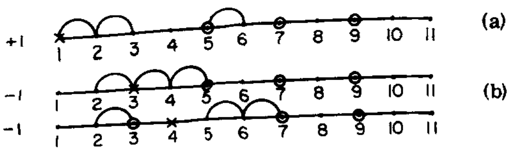

FIG. 14. Some typical terms contributing to J_{2k+1} for $k=5$. (a) Typical graph from $J_{11}^{(1)}$. (b) Two graphs from $J_{11}^{(3)}$.

$$\times (y_{2l+3}^2 - 1) y_{2l+1} \sum_{n_3=1}^{2l-2} y_{n_3} + \sum_{n_2=1}^{2k-2l-2} y_{2k+1-n_2}$$
$$\times [(y_{2l+1}^2 - 1)(y_{2l+3} - y_1) + y_1 y_{2l+1}(y_1 - y_{2l+1})$$
$$\times (y_{2l+3} y_{2k+1} - 1)] + \sum_{n_3=2}^{2l+1} y_{n_3}[(y_{2l+3}^2 - 1)(y_{2l+1} - y_{2k+1})$$
$$+ y_{2k+1} y_{2l+3}(y_1 y_{2l+1} - 1)(y_{2k+1} - y_{2l+3})] + (y_1 y_{2l+1} - 1)$$
$$\times (y_{2l+3} y_{2k+1} - 1) \sum_{n_3=2}^{2l+1} y_{n_3} \sum_{n_2=1}^{2k-2l-2} y_{2k+1-n_2} \Big\}. \tag{2.103}$$

3. $J_{2k+1}=0$

Recalling the discussion that resulted in the definition of the quantity J_7 [just before (2.66)], we see that an analogous argument for the terms appearing in (2.103) leads to the definition

$$J_{2k+1} = J_{2k+1}^{(1)} + J_{2k+1}^{(2)} + J_{2k+1}^{(3)} + J_{2k+1}^{(4)} \tag{2.104}$$

with

$$J_{2k+1}^{(1)} = (y_1 + y_2)(y_2 + y_3)(y_1 - y_3) \times \prod_{n_1=3}^{k} (y_{2n_1-1}^2 - 1) \sum_{n_2=1}^{2k-4} y_{2k+1-n_2}, \tag{2.105a}$$

$$J_{2k+1}^{(2)} = (y_{2k-1} + y_{2k})(y_{2k} + y_{2k+1})(y_{2k+1} - y_{2k-1}) \times \prod_{n_1=2}^{k-1} (y_{2n_1-1}^2 - 1) \sum_{n_3=2}^{2k-3} y_{n_3}, \tag{2.105b}$$

$$J_{2k+1}^{(3)} = -\sum_{l=1}^{k-2} (y_{2l+1} + y_{2l+2})(y_{2l+2} + y_{2l+3}) \prod_{\substack{n_1=2 \\ n_1 \neq l+1,\, l+2}}^{k} (y_{2n_1-1}^2 - 1) \times \sum_{n_2=2}^{2l+1} y_{n_2}(y_{2l+3}^2 - 1)(y_{2l+1} - y_{2k+1}), \tag{2.105c}$$

and

$$J_{2k+1}^{(4)} = -\sum_{l=1}^{k-2} (y_{2l+1} + y_{2l+2})(y_{2l+2} + y_{2l+3}) \times \prod_{\substack{n_1=2 \\ n_1 \neq l+1,\, l+2}}^{k} (y_{2n_1-1}^2 - 1) \sum_{n_2=2l+3}^{2k} y_{n_2}(y_{2l+1}^2 - 1)(y_{2l+3} - y_1) \tag{2.105d}$$

where we changed the labeling in the last sums appearing in (2.105c) and (2.105d). Furthermore the result $J_7 = 0$ of the previous section leads us to conjecture that

$$J_{2k+1} = 0 \tag{2.106}$$

where "=" is interpreted in the generalized sense.

We now prove that (2.106) is true. From an examination of the graphs associated with $J_{2k+1}^{(\alpha)}$, $\alpha = 1, 2, 3$, and 4, it is clear that

$$J_{2k+1}^{(1)} = J_{2k+1}^{(2)} \quad \text{and} \quad J_{2k+1}^{(3)} = J_{2k+1}^{(4)}. \tag{2.107}$$

A typical graph coming from the set of graphs associated with $J_{11}^{(1)}$ is shown in Fig. 14(a) and two types of graphs appearing in $J_{11}^{(3)}$ are displayed in Fig. 14(b). The important point to emphasize is that all graphs associated with $J_{2k+1}^{(1)}$ are such that the three loops appearing in the graph divide the line connecting "1" to "$2k+1$" into two disjoint lines [in Fig. 14(a) the disjoint lines are from 3 to 5 and from 6 to 11]. The graphs associated with $J_{2k+1}^{(3)}$ are of two basic types. There are the graphs that divide the line "1" to "$2k+1$" into two disjoint lines [the first graph in Fig. 14(b) is of this type] and there are the graphs that divide the line into three disjoint lines [the second graph of Fig. 14(b) is this type and the disjoint lines are 1 to 2, 3 to 5, and 7 to 11].

We now claim that the subset of graphs of $J_{2k+1}^{(3)}$ with two disjoint lines exactly cancels all the graphs of $J_{2k+1}^{(1)}$. The remaining graphs of $J_{2k+1}^{(3)}$ that are of the three-line type cancel amongst themselves to give zero. Once these two statements are demonstrated we will have proved (2.106).

We first count the number of two-line graphs in $J_{2k+1}^{(1)}$ and $J_{2k+1}^{(3)}$. In $J_{2k+1}^{(1)}$ there are clearly $2(k-2)$ terms [factor "2" comes from $(y_1 - y_k)$]. The two-line graphs of $J_{2k+1}^{(3)}$ come from the last two terms of the sum $\sum_{n_2=2}^{2l+1} y_{n_2}$ in (2.105c). Thus for fixed l there are two two-line graphs and hence $2(k-2)$ graphs in all. The three-line graphs of $J_{2k+1}^{(3)}$ result from the $\sum_{n_2=2}^{2l-1} y_{n_2}$ terms. Fix the integer l ($\leqslant k-2$) and let q be one of the values $1, 2, \ldots, l-1$. Then one term in (2.105c) can be written as

$$-(y_{2l+1} + y_{2l+2})(y_{2l+2} + y_{2l+3}) \times \prod_{\substack{n_1=2 \\ n_1 \neq l+1,\, l+2}}^{k} (y_{2n_1-1}^2 - 1)(y_{2q} + y_{2q+1})(y_{2l+3}^2 - 1)(y_{2l+1} - y_{2k+1}). \tag{2.108}$$

The graphs corresponding to (2.108) are shown in Fig. 15. To these two graphs we consider the complement graphs as shown in Fig. 16 [these are obtained from (2.108) by letting $q \to q$ and $l \to k-1-l+q$]. From Figs. 15 and 16 it is clear that the sum of the diagram and its complement gives zero (the first three-line graph in Fig. 15 is canceled by the second three-line graph in Fig. 16). Hence the sum of all three-line graphs in $J_{2k+1}^{(3)}$ gives zero.

The two-line graphs of $J_{2k+1}^{(3)}$ are of the form

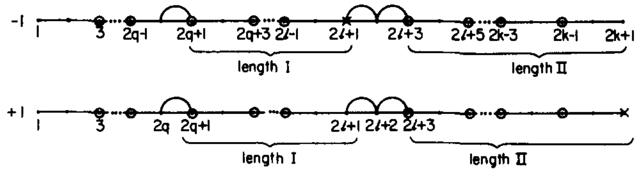

FIG. 15. Three-line graphs of $J_{2k+1}^{(3)}$ for q and l fixed.

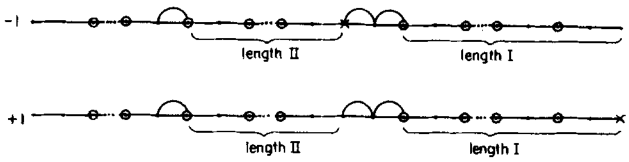

FIG. 16. Complement graph of Fig. 15. The second graph cancels the first graph of Fig. 15.

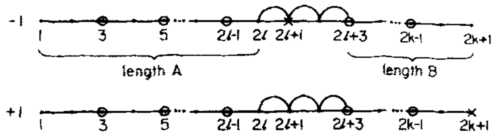

FIG. 17. Two-line graphs of $J^{(3)}_{2k+1}$.

$$-\sum_{l=1}^{k-2}(y_{2l}+y_{2l+1})(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})$$
$$\times \prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y^2_{2n_1-1}-1)(y_{2l+1}-y_{2k+1}) \qquad (2.109)$$

and are shown in Fig. 17. In Fig. 18 we draw a two-line graph associated with $J^{(1)}_{2k+1}$. If we choose the lengths as shown (which is always possible), then we conclude from a comparison of Figs. 17 and 18 that the two-line graphs of $J^{(1)}_{2k+1}$ and $J^{(3)}_{2k+1}$ cancel to give zero. Thus we have established (2.106).

4. Further cancellation in (2.103)

We examine the terms

$$A_1=\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\, l+2}}^{k}(y^2_{2n_1-1}-1)$$
$$\times\sum_{n_2=2l+3}^{2k} y_{n_2}\, y_1 y_{2l+1}(y_1-y_{2l+1})(y_{2l+3}y_{2k+1}-1) \qquad (2.110)$$

and

$$A_2=\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\, l+2}}^{k}(y^2_{2n_1-1}-1)$$
$$\times\sum_{n_2=2}^{2l+1} y_{n_2}\, y_{2k+1}\, y_{2l+3}(y_{2k+1}-y_{2l+1})(y_1 y_{2l+1}-1) \qquad (2.111)$$

that appear in (2.103) (note that $\sum_{n_2=1}^{2k-2l-2} y_{2k+1-n_2} =\sum_{n_2=2l+3}^{2k} y_{n_2}$). We now demonstrate that when A_1 and A_2 are used in (2.47) (A_1 and A_2 are parts of $\mathcal{L}_{2k+1}$) and the integration is performed the result is zero. That is to say, we show

$$A_1=A_2=0, \qquad (2.112)$$

where "=" is used in the generalized sense.

For fixed l we examine one term in (2.110). If we relabel the integration variables $1\to 2l+1$, $2\to 2l,\dots,2l\to 2$, and $2l+1\to 1$ while the remaining labels are fixed, then the integrand is antisymmetric in y_1 and y_{2l+1} [recall that we are always implicitly multiplying the factors A_1 and A_2 by $\prod_{j=1}^{2k}(y_j+y_{j+1})^{-1}$] and hence zero. For the term A_2 we relabel the variables $y_{2l+3}\to y_{2k+1}$, $y_{2l+4}\to y_{2k},\dots, y_{2k}\to y_{2l+4}$, and $y_{2k+1}\to y_{2l+3}$ and note that each term in A_2 is antisymmetric in y_{2k+1} and y_{2l+3}. Hence (2.112) is proved.

5. Final form for S_{2k+1}

Summarizing the results so far we have demonstrated that S_{2k+1} of (2.103) can be written as [this is the generalization of (2.70)]

$$S_{2k+1}=(y_1+y_2)(y_2+y_3)\prod_{n_1=3}^{k}(y^2_{2n_1-1}-1)\Big(-\tfrac{1}{2}(y_1y_{2k+1}-1)$$
$$\times(y_3+y_4)(y_4+y_5)-y_3(y_1y_{2k+1}-1)\sum_{n_2=6}^{2k+1}y_{n_2}\Big)$$
$$+(y_{2k-1}+y_{2k})(y_{2k}+y_{2k+1})\prod_{n_1=2}^{k-1}(y^2_{2n_1-1}-1)$$
$$\times\Big(-\tfrac{1}{2}(y_1y_{2k+1}-1)(y_{2k-2}+y_{2k-1})(y_{2k-3}+y_{2k-2})$$
$$-y_{2k-1}(y_1y_{2k+1}-1)\sum_{n_2=1}^{2k-4}y_{n_2}\Big)-\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})$$
$$\times(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\, l+2}}^{k}(y^2_{2n_1-1}-1)$$
$$\times\Big(\tfrac{1}{2}(y_1y_{2k+1}-1)(y^2_{2l+1}-1)(y_{2l+3}+y_{2l+4})(y_{2l+4}+y_{2l+5})$$
$$+\tfrac{1}{2}(y_1y_{2k+1}-1)(y^2_{2l+3}-1)(y_{2l}+y_{2l+1})(y_{2l-1}+y_{2l})$$
$$-y_{2l+1}y_{2l+3}(y_1-y_{2l+1})(y_{2k+1}-y_{2l+3})$$
$$+(y_1y_{2k+1}-1)(y^2_{2l+1}-1)y_{2l+3}$$
$$\times\sum_{n_2=2l+6}^{2k+1}y_{n_2}+(y_1y_{2k+1}-1)(y^2_{2l+3}-1)y_{2l+1}$$
$$\times\sum_{n_2=1}^{2l-2}y_{n_2}+(y_1y_{2l+1}-1)(y_{2l+3}y_{2k+1}-1)$$
$$\times\sum_{n_3=2}^{2l+1}y_{n_3}\sum_{n_2=2l+3}^{2k}y_{n_2}\Big). \qquad (2.113)$$

The generalization of the term in (2.71) is

$$\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\, l+2}}^{k}(y^2_{2n_1-1}-1)$$
$$\times\Big[y_{2l+1}y_{2l+3}(y_1-y_{2l+1})(y_{2k+1}-y_{2l+3})$$
$$-(y_1y_{2l+1}-1)(y_{2l+3}y_{2k+1}-1)\sum_{n_2=2}^{2l+1}y_{n_2}\sum_{n_3=2l+3}^{2k}y_{n_3}\Big]. \qquad (2.114)$$

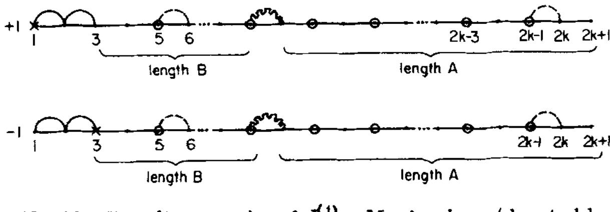

FIG. 18. Two-line graphs of $J^{(1)}_{2k+1}$. Moving loop (denoted by wavy line) starts at 5 and goes to $2k-1$ (shown in dotted lines).

As was done in going from (2.72) to (2.74), we want to write the first term in square brackets in (2.114) in a form so that the two sum-terms appearing in the second term in (2.114) become a common factor. To do this we write

$$(y_1 - y_{2l+1})(y_{2k+1} - y_{2l+3}) = \left(\sum_{n_2=1}^{2l} y_{n_2} - \sum_{n_2=2}^{2l+1} y_{n_2}\right)\left(\sum_{n_3=2l+4}^{2k+1} y_{n_3} - \sum_{n_3=2l+3}^{2k} y_{n_3}\right) \tag{2.115}$$

and examine the graphs of

$$y_{2l+1}y_{2l+3}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3}) \times \prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k} (y_{2n_1-1}^2 - 1)(y_1 - y_{2l+1})(y_{2k+1} - y_{2l+3}). \tag{2.116}$$

It is clear that we can relabel the integration variable subscripts so that the following is true [when used in (2.116), which in turn will be used in (2.47)]:

$$y_{2l+1}y_{2l+3}(y_1 - y_{2l+1})(y_{2k+1} - y_{2l+3}) = (y_1 - y_{2l+1})(y_{2k+1} - y_{2l+3})\sum_{n_2=2}^{2l+1} y_{n_2}\sum_{n_3=2l+3}^{2k} y_{n_3}. \tag{2.117}$$

Using (2.117) in (2.114) we obtain

$$\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k}(y_{2n_1-1}^2-1)\sum_{n_2=2}^{2l+1} y_{n_2} \times \sum_{n_3=2l+3}^{2k} y_{n_3}[-(y_1y_{2k+1}-1)(y_{2l+1}y_{2l+3}-1) - y_1(y_{2l+3}-y_{2l+1}) - y_{2k+1}(y_{2l+1}-y_{2l+3})], \tag{2.118}$$

where we used the algebraic identity

$$-(y_1-y_{2l+1})(y_{2k+1}-y_{2l+3}) + (y_1y_{2l+1}-1)(y_{2l+3}y_{2k+1}-1) = (y_1y_{2k+1}-1)(y_{2l+1}y_{2l+3}-1) + y_1(y_{2l+3}-y_{2l+1}) + y_{2k+1}(y_{2l+1}-y_{2l+3}). \tag{2.119}$$

It is clear that the last two terms in (2.118) give zero as the second and third terms will lead to an integrand that is antisymmetric in y_{2l+3} and y_{2l+1}. Also by relabeling we can let

$$\sum_{n_2=2}^{2l+1} y_{n_2}\sum_{n_3=2l+3}^{2k} y_{n_3} \to \sum_{n_2=1}^{2l} y_{n_2}\sum_{n_3=2l+4}^{2k+1} y_{n_3}.$$

Hence S_{2k+1} of (2.113) becomes

$$\begin{aligned}S_{2k+1} = &-(y_1y_{2k+1}-1)(y_1+y_2)(y_2+y_3)\prod_{n_1=3}^{k}(y_{2n_1-1}^2-1)\\ &\times\left[\tfrac{1}{2}(y_3+y_4)(y_4+y_5) + y_3\sum_{n_2=6}^{2k+1} y_{n_2}\right] - (y_1y_{2k+1}-1)\\ &\times(y_{2k-1}+y_{2k})(y_{2k}+y_{2k+1})\prod_{n_1=2}^{k-1}(y_{2n_1-1}^2-1)\\ &\times\left[\tfrac{1}{2}(y_{2k-3}+y_{2k-2})(y_{2k-2}+y_{2k-1}) + y_{2k-1}\sum_{n_2=1}^{2k-4} y_{n_2}\right]\\ &-(y_1y_{2k+1}-1)\sum_{l=1}^{k-2}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\\ &\times\prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k}(y_{2n_1-1}^2-1)\Big[\tfrac{1}{2}(y_{2l+1}^2-1)(y_{2l+3}+y_{2l+4})\\ &\times(y_{2l+4}+y_{2l+5}) + \tfrac{1}{2}(y_{2l+3}^2-1)(y_{2l-1}+y_{2l})(y_{2l}+y_{2l+1})\\ &+(y_{2l+1}^2-1)y_{2l+3}\sum_{n_2=2l+6}^{2k+1} y_{n_2} + (y_{2l+3}^2-1)y_{2l+1}\\ &\times\sum_{n_2=1}^{2l-2} y_{n_2} + (y_{2l+1}y_{2l+3}-1)\sum_{n_2=1}^{2l} y_{n_2}\sum_{n_3=2l+4}^{2k+1} y_{n_3}\Big].\end{aligned} \tag{2.120}$$

This can be written more compactly by combining the first two terms in (2.120) into the $l=0$ and $l=k-1$ terms of the third term. Doing this (2.120) becomes

$$\begin{aligned}S_{2k+1} = &-(y_1y_{2k+1}-1)\sum_{l=0}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})\\ &\times\Bigg[\prod_{\substack{n_1=2\\ n_1\neq l+2}}^{k}(y_{2n_1-1}^2-1)\tfrac{1}{2}(y_{2l+3}+y_{2l+4})(y_{2l+4}+y_{2l+5})\\ &+\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\tfrac{1}{2}(y_{2l-1}+y_{2l})(y_{2l}+y_{2l+1})\\ &+\prod_{\substack{n_1=2\\ n_1\neq l+2}}^{k}(y_{2n_1-1}^2-1)y_{2l+3}\sum_{n_2=2l+6}^{2k+1} y_{n_2} + \prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)\\ &\times y_{2l+1}\sum_{n_2=1}^{2l-2} y_{n_2} + \prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k}(y_{2n_1-1}^2-1)(y_{2l+1}y_{2l+3}-1)\\ &\times\sum_{n_2=1}^{2l} y_{n_2}\sum_{n_3=2l+4}^{2k+1} y_{n_3}\Bigg],\end{aligned} \tag{2.121}$$

where we must have the convention that any product term

$$\prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k}(y_{2n_1-1}^2-1)$$

is zero for $l=0$,

$$\prod_{\substack{n_1=2\\ n_1\neq l+2}}^{k}(y_{2n_1-1}^2-1)$$

is zero for $l=k-1$, and

$$\prod_{\substack{n_1=2\\ n_1\neq l+1,\,l+2}}^{k}(y_{2n_1-1}^2-1)$$

is zero for either $l=0$ or $l=k-1$.

Consider the term

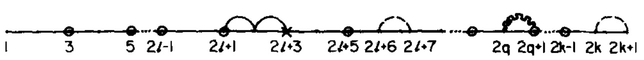

FIG. 19. Graph of (2.122) for a particular value of q. The range over which q varies is indicated by dotted loops.

$$(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3}) \prod_{\substack{n_1=2\\ n_1\neq l+2}}^{k} (y^2_{2n_1-1}-1)\, y_{2l+3} \times \sum_{q=l+3}^{k} (y_{2q}+y_{2q+1}) \qquad (2.122)$$

which is the third term in (2.121). For a particular q the graph of (2.122) is shown in Fig. 19. From the figure it is clear that the integrand is divided into three parts. We examine the integrand associated with the graph between the double loop and the single loop. It is

$$\left[y_{2l+3} \prod_{n=l+3}^{q} (y^2_{2n-1}-1)\right] \prod_{j=2l+3}^{2q-1} (y_j+y_{j+1})^{-1}. \qquad (2.123)$$

We claim that this integrand factor can be replaced by

$$\tfrac{1}{2}\Bigg[\sum_{p=0}^{q-l-2} (y_{2l+3+2p}+y_{2l+4+2p}) \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \prod_{n=l+3+p}^{q} (y^2_{2n-1}-1) - \sum_{p=0}^{q-l-3} (y_{2l+4+2p}+y_{2l+5+2p}) \prod_{\substack{n=l+3\\ n\neq l+3+p}}^{q} (y^2_{2n-1}-1) \times (y_{2l+3}y_{2l+5+2p}-1)\Bigg] \prod_{j=2l+3}^{2q-1} (y_j+y_{j+1})^{-1}, \qquad (2.124)$$

where the product symbol is to be interpreted as unity if the upper index is less than the lower index. To prove (2.124) we start with the algebraic identity

$$\begin{aligned} y_{2l+3} \prod_{n=l+3}^{q} (y^2_{2n-1}-1) &= \sum_{p=0}^{q-l-2} (y_{2l+3+2p}+y_{2l+4+2p}) \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \prod_{n=l+3+p}^{q} (y^2_{2n-1}-1) \\ &\quad - \sum_{p=0}^{q-l-3} (y_{2l+4+2p}+y_{2l+5+2p})(y_{2l+4+2p}y_{2l+5+2p}-1) \\ &\quad \times \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \prod_{n=l+p+4}^{q} (y^2_{2n-1}-1) - y_{2q} \prod_{n=l+2}^{q} (y^2_{2n}-1) \end{aligned} \qquad (2.125)$$

and note that when used as an integrand [multiplied by $\Pi_{j=2l+3}^{2q-1}(y_j+y_{k+1})^{-1}$] the last term in (2.125) is equivalent to the term on the left-hand side of (2.125). Hence (2.123) is equivalent to

$$\tfrac{1}{2}\Bigg[\sum_{p=0}^{q-l-2} (y_{2l+3+3p}+y_{2l+4+2p}) \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \prod_{n=l+3+p}^{q} (y^2_{2n-1}-1) - \sum_{p=0}^{q-l-3} (y_{2l+4+2p}+y_{2l+5+2p})(y_{2l+4+2p}y_{2l+5+2p}-1) \times \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \prod_{n=l+p+4}^{q} (y^2_{2n-1}-1)\Bigg] \prod_{j=2l+3}^{2q-1} (y_j+y_{j+1})^{-1}. \qquad (2.126)$$

By the change of variables $2l+4+2p \to 2l+3$, $2l+3+2p \to 2l+4, \ldots, 2l+4 \to 2l+3+2p$, and $2l+3 \to 2l+4+2p$ in the last term in (2.126) and remembering that the quantity $\Pi_{j=2l+3}^{2q-1}(y_j+y_{j+1})^{-1}(y_{2l+4+2p}+y_{2l+5+2p})$ remains invariant under this change of labels we obtain the equivalent expression (2.124).

Similarly we have that the quantity

$$\left[y_{2l+1} \prod_{n=q+1}^{l} (y^2_{2n-1}-1)\right] \prod_{j=2q}^{2l} (y_j+y_{j+1})^{-1} \qquad (2.127)$$

occurring in (2.121) can be replaced by

$$\tfrac{1}{2}\Bigg[\sum_{p=0}^{l-q} (y_{2q+2p}+y_{2q+2p+1}) \prod_{n=p+q+1}^{l} (y^2_{2n}-1) \prod_{n=q+1}^{q+p} (y^2_{2n-1}-1) - \sum_{p=0}^{l-q-1} (y_{2q+1+2p}+y_{2q+2+2p})(y_{2q+1+2p}y_{2l+1}-1) \times \prod_{\substack{n=q+1\\ n\neq p+q+1}}^{l} (y^2_{2n-1}-1)\Bigg] \prod_{j=2q}^{2l} (y_j+y_{j+1})^{-1}. \qquad (2.128)$$

Using these results S_{2k+1} of (2.121) becomes (see Fig. 20)

$$\begin{aligned} S_{2k+1} = &-(y_1y_{2k+1}-1) \sum_{l=0}^{k-1} (y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3}) \\ &\times \Bigg\{ \prod_{\substack{n_1=2\\ n_1\neq l+2}}^{k} (y^2_{2n_1-1}-1)\, \tfrac{1}{2}(y_{2l+3}+y_{2l+4})(y_{2l+4}+y_{2l+5}) \\ &+ \prod_{\substack{n_1=2\\ n_1\neq l+1}}^{k} (y^2_{2n_1-1}-1)\, \tfrac{1}{2}(y_{2l-1}+y_{2l})(y_{2l}+y_{2l+1}) \\ &+ \prod_{n=2}^{l+1} (y^2_{2n-1}-1) \sum_{q=l+3}^{k} (y_{2q}+y_{2q+1})\, \tfrac{1}{2} \\ &\times \Bigg[\sum_{p=0}^{q-l-2} (y_{2l+3+2p}+y_{2l+4+2p}) \prod_{n=l+2}^{l+1+p} (y^2_{2n}-1) \\ &\times \prod_{n=l+3+p}^{k} (y^2_{2n-1}-1) - \sum_{p=0}^{q-l-3} (y_{2l+4+2p}+y_{2l+5+2p}) \\ &\times \prod_{\substack{n=l+3\\ n\neq l+3+p}}^{k} (y^2_{2n-1}-1)(y_{2l+3}y_{2l+5+2p}-1)\Bigg] \\ &+ \prod_{n=l+2}^{k} (y^2_{2n-1}-1) \sum_{q=1}^{l-1} (y_{2q-1}+y_{2q})\, \tfrac{1}{2} \\ &\times \Bigg[\sum_{p=0}^{l-q} (y_{2q+2p}+y_{2q+2p+1}) \prod_{n=2}^{q+p} (y^2_{2n-1}-1) \\ &\times \prod_{n=p+q+1}^{l} (y^2_{2n}-1) - \sum_{p=0}^{l-q-1} (y_{2q+1+2p}+y_{2q+2+2p}) \\ &\times (y_{2q+1+2p}y_{2l+1}-1) \prod_{\substack{n=2\\ n\neq p+q+1}}^{l} (y^2_{2n-1}-1)\Bigg] \\ &+ \prod_{\substack{n=2\\ n\neq l+1,\, l+2}}^{k} (y^2_{2n-1}-1)(y_{2l+1}y_{2l+3}-1) \sum_{n=1}^{2l} y_n \sum_{n=2l+4}^{2k+1} y_n \Bigg\}. \end{aligned} \qquad (2.129)$$

We now consider the term

$$I_{2k+1} = \sum_{l=0}^{k-1} (y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3}) \Bigg\{ -\tfrac{1}{2} \prod_{n=2}^{l+1} (y^2_{2n-1}-1)$$

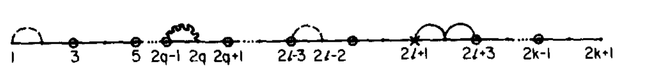

FIG. 20. Graph of one term of (2.129). Dotted loops indicate range of the wavy loop.

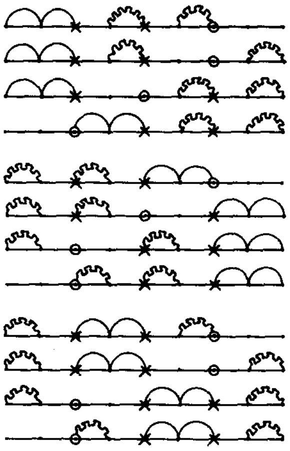

FIG. 21. Graphs associated with I_9. The first eight graphs are multiplied by $-\frac{1}{2}$ and the last four graphs by $+1$.

$$\times \sum_{q=l+3}^{k} (y_{2q}+y_{2q+1}) \sum_{p=0}^{q-l-3} (y_{2l+4+2p}+y_{2l+5+2p})$$
$$\times \prod_{\substack{n=l+3\\ n\neq l+3+p}}^{k} (y_{2n-1}^2-1)(y_{2l+3}y_{2l+5+2p}-1) - \tfrac{1}{2}\prod_{n=l+2}^{k}(y_{2n-1}^2-1)$$
$$\times \sum_{q=1}^{l-1}(y_{2q-1}+y_{2q}) \sum_{p=0}^{l-q-1}(y_{2q+1+2p}+y_{2q+2+2p})$$
$$\times \prod_{\substack{n=2\\ n\neq p+q+1}}^{l} (y_{2n-1}^2-1)(y_{2q+1+2p}y_{2l+1}-1) + \prod_{\substack{n=2\\ n\neq l+1,\, l+2}}^{k} (y_{2n-1}^2-1)$$
$$\times (y_{2l+1}y_{2l+3}-1)\sum_{n=1}^{2l} y_n \sum_{n=2l+4}^{2k+1} y_n \Bigg\} \tag{2.130}$$

which is part of S_{2k+1}. We claim that

$$I_{2k+1}=0, \tag{2.131}$$

where equality is in the generalized sense. We first examine a special case. Consider $k=4$, then there are twelve terms in (2.130) and the graphs of these terms are shown in Fig. 21.

Concerning the terms with the structure $(y_\alpha y_\beta - 1)$ we indicate by "×" the presence of the y_α and y_β terms. From an examination of Fig. 21 it is clear that graphs 5–8 are just a reversed labeling of the first four graphs. Hence we need only consider the first four graphs with weight -1 and the last four graphs. However, it is clear from Fig. 21 that the last four graphs have the same structure as do the first four graphs. Hence they add to give zero, i.e., $I_9=0$.

The general case proceeds along similar lines. Some typical graphs are shown in Fig. 22. As in the $k=4$ case, the terms arising from the second term in (2.130) can be combined with the first term as they are

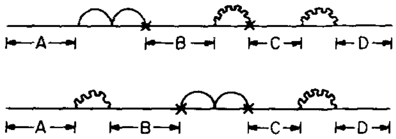

FIG. 22. Typical graphs of I_{2k+1}. The first graph comes from the first set of terms and the second graph from the third set of terms.

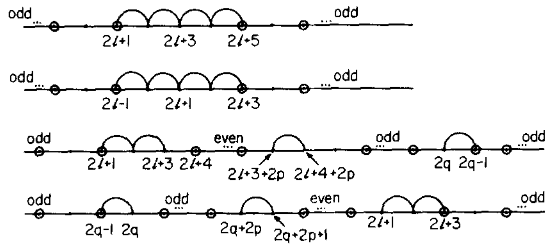

FIG. 23. Graphs of terms appearing in S_{2k+1} as defined by (2.132). The labels "odd" and "even" refer to whether (y_α^2-1) occurs at an odd site or an even site, respectively.

of the same structure. Furthermore, if the distances A, B, C, and D as depicted in Fig. 22 are made equal, then the two graphs cancel. One need only check that the (y_α^2-1) type terms are in the correct place. An examination of (2.130) convinces oneself that they are in the correct places for cancellation. Hence (2.131) follows, and incorporating this result into (2.129) results in our final form for S_{2k+1},

$$S_{2k+1} = -\tfrac{1}{2}(y_1y_{2k+1}-1)\sum_{l=0}^{k-1}(y_{2l+1}+y_{2l+2})(y_{2l+2}+y_{2l+3})$$
$$\times \Bigg\{ \prod_{\substack{n=2\\ n\neq l+2}}^{k}(y_{2n-1}^2-1)(y_{2l+3}+y_{2l+4})(y_{2l+4}+y_{2l+5})$$
$$+ \prod_{\substack{n=2\\ n\neq l+1}}^{k}(y_{2n-1}^2-1)(y_{2l-1}+y_{2l})(y_{2l}+y_{2l+1})$$
$$+ \prod_{n=2}^{l+1}(y_{2n-1}^2-1)\sum_{q=l+3}^{k}(y_{2q}+y_{2q+1})\sum_{p=0}^{q-l-2}(y_{2l+3+2p}+y_{2l+4+2p})$$
$$\times \prod_{n=l+2}^{l+1+p}(y_{2n}^2-1)\prod_{n=l+3+p}^{k}(y_{2n-1}^2-1) + \prod_{n=l+2}^{k}(y_{2n-1}^2-1)$$
$$\times \sum_{q=1}^{l-1}(y_{2q-1}+y_{2q})\sum_{p=0}^{l-q}(y_{2q+2p}+y_{2q+1+2p})$$
$$\times \prod_{n=2}^{p+q}(y_{2n-1}^2-1)\prod_{n=p+q+1}^{l}(y_{2n}^2-1)\Bigg\}. \tag{2.132}$$

In Fig. 23 we display a graph of a typical term from each of the four basic terms in (2.132). In the last two graphs the "even" and "odd" structure of (y_α^2-1) should be noted.

6. *Final form for* $\mathcal{L}_{2k+1}$

Equation (2.132) is the result of summing $\mathcal{L}_{2k+1}(l,m)$ subject to the restriction $l+m=k-1$. We now claim that $\mathcal{L}_{2k+1}$ [(defined by (2.86)] is in fact

$$\mathcal{L}_{2k+1} = -\tfrac{1}{2}(y_1y_{2k+1}-1)\sum_{l=0}^{k-1}(y_{2l+1}+y_{2l+2})\sum_{m=0}^{k-1-l}(y_{2l+2+2m}+y_{2l+3+2m})$$
$$\times \prod_{n=l+2}^{l+m+1}(y_{2n-1}^2-1)\Bigg\{\prod_{\substack{n=2\\ n\neq l+2,\ldots,l+m+2}}^{k}(y_{2n-1}^2-1)(y_{2l+3+2m}$$
$$+y_{2l+4+2m})(y_{2l+4+2m}+y_{2l+5+2m}) + \prod_{\substack{n=2\\ n\neq l+1,\, l+2,\ldots,\, l+m+1}}^{k}$$
$$(y_{2n-1}^2-1)(y_{2l-1}+y_{2l})(y_{2l}+y_{2l+1}) + \prod_{n=2}^{l+1}(y_{2n-1}^2-1)$$

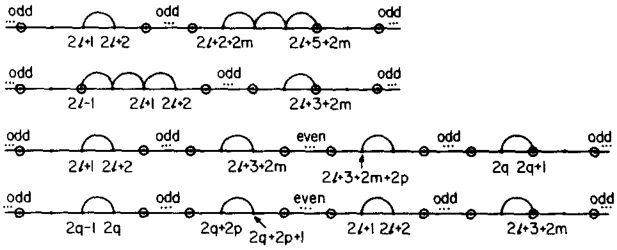

FIG. 24. Graphs of terms appearing in $\mathcal{L}_{2k+1}$ as given by (2.133).

$$\times \sum_{q=l+3}^{k} (y_{2q} + y_{2q+1}) \sum_{p=0}^{q-l-2} (y_{2l+3+2m+2p} + y_{2l+4+2m+2p})$$

$$\times \prod_{n=l+m+2}^{l+m+1+p} (y_{2n}^2 - 1) \prod_{n=l+m+3+p}^{k} (y_{2n-1}^2 - 1) + \prod_{n=l+2+m}^{k} (y_{2n-1}^2 - 1)$$

$$\times \sum_{q=1}^{l-1} (y_{2q-1} + y_{2q}) \sum_{p=0}^{l-q} (y_{2q+2p} + y_{2q+1+2p}) \prod_{n=2}^{q+p} (y_{2n-1}^2 - 1)$$

$$\times \prod_{n=p+q+1}^{l} (y_{2n}^2 - 1) \Bigg\}. \tag{2.133}$$

A graph of a typical term from each of the four basic terms of (2.133) is shown in Fig. 24. Figure 24 should be compared with Fig. 23, the $m=0$ case of Fig. 24.

To demonstrate (2.133) one can proceed in two ways. The first method is to repeat the analysis starting at (2.101) leading to (2.132) where now $l+m$ is fixed to be less than $k-1$. This will result in (2.133). Alternatively one can study special cases and note that the general case (2.133) is obtained from the specific case (2.132) by letting $l \to l+m$ in certain terms containing the index l. These special cases indicate the transition from Fig. 23 to Fig. 24.

7. Proof of (2.38)

Using (2.133) in (2.47) we obtain an integral representation of the left-hand side of (2.38). We now compare this with the integral representation of the right-hand side of (2.38) and demonstrate that the two representations are identical. This will establish (2.38) as an identity which in view of Lemma 2.1 proves Theorem 1.

Consider the right-hand side of (2.38). A graph of a typical term is shown in Fig. 25 (the labeling is first g_{2l+1}, then g_{2m+1}, and finally $L_\nu g_{2(k-m-l)-1}$ where we use Lemma 2.3 for this last term). This graph can be made equivalent to the last graph of Fig. 24 by rearranging the graph in the order 1—2—3—4—5 as indicated in the figure. The factor "2" on the right-hand side comes about since there are two graphs in Fig. 24 to each graph in Fig. 25. The first two graphs are a degenerate form of Fig. 25 graphs.

Thus Theorem 1 is proved.

III. THEOREM 2 AND THE FUNCTION $\psi(t;\nu,\lambda)$

A. Differential equation and the functions $\psi_{2n+1}(t;\nu)$

We define $\psi(t;\nu,\lambda)$ by Eq. (1.7). In terms of the function $G(t;\nu,\lambda)$ the definition of $\psi(t;\nu,\lambda)$ is

$$G(t;\nu,\lambda) = \tanh[\tfrac{1}{2}\psi(t;\nu,\lambda)]. \tag{3.1}$$

From either (1.7) or (3.1) and either (1.3) or (2.7) it follows that $\psi(t;\nu,\lambda)$ satisfies the differential equation

$$\psi'' + \frac{1}{t}\psi' = \tfrac{1}{2}\sinh(2\psi) + \frac{2\nu}{t}\sinh(\psi) \tag{3.2}$$

with

$$\psi(t;\nu,\lambda) \sim 2g_1(t;\nu)\lambda \tag{3.3}$$

as t approaches infinity along the positive t axis.

The λ expansion of the function $G(t;\nu,\lambda)$ [see Eq. (1.5)] induces a corresponding λ expansion for the function $\psi(t;\nu,\lambda)$,

$$\psi(t;\nu,\lambda) = \sum_{n=0}^{\infty} \lambda^{2n+1}\psi_{2n+1}(t;\nu). \tag{3.4}$$

The defining relation (3.1) in conjunction with (1.5) requires that

$$g_1(t;\nu) = \tfrac{1}{2}\psi_1(t;\nu), \tag{3.5a}$$

$$g_3(t;\nu) = \tfrac{1}{2}\psi_3(t;\nu) - \tfrac{1}{3}[\tfrac{1}{2}\psi_1(t;\nu)]^3, \tag{3.5b}$$

$$g_5(t;\nu) = \tfrac{1}{2}\psi_5(t;\nu) + [\tfrac{1}{2}\psi_1(t;\nu)]^2[\tfrac{1}{2}\psi_3(t;\nu)] + \tfrac{2}{15}[\tfrac{1}{2}\psi_1(t;\nu)]^5, \tag{3.5c}$$

etc.

The content of Theorem 2 is the assertion that the functions $\psi_{2n+1}(t;\nu)$ as defined by (3.1)–(3.5) possess the representation (1.9). To prove Theorem 2 we define $\psi(t;\nu,\lambda)$ by (1.8) and (1.9) and demonstrate that either (3.1) or (3.2) is true. We choose to demonstrate (3.1).

If (3.1) is true, then it certainly follows that

$$\frac{\partial G}{\partial\lambda} = \tfrac{1}{2}\operatorname{sech}^2[\tfrac{1}{2}\psi]\frac{\partial\psi}{\partial\lambda}$$

$$= \tfrac{1}{2}[1 - \tanh^2(\tfrac{1}{2}\psi)]\frac{\partial\psi}{\partial\lambda}$$

$$= \tfrac{1}{2}[1 - G^2(t;\nu,\lambda)]\frac{\partial\psi}{\partial\lambda}. \tag{3.6}$$

With the boundary condition

$$G(t;\nu,0) = 1 \tag{3.7}$$

and the assumption that (3.6) is true, it follows that (3.1) is true. Equation (3.6) can be written in the equivalent form

$$\tfrac{1}{2}(2k+1)\psi_{2k+1}(t;\nu)$$

$$= (2k+1)g_{2k+1}(t;\nu) + \sum_{m=0}^{k-1}\tfrac{1}{2}[2(k-m)-1]\psi_{2(k-m)-1}(t;\nu)$$

$$\times \sum_{l=0}^{m} g_{2l+1}(t;\nu)g_{2(m-l)+1}(t;\nu). \tag{3.8}$$

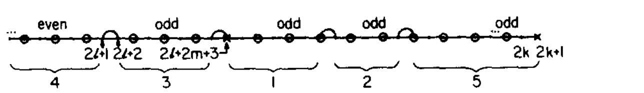

FIG. 25. Graph of a typical term from the right-hand side of (2.38). The numbers beneath the graph represent the ordering to be followed to show equivalence with the graphs of Fig. 24 (in this case the last graph).

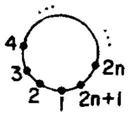

FIG. 26. Quantity (3.11).

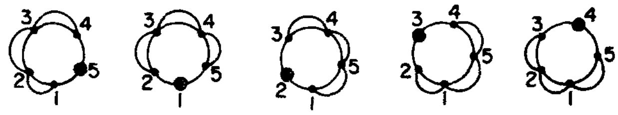

FIG. 28. Quantity $G_5(3)$.

B. Graphs and a lemma

The defining equation (1.9b) can be written in a slightly different form

$$\psi_{2n+1}(t;\nu)=\frac{2}{2n+1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n+1}\times\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\frac{1}{y_j+y_{j+1}}\times\left[\prod_{j=1}^{2n+1}(y_j+1)+\prod_{j=1}^{2n+1}(y_j-1)\right]. \quad (3.9)$$

To prove (3.8) it will be useful to rewrite the term

$$\prod_{j=1}^{2n+1}(y_j+1)+\prod_{j=1}^{2n+1}(y_j-1) \quad (3.10)$$

appearing in (3.9) in a different form. To help visualize the structure of these terms a graphical representation will now be introduced.

Since the factor $(y_{2n+1}+y_1)$ appears in the denominator of the integrand of (3.9), the linear graphs introduced in Sec. II are not the most convenient. We use circular graphs to emphasize the cyclic nature of the integrand in (3.9). Thus the factor

$$\prod_{j=1}^{2n+1}(y_j+y_{j+1})^{-1} \quad (3.11)$$

is represented by a circular graph of $2n+1$ points (see Fig. 26). We adopt the same rules as in Sec. II concerning "loops" and "circles."

Thus the integrand factor

$$(y_2^2-1)(y_5^2-1)(y_3+y_4)\prod_{j=1}^{5}(y_j+y_{j+1})^{-1} \quad (3.12)$$

has the graph shown in Fig. 27. As in Sec. II we omit the term $\Pi_{j=1}^{2n+1}(y_j+y_{j+1})^{-1}$ that multiplies the various factors in (3.9). Thus, for example, when we speak of the graph of the factor

$$(y_2^2-1)(y_5^2-1)(y_3+y_4) \quad (3.13)$$

that appears in an integrand with five variables we always mean (3.12).

Furthermore for the graphs considered in this section we make the additional restrictions:

(i) All graphs have an odd number of points.

(ii) All graphs have an odd number of loops.

(iii) Following any loop there immediately follows another loop or a circle. If a circle follows, then either a loop or a point must follow this circle. If a point follows, then a circle must follow this point.

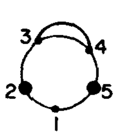

FIG. 27. Quantity (3.12).

(iv) The sign of a graph is defined to be $(-1)^{N_c}$ where N_c is the number of circles appearing in the graph. The integrand associated with the graph carries this sign. As a result of the above rule, $N_c=\frac{1}{2}(K-L)$ where K is the total number of points of the graph and L is the number of loops in the graph.

With these restrictions in mind we make the following definition:

$G_K(L)=$ the sum of all labeled graphs of K points with L loops. (3.14)

As an example the set of graphs $G_5(3)$ is shown in Fig. 28.

We use the word graph and the integrand associated with such a graph interchangeably. With this understanding we now prove

Lemma 3.1:

$$\prod_{j=1}^{2k+1}(y_j+1)+\prod_{j=1}^{2k+1}(y_j-1)=\sum_{j=0}^{k}G_{2k+1}(2j+1). \quad (3.15)$$

Proof: At any site in a graph of $2k+1$ points there are five different configurations at this site (see Fig. 29). We represent each possible configuration at a site by a vector:

$$|LL\rangle=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\quad |L_R\rangle=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\quad |L_l\rangle=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\quad |P\rangle=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\quad |C\rangle=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}. \quad (3.16)$$

Now consider the points j and $j+1$ in a graph. We assume j has the configuration $|\alpha\rangle$ and $j+1$ has the con-

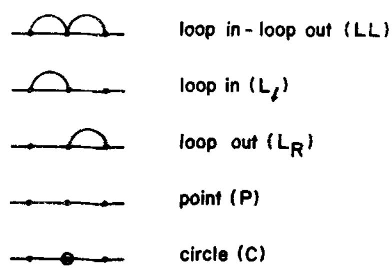

FIG. 29. Five distinct configurations at a site.

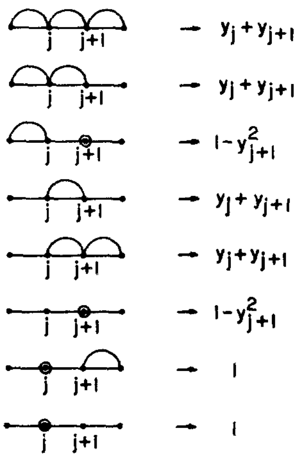

FIG. 30. All possible configurations at sites j and $j+1$ with their respective weights. From this Fig. (3.18) follows.

figuration $|\beta\rangle$ ($\alpha, \beta = LL$, L_R, L_l, C, or P). To this part of the graph we assign in accordance with the above rules an integrand factor. We denote this factor by

$$\langle\alpha|M(j,j+1)|\beta\rangle. \tag{3.17}$$

Using the graphical rules we have (see Fig. 30)

$$M(j,j+1)=\begin{bmatrix} y_j+y_{j+1} & 0 & y_j+y_{j+1} & 0 & 0 \\ y_j+y_{j+1} & 0 & y_j+y_{j+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-y_{j+1}^2 \\ 0 & 0 & 0 & 0 & 1-y_{j+1}^2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{3.18}$$

To avoid double counting the circles we assign the weight one when they occur at site j and the weight $1-y_{j+1}^2$ when they occur at site $j+1$. Then we have

$$\sum_{j=0}^{k} G_{2k+1}(2j+1)=\operatorname{Tr}\left\{\prod_{j=1}^{2k+1} M(j,j+1)\right\}, \tag{3.19}$$

where $y_{2k+2}\equiv y_1$.

If we make the similarity transformation

$$\tilde{M}(j,j+1)=UM(j,j+1)\,U^{-1}, \tag{3.20}$$

where

$$U=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.21}$$

then

$$\tilde{M}(j,j+1)=\begin{bmatrix} y_j+y_{j+1} & y_j+y_{j+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1-y_{j+1}^2) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}. \tag{3.22}$$

Thus (3.19) can be written as

$$\sum_{j=0}^{k} G_{2k+1}(2j+1)=\operatorname{Tr}\left\{\prod_{j=1}^{2k+1} V(j,j+1)\right\} \tag{3.23}$$

with

$$V(j,j+1)=\begin{pmatrix} y_j+y_{j+1} & y_j+y_{j+1} & 0 \\ 0 & 0 & 2(1-y_{j+1}^2) \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{3.24}$$

We write (3.23) as

$$\begin{aligned}\operatorname{Tr}&\left\{\prod_{j=1}^{2k+1} V(j,j+1)\right\}\\ &=\operatorname{Tr}\{[B(1)\,V(1,2)\,B^{-1}(2)][B(2)\,V(2,3)\,B^{-1}(3)]\cdots\\ &\quad\times[B(2k+1)\,V(2k+1,1)\,B^{-1}(1)]\},\end{aligned} \tag{3.25}$$

where we define

$$B(j)=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -y_j+1 \\ \frac{1}{2} & -\frac{1}{2} & -y_j \\ \frac{1}{2} & \frac{1}{2} & -y_j-1 \end{bmatrix}. \tag{3.26}$$

From (3.24) and (3.26) it follows that

$$\begin{aligned}&B(j)\,V(j,j+1)\,B^{-1}(j+1)\\ &=\begin{bmatrix} y_{j+1}+1 & 0 & 0 \\ y_{j+1}^2+\frac{1}{2}y_{j+1}-\frac{1}{2} & 0 & -y_{j+1}^2+\frac{1}{2}y_{j+1}+\frac{1}{2} \\ 0 & 0 & y_{j+1}-1 \end{bmatrix}.\end{aligned} \tag{3.27}$$

Since the second column of the matrix in (3.27) consists of all zeros, the matrix elements $(y_{j+1}^2+\frac{1}{2}y_{j+1}-\frac{1}{2})$ and $(-y_{j+1}^2+\frac{1}{2}y_{j+1}+\frac{1}{2})$ do not affect the value of (3.25). Hence these terms can be set equal to zero when evaluating the trace in (3.25). Doing this we see that (3.27), (3.25), and (3.23) imply that (3.15) is true.

C. Proof of (3.8)

If we use Lemma 3.1 and let $n\to k-m-1$ in (3.9) we have

$$\begin{aligned}\psi_{2(k-m)-1}(t;\nu)&=\frac{2}{2k-2m-1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k-2m-1}\left[\prod_{j=1}^{2k-2m-1}\frac{\exp(-ty_j)}{(y_j-1)^{1/2}}\right.\\ &\left.\quad\left(\frac{y_j-1}{y_j+1}\right)^{\nu}(y_j+y_{j+1})^{-1}\right]\left[\sum_{j=0}^{k} G_{2k+1}(2j+1)\right].\end{aligned} \tag{3.28}$$

Each term in $G_{2k+1}(2j+1)$ contains at least one loop. Since the first term in square brackets in (3.28) is invariant under cyclic permutations of the integration variable labels, each term in $G_{2k+1}(2j+1)$ may be cyclically permuted (by cyclically permuting the labels on the graph) so that one of the loops occurring in $G_{2k+1}(2j+1)$ connects the points "1" and "$2k-2m-1$." We denote this permuted version of (3.28) by placing a prime on $G_{2k+1}(2j+1)$.

Now consider the right-hand side of (3.8). If we use the definition of the functions g_{2l+1} and $g_{2(m-l)+1}$ (see

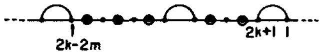

FIG. 31. Typical graph occurring in argument following (3.32).

Theorem 1), the permuted version of (3.28) just described, and the identity

$$\sum_{m=0}^{k-1}\sum_{j=0}^{k-1-m}=\sum_{j=0}^{k-1}\sum_{m=0}^{k-1-j}, \tag{3.29}$$

we have upon letting $j\to j+1$ in the first sum of the right-hand side of (3.29)

$$\begin{aligned}\sum_{m=0}^{k-1}\sum_{l=0}^{m}\frac{2(k-m)-1}{2}\,\psi_{2(k-m)-1}g_{2l+1}g_{2(m-l)+1}
&=\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}\left[\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right.\\
&\quad\left.\times(y_j+y_{j+1})^{-1}\right]\sum_{j=1}^{k}\left\{\sum_{m=0}^{k-j}(-1)^m\sum_{l=0}^{m}\frac{G'_{2k-2m-1}(2j-1)}{y_{2k-2m-1}+y_1}\right.\\
&\quad\times(y_{2k+1}+y_1)(y_{2k-2m-1}+y_{2k-2m})(y_{2k-2m+2}+y_{2k-2m+2l+1})\\
&\quad\left.\times\prod_{j=k-m}^{k-m+l-1}(y_{2j+1}^2-1)\prod_{j=k-m+l+1}^{k}(y_{2j}^2-1)\right\}.\end{aligned} \tag{3.30}$$

The primed graphs in $G'_{2k-2m-1}(2j-1)$ all contain a factor $(y_{2k-2m-1}+y_1)$ which is canceled in (3.30) by the same term that appears in the denominator. Hence to give a graphical representation of the term [valid when used in (3.30)]

$$\frac{G'_{2k-2m-1}(2j-1)}{y_{2k-2m-1}+y_1} \tag{3.31}$$

we imagine starting with the sum of graphs of $2k-2m-1$ points and $2j-1$ loops. Each graph's labels are cyclically permuted so that a term $(y_1+y_{2k-2m-1})$ appears (that is, a loop from "1" to "$2k-2m-1$"). When this loop is removed from each graph the result is (3.31).

We now claim that

$$\begin{aligned}G_{2k+1}(2j+1)
&=\sum_{m=0}^{k-j}(-1)^m\sum_{l=0}^{m}\left[\frac{G'_{2k-2m-1}(2j-1)}{y_{2k-2m-1}+y_1}(y_{2k+1}+y_1)\right.\\
&\quad\times(y_{2k-2m-1}+y_{2k-2m})(y_{2k-2m+2l}+y_{2k-2m+2l+1})\\
&\quad\left.\times\prod_{j=k-m}^{k-m+l-1}(y_{2j+1}^2-1)\prod_{j=k-m+l+1}^{k}(y_{2j}^2-1)\right],\end{aligned} \tag{3.32}$$

where equality in (3.32) is used in the sense that the left-hand side and the right-hand side lead to identical results when used in (3.30).

Consider the set of graphs $G_{2k+1}(2j+1)$. We cyclically permute the labels of the graphs such that a loop connects the points "1" and "$2k+1$." Imagine proceeding from $2k+1$ until two loops are encountered. This second loop must start at an even label which we denote by $2k-2m$ (see Fig. 31). Clearly the smallest m can be is zero which corresponds to the three loops together (see Fig. 32). The largest m can be is $k-j$ since the graph must contain $2j+1$ loops in all. Thus the set of permuted graphs with the points $2k-2m+1$ to $2k$ omitted with just one loop between these points and $2j+1$ loops in all is

$$\frac{G'_{2k-2m-1}(2j-1)}{y_{2k-2m-1}+y_1}(y_{2k+1}+y_1)(y_{2k-2m-1}+y_{2k-2m}). \tag{3.33}$$

The remaining terms in (3.32), i.e.,

$$\sum_{l=0}^{m}\prod_{j=k-m}^{k-m+l-1}(y_{2j+1}^2-1)(y_{2k-2m+2l}+y_{2k-2m+2l+1})\prod_{j=k-m+l+1}^{k}(y_{2j}^2-1)$$

are just all ways of putting in the final loop. The factor $(-1)^m$ gives the correct sign for the m inserted circles between $2k-2m$ and $2k+1$. Summing this from $m=0$ to $k-j$ gives all possible (permuted) graphs in $G_{2k+1}(2j+1)$. Hence (3.32) is true.

Using (3.32) in (3.30) we have that (3.8) can be written as

$$\begin{aligned}(2k+1)g_{2k+1}+\sum_{m=0}^{k-1}\frac{2(k-m)-1}{2}\,\psi_{2(k-m)-1}\sum_{l=0}^{m}g_{2l+1}g_{2(m-l)+1}
&=\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}\left[\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\right.\\
&\quad\left.\times\left(\frac{y_j-1}{y_j+1}\right)^\nu(y_j+y_{j+1})^{-1}\right]\sum_{j=0}^{k}G_{2k+1}(2j+1),\end{aligned} \tag{3.34}$$

where we identified $(2k+1)g_{2k+1}$ as $G_{2k+1}(1)$. Using Lemma 3.1 we see that (3.34) is just $\frac{1}{2}(2k+1)\psi_{2k+1}$. Thus we have proved (3.8) and hence Theorem 2.

From Theorem 2 we can prove that underlying the nonlinear differential equation (3.2) and hence the Painlevé equation (1.1) with the restriction (1.2) there is an associated *linear* integral equation.

Consider the integral operator $\mathcal{K}$ defined on $\mathcal{L}^2(1,\infty,d\sigma_\pm)$ by

$$(\mathcal{K}f)(x)=\int_1^\infty d\sigma_\pm(y)\exp[-\theta(x+y)](x+y)^{-1}f(y), \tag{3.35a}$$

where the measure $d\sigma_\pm$ is

$$d\sigma_\pm=d\sigma_\pm(y)=\left(\frac{y-1}{y+1}\right)^{\nu\pm1/2}dy. \tag{3.35b}$$

The scalar product is

$$(g,f)_\pm=\int_1^\infty d\sigma_\pm(y)\overline{g(y)}f(y). \tag{3.36}$$

The operator $\mathcal{K}$ is Hilbert–Schmidt for all real $\theta>0$. As $\theta\to0$ the Hilbert–Schmidt norm of $\mathcal{K}$ approaches infinity (the approach is $\sim\ln\theta^{-1}$). We denote by $\lambda_j^\pm(\theta,\nu)$

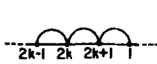

FIG. 32. Case $m=0$ in argument following (3.32).

the eigenvalues and by $\phi_j^{\pm}(x;\theta,\nu)$ the orthonormal eigenfunctions of $\mathcal{K}$. For brevity we sometimes write $\lambda_j^{\pm}$ for $\lambda_j^{\pm}(\theta,\nu)$. Thus we have

$$(\mathcal{K}\phi_j^{\pm})(x)=\lambda_j^{\pm}(\theta,\nu)\,\phi_j^{\pm}(x;\theta,\nu), \tag{3.37}$$

where + (−) refers to the measure $d\sigma_+$ $(d\sigma_-)$.

We now prove

Corollary: The Painlevé transcendents $\eta(\theta;\nu,\lambda)$ of Theorem 1 possess the representation

$$\eta(\theta;\nu,\lambda)=\prod_{j=1}^{\infty}\left(\frac{1-\lambda_j^+\lambda}{1+\lambda_j^+\lambda}\right)^{a_j^+}\prod_{j=1}^{\infty}\left(\frac{1-\lambda_j^-\lambda}{1+\lambda_j^-\lambda}\right)^{a_j^-}, \tag{3.38}$$

where

$$a_j^{\pm}=a_j^{\pm}(\theta,\nu)=(\lambda_j^{\pm})^{-1}\int_\theta^\infty d\zeta\,\Big|\int_1^\infty d\sigma_{\pm}(y)\times\exp(-\zeta y)\,\phi_j^{\pm}(y;\theta,\nu)\Big|^2. \tag{3.39}$$

Proof: Using

$$(y_{2n+1}+y_1)^{-1}=\int_0^\infty d\zeta\,\exp[-\zeta(y_1+y_{2n+1})]$$

in the representation (1.9b) of the functions $\psi_{2n+1}(t;\nu)$ we see we can write $\psi_{2n+1}(t;\nu)$ as

$$\psi_{2n+1}(t;\nu)=\frac{2}{2n+1}\int_0^\infty d\zeta\,[(e,\mathcal{K}^{2n}e)_+ +(e,\mathcal{K}^{2n}e)_-], \tag{3.40}$$

where

$$e(y)=\exp[-(\zeta+\theta)y]. \tag{3.41}$$

Using Mercer's theorem we can write (3.40) as

$$\psi_{2n+1}(t;\nu)=\frac{2}{2n+1}\int_0^\infty d\zeta\left\{\sum_{j=1}^\infty(\lambda_j^+)^{2n}|(e,\phi_j^+)_+|^2+\sum_{j=1}^\infty(\lambda_j^-)^{2n}|(e,\phi_j^-)_-|^2\right\}. \tag{3.42}$$

Recalling the elementary relation (valid for $|x|<1$)

$$\sum_{n=1}^\infty\frac{2}{2n+1}x^{2n+1}=\ln\left(\frac{1+x}{1-x}\right),$$

we can conclude from (3.42) and (1.8) that for $|\lambda|<\min[(\lambda_1^+)^{-1},(\lambda_1^-)^{-1}]$ (where $\lambda_1^+\geq\lambda_2^+\geq\cdots$ and $\lambda_1^-\geq\lambda_2^-\geq\cdots$) $\psi(t;\nu,\lambda)$ has the representation

$$\psi(t;\nu,\lambda)=\int_0^\infty d\zeta\left\{\sum_{j=1}^\infty(\lambda_j^+)^{-1}|(e,\phi_j^+)_+|^2\ln\left(\frac{1+\lambda_j^+\lambda}{1-\lambda_j^+\lambda}\right)+\sum_{j=1}^\infty(\lambda_j^-)^{-1}|(e,\phi_j^-)_-|^2\ln\left(\frac{1+\lambda_j^-\lambda}{1-\lambda_j^-\lambda}\right)\right\}. \tag{3.43}$$

Defining $a_j^{\pm}$ by (3.39) and recalling (1.7) we conclude that the Painlevé transcendent $\eta(\theta;\nu,\lambda)$ is given by (3.38).

From (3.38) we see that the closest singularity in the complex λ plane occurs at $\min[(\lambda_1^+)^{-1},(\lambda_1^-)^{-1}]$. This gives the radius of convergence of (1.7) in the complex λ plane. The restriction $|\lambda|<\min[(\lambda_1^+)^{-1},(\lambda_1^-)^{-1}]$ can be lifted in (3.38). From the theory of analytic continuation we know that, for fixed θ and ν, $\eta(\theta;\nu,\lambda)$ is given by the right-hand side of (3.38) whenever the infinite products converge. A necessary condition that (3.38) converge in the complex λ plane is $\lambda\neq\pm(\lambda_j^+)^{-1}$ and $\lambda\neq\pm(\lambda_j^-)^{-1}$ for all j. We conjecture this is also sufficient.

It is an open problem to compute the quantities $\lambda_j^{\pm}$ and $a_j^{\pm}$ appearing in (3.38).

IV. THEOREM 3 AND COROLLARIES

A. Formal small-t expansion

A formal small-t expansion of the differential equation (3.2) is

$$\psi(t)\sim-\sigma\ln t-\ln B+\sum_{j=1}^\infty\sum_{k=1}^{j+1}a_{j,k}t^{j-\sigma(j+2-2k)}. \tag{4.1}$$

The coefficients $a_{j,k}$ are determined from (3.2) by equating like powers of t and are unique functions of σ and B (and ν). The requirement that (4.1) be asymptotic as $t\to0$ requires that

$$-1<\mathrm{Re}\,\sigma<1, \tag{4.2}$$

but otherwise the coefficients σ and B are arbitrary. If we define

$$w(t)=\exp[-\psi(t)], \tag{4.3}$$

then

$$w(t)\sim Bt^\sigma\left\{1+\sum_{j=1}^\infty\sum_{k=1}^{j+1}b_{j,k}t^{j-\sigma(j+2-2k)}\right\} \tag{4.4}$$

is a formal small-t expansion of (1.3) where we again assume (4.2). The coefficients $b_{j,k}$ can be determined from either (4.1) and (4.3) (assuming $a_{j,k}$ are known) or directly from the differential equation (1.3). The first few coefficients are

$$\begin{aligned}
b_{1,1}&=-\nu B^{-1}(1-\sigma)^{-2},\\
b_{1,2}&=B\nu(1+\sigma)^{-2},\\
b_{2,1}&=\tfrac14\nu^2B^{-2}(1-\sigma)^{-4}-\tfrac1{16}B^{-2}(1-\sigma)^{-2},\\
b_{2,2}&=-\nu^2(1+\sigma)^{-1}(1-\sigma)^{-2},\\
b_{2,3}&=\tfrac1{16}B^2(1+\sigma)^{-2}+\tfrac34\nu^2B^2(1+\sigma)^{-4},
\end{aligned} \tag{4.5}$$

etc.

Computation of the coefficients of the terms $t^{3-3\sigma}$ and $t^{4-4\sigma}$ ($b_{3,1}$ and $b_{4,1}$, respectively) in the expansion (4.4) shows that these terms are zero. This is a general result, i.e.,

$$b_{n,1}=0,\quad n=3,4,5,\cdots. \tag{4.6}$$

To prove (4.6) we can proceed by induction. Since the argument is straightforward we omit the proof. Thus for $n\geq3$ there are no terms of the form $t^{n-n\sigma}$ in (4.4).

When $\sigma=0$ (4.4) becomes a formal power series expansion in the variable t about the point $t=0$. This formal power series can be shown to converge. The result that there exists a one-parameter family of solutions to (1.3) such that the point $t=0$ is an analytic point is known.[1,2,5] Furthermore when $t=0$ ($\theta=0$) is an analytic point, the solution to (1.3) is known to be a meromorphic function.[1,2,5]

B. $\psi_{2n+1}(t;\nu)$ and $\psi(t;\nu,\lambda)$ as $t\to 0$

We define for $n\geq 2$

$$\psi_n(t;\nu)=\frac{2}{n}\int_1^\infty dy_1\cdots\int_1^\infty dy_n\prod_{j=1}^n\frac{\exp(-ty_j)}{y_j+y_{j+1}}\times\left[\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}+\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu+1/2}\right] \quad (4.7)$$

with $y_{n+1}\equiv y_1$ (this merely defines ψ_n for even integers and coincides with Theorem 2 for odd integers).

Lemma 4.1: As $t\to 0$ along the positive real axis

$$\psi_n(t;\nu)=\sigma_n\ln\left(\frac{1}{t}\right)+B_n+o(1) \quad (4.8)$$

where

$$\sigma_n=\frac{4}{n}\int_0^1 dx_1\cdots\int_0^1 dx_n\prod_{j=1}^n(x_j+x_{j+1})^{-1}\delta\left(1-\sum_{j=1}^n x_j\right), \quad (4.9)$$

$x_{n+1}\equiv x_1$, and

$$B_n=B_n^{(1)\prime}+B_n^{(1)\prime\prime}, \quad (4.10)$$

with

$$B_n^{(1)\prime}=\frac{4}{n}\int_0^1 dx_1\cdots\int_0^1 dx_n\prod_{j=1}^n(x_j+x_{j+1})^{-1}\times\ln x_1\,\delta\left(1-\sum_{j=1}^n x_j\right) \quad (4.11)$$

and

$$B_n^{(1)\prime\prime}=\frac{2}{n}\lim_{t\to 0}\left\{\int_1^\infty dy_1\cdots\int_1^\infty dy_n\prod_{j=1}^n\exp(-ty_j)(y_j+y_{j+1})^{-1}\times\left[\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu+1/2}+\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}\right]-2\int_0^\infty dy_1\cdots\int_0^\infty dy_n\prod_{n=1}^n\exp(-ty_j)(y_j+y_{j+1})^{-1}\times[1-\exp(-y_1)]\right\}, \quad (4.12)$$

with $\delta(x)$ denoting the Dirac delta function.

Proof: Let $F(y)$ be such that $F(y)/y$ is bounded for all $y>0$ and $F(y)\to\infty$. Define

$$\Theta=\Theta(y_1,y_2,\ldots,y_n)=\prod_{j=1}^n\theta(y_j-1), \quad (4.13)$$

where

$$\theta(x)=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x\leq 0.\end{cases} \quad (4.14)$$

We write

$$\psi_n=\frac{2}{n}\int_0^\infty dy_1\cdots\int_0^\infty dy_n\prod_{j=1}^n\exp(-ty_j)(y_j+y_{j+1})^{-1}\times\left[\Theta(y_1,\ldots,y_n)\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu+1/2}+\Theta(y_1,\ldots,y_n)\prod_{j=1}^n\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}-2F(y_1)\right]+I_n, \quad (4.15)$$

where

$$I_n=\frac{4}{n}\int_0^\infty dy_1\cdots\int_0^\infty dy_nF(y_1)\prod_{j=1}^n\exp(-ty_j)(y_j+y_{j+1})^{-1}. \quad (4.16)$$

The limit $t\to 0$ exists for the first term in (4.15) (that is for the quantity ψ_n-I_n).

We choose

$$F(y)=1-\exp(-y) \quad (4.17)$$

which clearly satisfies the above two requirements of $F(y)$. Thus $B_n^{(1)\prime\prime}$ is just the $t\to 0$ limit of ψ_n-I_n with the choice (4.17) for $F(y)$.

We make the change of variables

$$\rho=\sum_{j=1}^n x_j,\quad x_j=\rho^{-1}y_j,\quad j=1,2,\ldots,n, \quad (4.18)$$

in (4.16) with the choice (4.17).

Then

$$I_n=\int_0^\infty\frac{d\rho}{\rho}\exp(-t\rho)\int_0^1 dx_1\cdots\int_0^1 dx_n\,\delta\left(1-\sum_{j=1}^n x_j\right)\times[1-\exp(-\rho x_1)]\prod_{j=1}^n(x_j+x_{j+1})^{-1}. \quad (4.19)$$

If we make use of the identity

$$\ln\left(\frac{x}{y}\right)=\int_0^\infty\frac{d\xi}{\xi}[\exp(-\xi y)-\exp(-\xi x)], \quad (4.20)$$

then I_n becomes

$$\begin{aligned}I_n&=\int_0^1 dx_1\cdots\int_0^1 dx_n\delta\left(1-\sum_{j=1}^n x_j\right)\times\prod_{j=1}^n(x_j+x_{j+1})^{-1}\ln\left(\frac{t+x_1}{t}\right)\\&=\ln\left(\frac{1}{t}\right)\int_0^1 dx_1\cdots\int_0^1 dx_n\delta\left(1-\sum_{j=1}^n x_j\right)\prod_{j=1}^n(x_j+x_{j+1})^{-1}\\&\quad+\int_0^1 dx_1\cdots\int_0^1 dx_n\delta\left(1-\sum_{j=1}^n x_j\right)\prod_{j=1}^n(x_j+x_{j+1})^{-1}\ln x_1\\&\quad+o(1)\quad(\text{as } t\to 0).\end{aligned} \quad (4.21)$$

This proves the Lemma.

From (4.8) and the fact that

$$\psi(t;\nu,\lambda)=\sum_{n=0}^\infty\lambda^{2n+1}\psi_{2n+1}(t;\nu), \quad (4.22)$$

we conclude for $|\lambda|<1/\pi$

$$\psi(t;\nu,\lambda)=+\sigma\ln t^{-1}-\ln B+o(1) \quad (4.23)$$

as $t\to 0^+$ where

$$\sigma=\sum_{n=0}^\infty\lambda^{2n+1}\sigma_{2n+1} \quad (4.24)$$

and

$$-\ln B=\sum_{n=0}^\infty\lambda^{2n+1}B_{2n+1}, \quad (4.25)$$

where σ_{2n+1} and B_{2n+1} are given by Lemma 4.1. For the steps (4.23)–(4.25) to be completely rigorous we must ensure that the error estimate in (4.8) remains $o(1)$

when summed over n in (4.22). We do not present a rigorous proof of this point. Heuristically, if one sums the *leading* term in the $o(1)$ term in (4.8), then the result is still $o(1)$. Also the function $B(\sigma, \nu=0)$ was computed numerically by a procedure independent of the steps (4.23)–(4.25) and to within numerical accuracy (five to six decimal places) the result agrees with that given by (1.12).

C. Computation of σ_n

Lemma 4.2: If we denote by σ_n the quantity defined in (4.9) then

$$\sigma_n = (2/n)\, \pi^{n-2} B(\tfrac{1}{2}, n/2), \tag{4.26}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function.

Proof: Consider the integral

$$J_n = \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \prod_{j=1}^{n-1} (x_j + x_{j+1})^{-1} \exp(-x_1 - x_n). \tag{4.27}$$

Let

$$\lambda = \sum_{j=1}^{n} x_j, \quad \alpha_j = \frac{x_j}{\lambda}, \quad j = 1, 2, \ldots, n-1, \tag{4.28}$$

then the Jacobian is

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \lambda)} = \lambda^{n-1}. \tag{4.29}$$

Since there are $(n-1)$ factors in the denominator of J_n, we get

$$\begin{aligned} J_n &= \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \cdots \int_0^{1-\alpha_1-\cdots-\alpha_{n-2}} d\alpha_{n-1} \int_0^\infty d\lambda \\ &\quad \times \exp\left[-\lambda\alpha_1 - \lambda\left(1 - \sum_{j=1}^{n-1}\alpha_j\right)\right] \\ &\quad \times \left\{\prod_{j=1}^{n-2}(\alpha_j + \alpha_{j+1})\left[\alpha_{n-1} + \left(1 - \sum_{j=1}^{n-1}\alpha_j\right)\right]\right\}^{-1} \\ &= \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \cdots \int_0^1 d\alpha_n \delta\left(1 - \sum_{j=1}^{n}\alpha_j\right) \prod_{j=1}^{n}(\alpha_j + \alpha_{j+1})^{-1}, \end{aligned} \tag{4.30}$$

where $\alpha_{n+1} \equiv \alpha_1$. Hence in view of (4.9) we have shown that

$$\sigma_n = (4/n) J_n. \tag{4.31}$$

To evaluate (4.27) we use the method of Mellin transforms. If we define

$$F(\zeta) = \int_0^\infty x^{-\zeta} f(x)\, dx, \tag{4.32}$$

then for f and $g \in L^2$ we have the Mellin convolution formula

$$\int_0^\infty f(x) g(x)\, dx = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(\zeta) G(1-\zeta)\, d\zeta. \tag{4.33}$$

Now the Mellin transform of $(x+y)^{-1}$ is

$$\int_0^\infty x^{-\zeta} \frac{1}{x+y}\, dx = \frac{\pi}{\sin\pi\zeta} y^{-\zeta}, \tag{4.34}$$

where $0 < \mathrm{Re}\,\zeta < 1$ while that of $\exp(-x)$ is of course

$$\int_0^\infty x^{-\zeta} \exp(-x)\, dx = \Gamma(1-\zeta). \tag{4.35}$$

Therefore,

$$\begin{aligned} J_n &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\zeta \left(\frac{\pi}{\sin\pi\zeta}\right)^{n-1} \Gamma(1-\zeta)\Gamma(\zeta) \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\zeta \left(\frac{\pi}{\sin\pi\zeta}\right)^{n} \\ &= \tfrac{1}{2}\pi^{n-1} \int_{-\infty}^{\infty} d\xi \frac{1}{(\cosh\pi\xi)^n} = \tfrac{1}{2}\pi^{n-2} B(\tfrac{1}{2}, n/2), \end{aligned} \tag{4.36}$$

Therefore, (4.31) and (4.36) prove the Lemma.

The result (1.11) of Theorem 3 now follows from Lemma 4.2. We note that for n odd, (4.26) can be written as

$$\sigma_{2n+1} = \frac{1}{2n+1} \frac{\pi^{2n}}{2^{n-1}} \frac{(2n-1)!!}{n!} \tag{4.37}$$

D. Relating B_n to integral equations

From (4.11) it follows that

$$\begin{aligned} B_n^{(1)\prime} &= \frac{4}{n} \int_0^\infty d\lambda \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta\left(1 - \sum_{j=1}^{n} x_j\right) \\ &\quad \times \ln x_1 \prod_{j=1}^{n-1} (x_j + x_{j+1})^{-1} \exp[-\lambda(x_1 + x_n)]. \end{aligned} \tag{4.38}$$

Reversing the steps that went from (4.27) to (4.30) we see that (4.38) can be written as

$$\begin{aligned} B_n^{(1)\prime} &= \frac{4}{n} \int_0^\infty d\lambda \int_0^\infty dy_1 \cdots \int_0^\infty dy_n\, \delta(\lambda - \Sigma y_j) \\ &\quad \times \ln(y_1 \lambda^{-1}) \prod_{j=1}^{n-1} (y_j + y_{j+1})^{-1} \exp(-y_1 - y_n) \\ &= \frac{4}{n} \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \ln\left(\frac{y_1}{y_1 + \cdots + y_n}\right) \\ &\quad \times \prod_{j=1}^{n-1} (y_j + y_{j+1})^{-1} \exp(-y_1 - y_n). \end{aligned} \tag{4.39}$$

Using identity (4.20) for the logarithm term in (4.39) we conclude

$$\begin{aligned} B_n^{(1)\prime} &= \frac{4}{n} \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \int_0^\infty \frac{d\xi}{\xi} \\ &\quad \times \{\exp[-\xi(y_1 + \cdots + y_n)] - \exp(-\xi y_1)\} \\ &\quad \times \prod_{j=1}^{n-1} (y_j + y_{j+1})^{-1} \exp(-y_1 - y_n). \end{aligned} \tag{4.40}$$

We now wish to split the above integral into two parts. However as it stands, the integrals taken separately are divergent. Thus we write

$$B_n^{(1)\prime} = \lim_{\epsilon \to 0} [B_n^{(1)}(\epsilon) + B_n^{(2)}(\epsilon)], \tag{4.41}$$

where

$$\begin{aligned} B_n^{(1)}(\epsilon) &= -\frac{4}{n} \int_\epsilon^{1/\epsilon} \frac{d\xi}{\xi} \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \\ &\quad \times \exp[-(1+\xi)y_1] \prod_{j=1}^{n-1} (y_j + y_{j+1})^{-1} \exp(-y_n) \end{aligned} \tag{4.42}$$

and

$$B_n^{(2)}(\epsilon)=\frac{4}{n}\int_\epsilon^\infty\frac{d\xi}{\xi}\int_0^\infty dy_1\cdots\int_0^\infty dy_n \times\exp(-y_1)\prod_{j=1}^{n-1}\exp(-\xi y_j)(y_j+y_{j+1})^{-1} \times\exp[-y_n(1+\xi)]. \quad (4.43)$$

We let $x_j=\xi y_j$ in this last expression (and then followed by $\zeta=1/\xi$) to obtain

$$B_n^{(2)}(\epsilon)=\frac{4}{n}\int_0^{1/\epsilon}d\zeta\int_0^\infty dx_1\cdots\int_0^\infty dx_n \times\exp(-\zeta x_1)\prod_{j=1}^{n-1}\frac{\exp(-x_j)}{x_j+x_{j+1}}\exp[-(1+\zeta)x_n]. \quad (4.44)$$

Since we are interested only in the $t\to 0$ limit of the integrals occuring in (4.12), we may write $B_n^{(1)\prime\prime}$ as

$$B_n^{(1)\prime\prime}=\frac{2}{n}\lim_{t\to 0}\left\{\int_1^\infty dy_1\cdots\int_1^\infty dy_n \times\exp[-t(y_1+y_n)]\prod_{j=1}^{n}(y_j+y_{j+1})^{-1}\left[\prod_{j=1}^{n}\left(\frac{y_j-1}{y_j+1}\right)^{\nu+1/2} + \prod_{j=1}^{n}\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}\right]-2\int_0^\infty dy_1\cdots\int_0^\infty dy_n \times\exp[-t(y_1+y_n)]\prod_{j=1}^{n}(y_j+y_{j+1})^{-1}[1-\exp(-y_1)]\right\}. \quad (4.45)$$

That is, we do not change the value of (4.12) if we set $t=0$ in $\exp(-ty_2),\ldots,\exp(-ty_{n-1})$ and leave only the factor $\exp[-t(y_1+y_n)]$. As we did for $B_n^{(1)\prime}$, we break $B_n^{(1)\prime\prime}$ into a sum of terms. As (4.45) stands, the individual integrals are divergent. First we use

$$\frac{\exp[-t(y_1+y_n)]}{y_1+y_n}=\int_t^\infty d\xi\exp[-\xi(y_1+y_n)] \quad (4.46)$$

in (4.45) and then write (also let $t\to\epsilon$)

$$B_n^{(1)\prime\prime}=\lim_{\epsilon\to 0}[B_n^{(3)}+B_n^{(4)}(\epsilon)+B_n^{(5)}(\epsilon)], \quad (4.47)$$

where

$$B_n^{(3)}(\epsilon)=-\frac{4}{n}\int_\epsilon^{\epsilon^{-1}}d\xi\int_0^\infty dy_1\cdots\int_0^\infty dy_n[1-\exp(-y_1)] \times\exp[-\xi(y_1+y_n)]\prod_{j=1}^{n-1}(y_j+y_{j+1})^{-1}$$
$$=-\frac{4}{n}\int_\epsilon^{\epsilon^{-1}}\frac{d\zeta}{\zeta}\int_0^\infty dx_1\cdots\int_0^\infty dx_n \times\{\exp(-x_1)-\exp[-(1+\zeta)x_1]\} \times\prod_{j=1}^{n-1}(x_j+x_{j+1})^{-1}\exp(-x_n), \quad (4.48)$$

$$B_n^{(4)}(\epsilon)=\frac{2}{n}\int_\epsilon^\infty d\xi\int_1^\infty dy_1\cdots\int_1^\infty dy_n \times\exp[-\xi(y_1+y_n)]\prod_{j=1}^{n-1}(y_j+y_{j+1})^{-1}\prod_{j=1}^{n}\left(\frac{y_j-1}{y_j+1}\right)^{\nu-1/2}, \quad (4.49)$$

and $B_n^{(5)}(\epsilon)$ is just $B_n^{(4)}(\epsilon)$ with $\nu-\frac{1}{2}$ replaced by $\nu+\frac{1}{2}$ in (4.49).

Comparing (4.42) and (4.48) we see that

$$B_n^{(1)}(\epsilon)+B_n^{(3)}(\epsilon)=-\frac{4}{n}\int_\epsilon^{\epsilon^{-1}}\frac{d\xi}{\xi}\int_0^\infty dx_1\cdots\int_0^\infty dx_n \times\exp(-x_1)\prod_{j=1}^{n-1}(x_j+x_{j+1})^{-1}\exp(-x_n). \quad (4.50)$$

The ξ integration is now decoupled from the x_j variables so that (4.50) is simply

$$B_n^{(1)}(\epsilon)+B_n^{(3)}(\epsilon)=-2\sigma_n\ln(1/\epsilon), \quad (4.51)$$

where we used the results of Lemmas 4.1 and 4.2.

The quantities $B_n^{(2)}(\epsilon)$, $B_n^{(4)}(\epsilon)$, and $B_n^{(5)}(\epsilon)$ remain to be computed. Equations (4.44) and (4.49) are in the form of an iterated kernel. Therefore, we now examine the integral equations associated with these kernels.

E. Integral equations

Lemma 4.3:

$$\int_1^\infty dy\left(\frac{y-1}{y+1}\right)^\nu\frac{\phi_{p,\nu}(y)}{x+y}=\lambda_p\phi_{p,\nu}(x) \quad (4.52)$$

with

$$\lambda_p=\pi\,\mathrm{sech}\,\pi p,\quad 0\le p<\infty, \quad (4.53)$$

$$\phi_{p,\nu}(x)=C_{p,\nu}F(\tfrac{1}{2}+ip,\tfrac{1}{2}-ip;1+\nu;\tfrac{1}{2}-\tfrac{1}{2}x), \quad (4.54)$$

where $F(a,b;c;x)$ is the hypergeometric function and

$$C_{p,\nu}=[\pi^{-1}\Gamma^{-2}(\nu+1)p\sinh\pi p\,\Gamma(\tfrac{1}{2}+\nu+ip)\Gamma(\tfrac{1}{2}+\nu-ip)]^{1/2}. \quad (4.55)$$

Furthermore the $\phi_{p,\nu}(x)$ are orthogonal, i.e.,

$$\int_1^\infty dy\left(\frac{y-1}{y+1}\right)^\nu\phi_{p,\nu}(y)\phi_{p',\nu}(y)=\delta(p-p'), \quad (4.56)$$

where $\delta(x)$ is the Dirac delta function.

The functions $\phi_{p,\nu}(x)$ can alternatively be expressed in terms of Legendre functions[6]

$$\phi_{p,\nu}(x)=C_{p,\nu}\Gamma(1+\nu)\left(\frac{x+1}{x-1}\right)^{\nu/2}P^{-\nu}_{-1/2+ip}(x). \quad (4.57)$$

Lemma 4.3 is a special case of the inversion formulas for the generalized Mehler–Fock transform.[7,8]

To compute $B_n^{(2)}(\epsilon)$ we need

Lemma 4.4:

$$\int_0^\infty\frac{\exp(-y)}{x+y}\chi_p(y)\,dy=\lambda_p\chi_p(x), \quad (4.58)$$

where

$$\chi_p(x)=(2\lambda_p)^{-1/2}\int_1^\infty d\xi\exp[-(\xi-1)x/2]\phi_{p,0}(\xi) \quad (4.59)$$

and $\phi_{p,0}(\xi)$ is the $\nu=0$ case of (4.54) and λ_p is given by (4.53).

Proof: Consider the integral equation

$$\int_0^\infty \frac{\exp[-(u+v)]}{u+v} g(v)\, dv = \lambda g(u). \tag{4.60}$$

We can write this as

$$\lambda g(u) = \int_0^\infty dv\, g(v) \int_1^\infty d\xi' \exp[-\xi'(u+v)]. \tag{4.61}$$

If we multiply both sides of this equation by $\exp(-\xi u)$, and if we integrate the result over u from zero to infinity, then (4.61) becomes

$$\lambda G(\xi) = \int_1^\infty d\xi' \frac{1}{\xi+\xi'} G(\xi'), \tag{4.62}$$

where $G(\xi)$ is the Laplace transform of $g(u)$, i.e.,

$$G(\xi) = \int_0^\infty \exp(-\xi u)\, g(u)\, du. \tag{4.63}$$

From Lemma 4.3,

$$G(\xi) = \phi_{p,0}(\xi)$$

and

$$\lambda = \lambda_p = \pi \operatorname{sech} \pi p. \tag{4.64}$$

From (4.61) and (4.63),

$$\begin{aligned}\lambda g(u) &= \int_1^\infty d\xi' \exp(-\xi' u) G(\xi') \\ &= \int_1^\infty d\xi' \exp(-\xi' u) \phi_{p,0}(\xi').\end{aligned} \tag{4.65}$$

Letting $f(x) = \exp(x/2) g(u)$, $x = 2u$ we see that $f(x)$ satisfies (4.58). The overall constant in (4.59) has been chosen so that

$$\int_0^\infty \exp(-x) \chi_p(x) \chi_{p'}(x)\, dx = \delta(p - p'). \tag{4.66}$$

From Lemma 4.4 and (4.4) it follows that

$$B_n^{(2)}(\epsilon) = \frac{4}{n} \int_0^{\epsilon^{-1}} d\zeta \int_0^\infty dp\, \lambda_p^{n-1} \left| (\exp(-\zeta x), \chi_p(x)) \right|^2, \tag{4.67}$$

where

$$\begin{aligned}(\exp(-\zeta x), \chi_p(x)) &= \int_0^\infty \exp[-(\zeta+1)x]\, \chi_p(x)\, dx \\ &= (2\lambda_p)^{-1/2} \int_0^\infty d\xi\, \phi_{p,0}(\xi) \\ &\quad \times \int_0^\infty dx \exp[-(\zeta + \xi/2 + \tfrac{1}{2})x] \\ &= (2\lambda_p)^{-1/2} \int_1^\infty d\xi \frac{\phi_{p,0}(\xi)}{\zeta + \xi/2 + \frac{1}{2}} \\ &= (2\lambda_p)^{1/2} \phi_{p,0}(2\zeta+1).\end{aligned} \tag{4.68}$$

Thus

$$B_n^{(2)}(\epsilon) = \frac{8}{n} \int_0^{\epsilon^{-1}} d\zeta \int_0^\infty dp \left(\frac{\pi}{\cosh \pi p}\right)^n \left| \phi_{p,0}(2\zeta+1) \right|^2. \tag{4.69}$$

Using (4.57) for $\nu = 0$ we have

$$B_n^{(2)}(\epsilon) = \frac{8}{n} \int_0^{\epsilon^{-1}} d\zeta \int_0^\infty dp \left(\frac{\pi}{\cosh \pi p}\right)^n \times p \tanh p\pi [P_{-1/2+ip}(1+2\zeta)]^2. \tag{4.70}$$

From Lemma 4.3 and (4.49) it follows that

$$B_n^{(4)}(\epsilon) = \frac{2}{n} \int_\epsilon^\infty d\xi \int_0^\infty dp \left(\frac{\pi}{\cosh \pi p}\right)^{n-1} \times \left| (\exp(-\xi x), \phi_{p,\nu-1/2}(x)) \right|^2 \tag{4.71}$$

and

$$B_n^{(5)}(\epsilon) = \frac{2}{n} \int_\epsilon^\infty d\xi \int_0^\infty dp \left(\frac{\pi}{\cosh \pi p}\right)^{n-1} \times \left| (\exp(-\xi x), \phi_{p,\nu+1/2}(x)) \right|^2, \tag{4.72}$$

where the scalar product in (4.71) and (4.72) is

$$(\exp(-\xi x), \phi_{p,\mu}(x)) = \int_1^\infty dx \left(\frac{x-1}{x+1}\right)^\mu \exp(-\xi x) \phi_{p,\mu}(x). \tag{4.73}$$

Thus to prove Theorem 3 we need to compute the integrals (4.70)–(4.72).

F. $B_n^{(2)}(\epsilon)$

In (4.70) we do the ζ integration first. Now

$$\int_0^{\epsilon^{-1}} [P_{-1/2+ip}(1+2\zeta)]^2 d\zeta = \tfrac{1}{2} \int_1^\Lambda [P_{-1/2+ip}(z)]^2 dz, \tag{4.74}$$

where $z = 1 + 2\zeta$ and $\Lambda = 1 + 2\epsilon^{-1}$. We are interested in computing (4.74) in the limit $\Lambda \to \infty$.

For any two Legendre functions w_ν and w_σ on the cut, we have[9]

$$\int_a^b w_\nu(Z) w_\sigma(Z)\, dZ = [(\nu-\sigma)(\nu+\sigma+1)]^{-1} \times \left[(1-Z^2)\left(w_\nu \frac{d}{dZ} w_\sigma - w_\sigma \frac{d}{dZ} w_\nu\right)\right]_a^b. \tag{4.75}$$

Letting $\nu = -\frac{1}{2} + ip$, $\sigma = -\frac{1}{2} + ip'$, then we have in particular

$$\begin{aligned}&\int_1^\Lambda dz\, P_{-1/2+ip}(z) P_{-1/2+ip'}(z) \\ &= (p^2 - p'^2)^{-1} (\Lambda^2 - 1)[P_{-1/2+ip}(\Lambda) P'_{-1/2+ip'}(\Lambda) \\ &\quad - P_{-1/2+ip'}(\Lambda) P'_{-1/2+ip}(\Lambda)].\end{aligned} \tag{4.76}$$

Writing

$$P_{-1/2+ip'}(z)(p-p')^{-1} = [P_{-1/2+ip'}(z) - P_{-1/2+ip}(z)](p-p')^{-1} + P_{-1/2+ip}(z)(p-p')^{-1}$$

in (4.76) we obtain

$$\int_1^\Lambda dz\, P_{-1/2+ip}(z) P_{-1/2+ip'}(z) = \frac{\Lambda^2 - 1}{p+p'} \left\{ P_{-1/2+ip}(\Lambda) \frac{\partial}{\partial \Lambda} \frac{P_{-1/2+ip'}(\Lambda) - P_{-1/2+ip}(\Lambda)}{p - p'} \right.$$

$$-\frac{\partial P_{-1/2+ip}(\Lambda)}{\partial\Lambda}\left[\frac{P_{-1/2+ip'}(\Lambda)-P_{-1/2+ip}(\Lambda)}{p-p'}\right]\Bigg\}. \qquad (4.77)$$

We now let $p'\to p$ in (4.77),

$$\int_1^\Lambda dz\,[P_{-1/2+ip}(z)]^2=\frac{\Lambda^2-1}{2p}\left(\frac{\partial P_{-1/2+ip}(\Lambda)}{\partial\Lambda}\frac{\partial P_{-1/2+ip}(\Lambda)}{\partial p}-P_{-1/2+ip}(\Lambda)\frac{\partial^2}{\partial p\,\partial\Lambda}P_{-1/2+ip}(\Lambda)\right). \qquad (4.78)$$

We need to compute the right-hand side of (4.78) to order $o(1)$ as $\Lambda\to\infty$. Using Eq. (3.2.9) of Ref. 6, one can show for $\Lambda\to\infty$ (along the positive real axis)

$$P_{-1/2+ip}(\Lambda)=(2/\pi)^{1/2}\,\mathrm{Re}\left(\frac{2^{ip}\Gamma(ip)}{\Gamma(\frac12+ip)}\Lambda^{-1/2+ip}\right)+O\left(\frac{1}{\Lambda}\right). \qquad (4.79)$$

We have also

$$\frac{\partial P_{-1/2+ip}(\Lambda)}{\partial\Lambda}=\frac{2^{-1/2+ip}\Gamma(ip)}{\Gamma(\frac12+ip)}\pi^{-1/2}(-\tfrac12+ip)\Lambda^{-3/2+ip}+\text{complex conj.}+O(\Lambda^{-2}), \qquad (4.80)$$

$$\frac{\partial P_{-1/2+ip}(\Lambda)}{\partial p}=[i\ln 2+i\psi(ip)-i\psi(\tfrac12+ip)+i\ln\Lambda]\times\frac{2^{-1/2+ip}\Gamma(ip)}{\pi^{1/2}\Gamma(\frac12+ip)}\Lambda^{-1/2+ip}+\text{complex conj.}+O(\Lambda^{-1}), \qquad (4.81)$$

and

$$\frac{\partial^2 P_{-1/2+ip}(\Lambda)}{\partial p\,\partial\Lambda}=i\frac{2^{-1/2+ip}\Gamma(ip)}{\pi^{1/2}\Gamma(\frac12+ip)}[(\ln 2+\psi(ip)-\psi(\tfrac12+ip)+\ln\Lambda)(-\tfrac12+ip)+1]\Lambda^{-3/2+ip}+\text{complex conj.}+O(\Lambda^{-2}), \qquad (4.82)$$

where $\psi(x)=(d/dx)\ln\Gamma(x)$ is the psi function.

Substituting (4.79)–(4.82) into (4.78) and using the relations $\Gamma(ip)\Gamma(-ip)=\pi p^{-1}\sinh^{-1}(\pi p)$ and $\Gamma(\frac12+ip)\times\Gamma(\frac12-ip)=\pi\,\mathrm{sech}(\pi p)$ the result

$$\int_1^\Lambda dz\,[P_{-1/2+ip}(z)]^2=(\pi p\tanh\pi p)^{-1}[\ln 2+\ln\Lambda+\mathrm{Re}\psi(ip)-\mathrm{Re}\psi(\tfrac12+ip)]+\frac{1}{2\pi p}\mathrm{Im}\left(\frac{\Gamma^2(ip)}{\Gamma^2(\frac12+ip)}(2\Lambda)^{2ip}\right)+o(1) \qquad (4.83)$$

follows.

Using (4.83) and (4.74) in (4.70) we obtain

$$B_n^{(2)}(\epsilon)=\sigma_n\ln\left(\frac{1}{\epsilon}\right)+\frac{4}{\pi n}\int_0^\infty dp\left(\frac{\pi}{\cosh\pi p}\right)^n\times[\ln 4+\mathrm{Re}\psi(ip)-\mathrm{Re}\psi(\tfrac12+ip)]-\frac{1}{n}\pi^n, \qquad (4.84)$$

where we used the result

$$\lim_{\Lambda\to\infty}\int_0^\infty dp\left(\frac{\pi}{\cosh\pi p}\right)^n\tanh\pi p\;\mathrm{Im}\left(\frac{\Gamma^2(ip)}{\Gamma^2(\frac12+ip)}(2\Lambda)^{2ip}\right)=-\tfrac12\pi^{n+1}. \qquad (4.85)$$

G. $B_n^{(4)}(\epsilon)$, $B_n^{(5)}(\epsilon)$, and B_n

The matrix element needed in (4.71) and (4.72) is [recall (4.57) and (4.73)]

$$(\exp(-\xi x),\phi_{p,\nu}(x))=C_{p,\nu}\Gamma(1+\nu)\int_1^\infty dx\left(\frac{x-1}{x+1}\right)^{\nu/2}\exp(-\xi x)P^{-\nu}_{-1/2+ip}(x). \qquad (4.86)$$

The integral (4.86) is known[7,10] and the result is

$$(\exp(-\xi x),\phi_{p,\nu}(x))=C_{p,\nu}\Gamma(1+\nu)\xi^{-1}W_{-\nu,ip}(2\xi), \qquad (4.87)$$

where $W_{k,\mu}(x)$ is a Whittaker function.[11] Using (4.87) in (4.71) and (4.72) we have

$$B_n^{(4)}(\epsilon)+B_n^{(5)}(\epsilon)=\frac{2}{n}\int_\epsilon^\infty d\xi\int_0^\infty dp\left(\frac{\pi}{\cosh\pi p}\right)^{n-1}\frac{p\sinh\pi p}{\pi}\times\xi^{-2}[\Gamma(\nu+1+ip)\Gamma(\nu+1-ip)(W_{-1/2-\nu,ip}(2\xi))^2+\Gamma(\nu+ip)\Gamma(\nu-ip)(W_{1/2-\nu,ip}(2\xi))^2]. \qquad (4.88)$$

We first examine the ξ integration. Define

$$F_1(\epsilon)=\int_\epsilon^\infty d\xi\,\xi^{-2}[W_{1/2-\nu,ip}(2\xi)]^2 \qquad (4.89)$$

and

$$F_2(\epsilon)=\int_\epsilon^\infty d\xi\,\xi^{-2}[W_{-1/2-\nu,ip}(2\xi)]^2. \qquad (4.90)$$

Let $\hat F_{1,2}(Z)$ be the respective Mellin transforms, i.e.,

$$\hat F_{1,2}(Z)=\int_0^\infty \epsilon^{Z-1}F_{1,2}(\epsilon)\,d\epsilon,\quad \mathrm{Re}Z>0. \qquad (4.91)$$

Using (4.89) and (4.90) in (4.91) and interchanging the orders of integration so that the ϵ integration can be trivially performed, we obtain

$$\hat F_1(Z)=Z^{-1}\int_0^\infty d\xi\,\xi^{Z-2}[W_{1/2-\nu,ip}(2\xi)]^2=Z^{-1}2^{-Z+1}\int_0^\infty d\xi\,\xi^{Z-2}[W_{1/2-\nu,ip}(\xi)]^2 \qquad (4.92)$$

and

$$\hat F_2(Z)=Z^{-1}2^{-Z+1}\int_0^\infty d\xi\,\xi^{Z-2}[W_{-1/2-\nu,ip}(\xi)]^2. \qquad (4.93)$$

The integrals appearing in (4.92) and (4.93) are known[12] and we have for $\mathrm{Re}Z>0$

$$\int_0^\infty \xi^{Z-2}[W_{1/2-\nu,ip}(\xi)]^2\,d\xi=\frac{\Gamma(Z+2ip)\Gamma(Z)\Gamma(-2ip)}{\Gamma(\nu-ip)\Gamma(\nu+ip+Z)}\,{}_3F_2(2ip+Z,Z,\nu+ip;1+2ip,\nu+ip+Z;1)+\frac{\Gamma(Z-2ip)\Gamma(Z)\Gamma(2ip)}{\Gamma(\nu+ip)\Gamma(\nu-ip+Z)}\times{}_3F_2(Z,Z-2ip,\nu-ip;1-2ip,\nu-ip+Z;1), \qquad (4.94)$$

where ${}_3F_2(a_1,a_2,a_3;b_1,b_2;Z)$ is a generalized hypergeo-

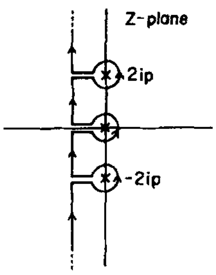

FIG. 33. Contour C used in (4.95b).

metric function.[13] From (4.92)–(4.94) we see that $\hat{F}_{1,2}(Z)$ has poles on the line $\mathrm{Re}Z=0$ at $Z=\pm 2ip$ and $Z=0$. To compute the small-ϵ behavior of $F_{1,2}(\epsilon)$ to order $o(1)$ it is sufficient to study the behavior of $\hat{F}_{1,2}(Z)$ on the line $\mathrm{Re}Z=0$ since

$$F_{1,2}(\epsilon)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\epsilon^{-Z}\hat{F}_{1,2}(Z)\,dZ \tag{4.95a}$$

$$=\frac{1}{2\pi i}\int_C \epsilon^{-Z}\hat{F}_{1,2}(Z)\,dZ, \tag{4.95b}$$

where (4.95a) is the Mellin inversion formula and the countour C in (4.95b) is shown in Fig. 33. The integral along the straight line lying in the $\mathrm{Re}Z<0$ plane is $o(1)$ as $\epsilon\to 0$.

We now examine $\hat{F}_{1,2}(Z)$ at $Z=\pm 2ip$ and $Z=0$. We first expand (4.94) about $Z=0$. For $p>0$

$$\frac{\Gamma(Z+2ip)\Gamma(Z)\Gamma(-2ip)}{\Gamma(\nu-ip)\Gamma(\nu+ip+Z)}=Z^{-1}\frac{\Gamma(2ip)\Gamma(-2ip)}{\Gamma(\nu+ip)\Gamma(\nu-ip)}\{1+Z[\psi(2ip)-\gamma-\psi(\nu+ip)]+O(Z^2)\}, \tag{4.96}$$

where $\gamma=0.5772\cdots$ in Euler's constant. By definition

$${}_3F_2(2ip+Z,Z,\nu+ip;1+2ip,\nu+ip+Z;1)$$

$$=\sum_{n=0}^{\infty}\frac{(2ip+Z)_n(Z)_n(\nu+ip)_n}{(1+2ip)_n(\nu+ip+Z)_n n!}, \tag{4.97}$$

where $(a)_n\equiv\Gamma(a+n)/\Gamma(a)$. Expanding (4.97) about $Z=0$ we have for $p>0$

$${}_3F_2(2ip+Z,Z,\nu+ip;1+2ip,\nu+ip+Z;1)$$

$$=1+Z\sum_{n=1}^{\infty}\frac{2ip}{n(2ip+n)}+O(Z^2)$$

$$=1+Z[\psi(1+2ip)+\gamma]+O(Z^2). \tag{4.98}$$

Thus we have shown that for $Z\to 0$

$$\int_0^{\infty}\xi^{Z-2}[W_{1/2-\nu,ip}(\xi)]^2\,d\xi$$

$$=2Z^{-1}\frac{\Gamma(-2ip)\Gamma(2ip)}{\Gamma(\nu-ip)\Gamma(\nu+ip)}\{1+Z[\mathrm{Re}\psi(2ip)-\mathrm{Re}\psi(\nu+ip)+\mathrm{Re}\psi(1+2ip)]+O(Z^2)\}, \tag{4.99}$$

and hence for $Z\to 0$

$$\hat{F}_1(Z)=f_{-2}Z^{-2}+f_{-1}Z^{-1}+O(1), \tag{4.100a}$$

where

$$f_{-2}=\pi[\Gamma(\nu+ip)\Gamma(\nu-ip)\,p\cosh(\pi p)\sinh(\pi p)]^{-1} \tag{4.100b}$$

and

$$f_{-1}=f_{-2}[2\,\mathrm{Re}\psi(2ip)-\ln 2-\mathrm{Re}\psi(\nu+ip)]. \tag{4.100c}$$

To obtain the Laurent expansion of $\hat{F}_2(Z)$ about $Z=0$, one replaces ν by $\nu+1$ in (4.100).

From (4.92), (4.94), and (4.97) we have for $Z\to\pm 2ip$

$$\hat{F}_1(Z)=\pm\frac{1}{2ip}2^{\mp 2ip+1}\left[\frac{\Gamma(\pm 2ip)}{\Gamma(\nu\pm ip)}\right]^2\frac{1}{Z\mp 2ip}+O(1). \tag{4.101}$$

Using (4.100) and (4.101) in (4.95b) and recalling (4.88)–(4.90) we have for $\epsilon\to 0^+$

$$B_n^{(4)}(\epsilon)+B_n^{(5)}(\epsilon)$$

$$=\sigma_n\ln\left(\frac{1}{\epsilon}\right)-\sigma_n\ln 2+\frac{4}{n\pi}\int_0^{\infty}dp\left(\frac{\pi}{\cosh\pi p}\right)^n$$

$$\times[2\,\mathrm{Re}\psi(2ip)-\tfrac{1}{2}\mathrm{Re}\psi(\nu+ip)-\tfrac{1}{2}\mathrm{Re}\psi(\nu+1+ip)]$$

$$+\lim_{\epsilon\to 0}\frac{4}{n\pi}\int_0^{\infty}dp\left(\frac{\pi}{\cosh\pi p}\right)^{n-1}\sinh\pi p\left\{\mathrm{Im}\left[2^{-2ip}\left(\frac{\Gamma(2ip)}{\Gamma(\nu+ip)}\right)^2\exp(-2ip\ln\epsilon)\right]\Gamma(\nu+ip)\,\Gamma(\nu-ip)\right.$$

$$\left.+\mathrm{Im}\left[2^{-2ip}\left(\frac{\Gamma(2ip)}{\Gamma(\nu+1+ip)}\right)^2\times\exp(-2ip\ln\epsilon)\right]\Gamma(\nu+1+ip)\,\Gamma(\nu+1-ip)\right\}. \tag{4.102}$$

In deriving (4.102) we made the identification [see (4.31) and (4.36)]

$$\sigma_n=\frac{4}{\pi n}\int_0^{\infty}dp\left(\frac{\pi}{\cosh\pi p}\right)^n. \tag{4.103}$$

The only nonzero contribution in the limit $\epsilon\to 0$ to the last integral in (4.102) is in the region $p\sim 0$. A computation shows this integral is $-(1/n)\pi^n$.

We now use (4.10), (4.41), (4.51), (4.84), and (4.102) to obtain [note that the $\ln(1/\epsilon)$ terms cancel]

$$B_n=3\ln 2\sigma_n-\frac{2}{n}\pi^n+\frac{4}{n\pi}\int_0^{\infty}dp\left(\frac{\pi}{\cosh\pi p}\right)^n\times[2\,\mathrm{Re}\psi(ip)-\tfrac{1}{2}\mathrm{Re}\psi(\nu+ip)-\tfrac{1}{2}\mathrm{Re}\psi(\nu+1+ip)], \tag{4.104}$$

where we used the functional equation[14]

$$\psi(2\psi)=\tfrac{1}{2}\psi(x)+\tfrac{1}{2}\psi(x+\tfrac{1}{2})+\ln 2.$$

H. $B(\sigma,\nu)$

To complete the proof of Theorem 3 we must compute the sum (4.25) where we have shown that the coefficients are given by (4.104). Rather than regard B as a function of λ and ν, it will prove more natural to think of B as a function of σ and ν where $\sigma=\sigma(\lambda)=2\pi^{-1}\arcsin(\pi\lambda)$.

Then for $\sigma < 1$ ($\lambda < 1/\pi$) it follows from (4.104) that

$$-\ln B = \sum_{n=0}^{\infty} B_{2n+1}\lambda^{2n+1}$$

$$= 3\sigma\ln 2 - \ln\left(\frac{1+\sin\pi\sigma/2}{1-\sin\pi\sigma/2}\right) + I_1 + I_2 + I_3, \qquad (4.105a)$$

where

$$I_1 = \frac{4}{\pi}\int_0^{\infty} dp \ln\left(\frac{\cosh\pi p + \sin\pi\sigma/2}{\cosh\pi p - \sin\pi\sigma/2}\right)\mathrm{Re}\psi(ip), \qquad (4.105b)$$

$$I_2 = -\frac{2}{\pi}\int_0^{\infty} dp \ln\left(\frac{\cosh\pi p + \sin\pi\sigma/2}{\cosh\pi p - \sin\pi\sigma/2}\right)\mathrm{Re}\psi(\nu + ip), \quad (4.105c)$$

and

$$I_3 = -\frac{\nu}{\pi}\int_0^{\infty} dp \ln\left(\frac{\cosh\pi p + \sin\pi\sigma/2}{\cosh\pi p - \sin\pi\sigma/2}\right)(\nu^2 + p^2)^{-1}. \qquad (4.105d)$$

From (4.105b)

$$\frac{\partial I_1}{\partial\sigma} = 4\cos\pi\sigma/2\int_0^{\infty} dp \frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}\mathrm{Re}\psi(ip)$$

$$= 2\cos\pi\sigma/2\int_{-\infty}^{\infty} dp \frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}\mathrm{Re}\psi(-ip), \qquad (4.106)$$

where the second equality follows from the fact that $\mathrm{Re}\psi(ip)$ is an even function of p with no singularities on the real p axis. $\mathrm{Im}\psi(-ip)$ is an odd function of p with a pole with residue -1 at $p = 0$. Hence

$$2\cos\pi\sigma/2\int_{\Omega} dp \frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}\mathrm{Im}\psi(-ip)$$

$$= 2\pi i\cos\pi\sigma/2(1 - \sin^2\pi\sigma/2)^{-1}, \qquad (4.107)$$

where the contour of the integration Ω is the real p axis from $-\infty$ to $-\epsilon$, a semicircle lying in the upper half-plane centered at the origin with radius ϵ, and the real axis from $+\epsilon$ to $+\infty$. The limit $\epsilon \to 0^+$ is then understood. Multiplying (4.107) by $+i$ and adding the result to (4.106) we have

$$2\cos\pi\sigma/2\int_{\Omega} dp\, \psi(-ip)\frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}$$

$$= \frac{\partial I_1}{\partial\sigma} - 2\frac{d}{d\sigma}\ln\left(\frac{1+\sin\pi\sigma/2}{1-\sin\pi\sigma/2}\right). \qquad (4.108)$$

The integral

$$J_1 \equiv 2\cos\pi\sigma/2\int_{\Omega} dp\, \psi(-ip)\frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}, \qquad (4.109)$$

can be evaluated by applying Cauchy's theorem to

$$2\cos\pi\sigma/2\int_{C_R} dz\, \psi(-iz)\frac{\cosh\pi z}{\cosh^2\pi z - \sin^2\pi\sigma/2}, \quad (4.110)$$

where the contour C_R is shown in Fig. 34. Letting $R \to \infty$ in (4.110) results in

$$J_1 = \psi\left(\frac{1+\sigma}{2}\right) + \psi\left(\frac{1-\sigma}{2}\right) - \pi\cos\pi\sigma/2(1 - \sin^2\pi\sigma/2). \qquad (4.111)$$

Hence from (4.108), (4.109), and (4.111)

$$\frac{\partial I_1}{\partial\sigma} = \psi\left(\frac{1+\sigma}{2}\right) + \psi\left(\frac{1-\sigma}{2}\right) + \frac{d}{d\sigma}\ln\left(\frac{1+\sin^2\pi\sigma/2}{1-\sin^2\pi\sigma/2}\right). \qquad (4.112)$$

Since I_1 $(\sigma = 0) = 0$ it follows from (4.112) that

$$I_1 = 2\ln\Gamma\left(\frac{1+\sigma}{2}\right) - 2\ln\Gamma\left(\frac{1-\sigma}{2}\right) + \ln\left(\frac{1+\sin\pi\sigma/2}{1-\sin\pi\sigma/2}\right). \quad (4.113)$$

The evaluation of I_2 is similar. We have for $\nu > 0$

$$\frac{\partial I_2}{\partial\sigma} = -\cos\pi\sigma/2\int_{-\infty}^{\infty} dp\, \psi(\nu - ip)\frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}, \qquad (4.114)$$

since $\mathrm{Im}\psi(\nu - ip)$ is an odd function of p with no singularities on the real axis. We again use the contour of Fig. 34 (the semicircles are no longer necessary) with the result

$$\frac{\partial I_2}{\partial\sigma} = -\tfrac{1}{2}\psi\left(\frac{1+\sigma}{2} + \nu\right) - \tfrac{1}{2}\psi\left(\nu + \frac{1-\sigma}{2}\right) + \tfrac{1}{2}\cos\pi\sigma/2$$

$$\times\int_{-\infty}^{\infty}\frac{dp}{\nu + ip}\frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}. \qquad (4.115)$$

The integral appearing in (4.115) is unchanged if we replace $(\nu + ip)^{-1}$ by $(\nu - ip)^{-1}$. Hence

$$\frac{\partial I_2}{\partial\sigma} = -\tfrac{1}{2}\psi\left(\frac{1+\sigma}{2} + \nu\right) - \tfrac{1}{2}\psi\left(\nu + \frac{1-\sigma}{2}\right) + \frac{\nu}{2}\cos\frac{\pi}{2}\sigma$$

$$\times\int_{-\infty}^{\infty} dp\, (\nu^2 + p^2)^{-1}\frac{\cosh\pi p}{\cosh^2\pi p - \sin^2\pi\sigma/2}. \qquad (4.116)$$

Integrating (4.116) $[I_2(\sigma = 0) = 0]$ we have

$$I_2 = -\ln\left(\frac{1+\sigma}{2} + \nu\right) + \ln\Gamma\left(\frac{1-\sigma}{2} + \nu\right) - I_3. \qquad (4.117)$$

It follows from (4.105a), (4.113), and (4.117) that $B(\sigma, \nu)$ is given by (1.12). The small-t behavior (1.17) of the functions $g_{2n+1}(t;\nu)$ now follows from Theorem 3 and Eq. (3.5).

I. Small-t behavior of $\eta(t/2;\nu,\lambda)$ for $\lambda \geq \pi^{-1}$

As $\sigma \to 1$ ($\lambda \to \pi^{-1}$) we have from (1.12)

$$B(\sigma, \nu) = b_{-2}(1-\sigma)^{-2} + b_{-1}(1-\sigma)^{-1} + b_0 + O(1-\sigma) \qquad (4.118a)$$

with

$$b_{-2} = \tfrac{1}{2}\nu, \qquad (4.118b)$$

$$b_{-1} = \tfrac{3}{2}\nu\ln 2 - \tfrac{1}{2}\nu\psi(\nu + 1) - \gamma\nu + \tfrac{1}{4}, \qquad (4.118c)$$

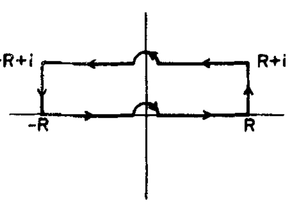

FIG. 34. Contour C_R used in (4.110) and (4.114).

and

$$b_0=\frac{9}{4}(\ln 2)^2\nu-\frac{3}{4}\nu\ln 2[\psi(\nu)+\psi(\nu+1)+4\gamma]$$
$$+\frac{1}{16}\nu\{[\psi(\nu)+\psi(\nu+1)]^2+\psi'(\nu+1)-\psi'(\nu)\}$$
$$-\frac{1}{2}\gamma+\gamma\nu\psi(\nu+1)+\gamma^2\nu. \qquad (4.118\text{d})$$

We now use (4.118) in (4.4) [also use (4.5)] and recall that there are no terms of the form $t^{n-n\sigma}$ for $n \geq 3$. The limit $\sigma \to 1$ exists with the result that for $t \to 0$ (along positive real axis)

$$\eta(t/2;\nu,\pi^{-1})\sim\frac{1}{2}t\{\nu\ln^2 t-C(\nu)\ln t+1/(4\nu)[C^2(\nu)-1]\}, \qquad (4.119)$$

where

$$C(\nu)=1+2\nu[3\ln 2-2\gamma-\psi(\nu+1)]. \qquad (4.120)$$

We note that $\lim_{\nu\to 0}(4\nu)^{-1}[C^2(\nu)-1]=3\ln 2-\gamma$.

The correction terms to (4.119) are most easily determined by using the differential equation (1.3). For example, for the special case $\nu=0$ we find[3,4]

$$\eta(\theta;0,\pi^{-1})=-\theta\Omega-\frac{\theta^5}{128}(8\Omega^3-8\Omega^2+4\Omega-1)+O(\theta^9\Omega^5), \qquad (4.121)$$

where $\Omega\equiv\ln(\theta/4)+\gamma$.

The case $\lambda>1/\pi$ can be similarly examined. We write for real positive μ

$$\lambda=(1/\pi)\cosh(\pi\mu), \qquad (4.122)$$

so that [see (1.10)]

$$\sigma=1+2i\mu. \qquad (4.123)$$

We examine here the case $\nu=0$. Then using (4.123) in (1.11) for $\nu=0$ we see that (4.4) becomes for $\mu>0$, $t\to 0^+$

$$\eta(t/2;0,\lambda)\sim\frac{1}{4\pi}t\sinh(\pi\mu)\,\mathrm{Im}\{\Gamma^2(-i\mu)\exp[2i\mu\ln(t/8)]\}. \qquad (4.124)$$

If we write

$$\Gamma(iy)=|\Gamma(iy)|\exp[i\phi(y)]$$
$$=\left[\frac{\pi}{y\sinh\pi y}\right]^{1/2}\exp[i\phi(y)], \qquad (4.125)$$

then (4.124) becomes ($t\to 0$, $\mu>0$)

$$\eta(t/2;0,\lambda)\sim-\frac{1}{4\mu}t\sin[2\mu\ln(t/8)+2\phi(\mu)]. \qquad (4.126)$$

Thus for $\lambda>\pi^{-1}$ there are an infinite number of zeros of the function $\eta(t/2;0,\lambda)$ lying on the positive t axis with $t=0$ being a limit point of these zeros. The asymptotic spacing of these zeros follows from (4.126). The correction terms to (4.124) [or (4.126)] can be found from the differential equation.

The case $\lambda<0$ can also be studied. From (1.4) and (1.5) it follows that

$$\eta(t/2;\nu,-\lambda)=\frac{1}{\eta(t/2;\nu,\lambda)}. \qquad (4.127)$$

Hence we see that for $\lambda<-\pi^{-1}$, $\eta(t/2;\nu,\lambda)$ has an infinite number of poles clustering to zero on the positive t axis.

V. THEOREM 4

We commence the proof of Theorem 4 by using (1.4a) to rewrite the left-hand side of (1.14a) in terms of G as

$$[1-G^2(t)]^{-1/2}\exp\int_t^\infty dt'\left\{t'\frac{[G^2(t')-G'^2(t')]}{[1-G^2(t')]^2}+\frac{2\nu}{1-G^2(t')}\right\}, \qquad (5.1)$$

where $G'(t)=(d/dt)G(t)$. The first factor may be written in the form

$$[1-G^2(t)]^{-1/2}=\exp\left(-\frac{1}{2}\int_t^\infty dt\,\frac{2G(t')G'(t')}{[1-G^2(t')]}\right) \qquad (5.2)$$

and therefore Theorem 4 is established if we can demonstrate

$$-\sum_{n=1}^\infty\lambda^{2n}f_{2n}(t;\nu)=\int_t^\infty dt'\left\{\frac{t'[G^2(t')-G'^2(t')]}{[1-G^2(t')]^2}-\frac{G(t)G'(t')}{[1-G^2(t')]}+\frac{2\nu}{1-G^2(t')}\right\}. \qquad (5.3)$$

Furthermore, because $f_{2n}(t;\nu)$ and $G^2(t)$ vanish exponentially rapidly as $t\to\infty$ (5.3) will be demonstrated if we can show

$$\sum_{n=1}^\infty\lambda^{2n}f'_{2n}(t;\nu)=\frac{t[G^2(t)-G'^2(t)]}{[1-G^2(t)]^2}-\frac{G(t)G'(t)}{[1-G^2(t)]}+\frac{2\nu}{1-G^2(t)} \qquad (5.4)$$

or, using the differential equation for $G(t)$ (2.7)

$$(1-G^2)^2\sum_{n=1}^\infty\lambda^{2n}f''_{2n}(t;\nu)=G^2(1-G'^2)-GG''(1-G^2). \qquad (5.5)$$

Here, all factors of t have been removed by use of the differential equation.

To demonstrate (5.5) we first define in analogy to g_{2k+1} (1.6b)

$$h_{2k+1}=(-1)^{k+1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2k+1}\prod_{j=1}^{2k+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}$$
$$\times\left(\frac{y_j-1}{y_j+1}\right)^\nu\prod_{j=1}^{2k}\frac{1}{y_j+y_{j+1}}\prod_{j=1}^{k+1}(y_{2j-1}^2-1). \qquad (5.6)$$

Then it is seen from the definition (1.13) that

$$f''_{2n}=\sum_{l=1}^n g_{2(n-l)+1}h_{2l-1}. \qquad (5.7)$$

Thus, if we define

$$H(t;\nu,\lambda)=\sum_{n=1}^\infty\lambda^{2n+1}h_{2n+1}, \qquad (5.8)$$

(5.5) reduces to

$$(1-G^2)^2H=G(1-G'^2)-G''(1-G^2) \qquad (5.9)$$

and therefore our theorem will be proven if we can demonstrate that

$$H-(G-G'')=G^2G''-GG'^2+2G^2H-G^4H. \qquad (5.10)$$

The coefficient of λ^{2n+1} of the left-hand side of (5.10) is

$$h_{2n+1}-(g_{2n+1}-g''_{2n+1})$$

$$=(-1)^{n+1}\int_1^\infty dy_1\cdots\int_1^\infty dy_{2n+1}\prod_{j=1}^{2n+1}\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}$$

$$\times\left(\frac{y_j-1}{y_j+1}\right)^\nu\prod_{j=1}^{2n}\frac{1}{y_j+y_{j+1}}\left\{\prod_{j=1}^{n+1}(y_{2j-1}^2-1)\right.$$

$$\left.+\prod_{j=1}^{n}(y_{2j}^2-1)\left[1-\left(\sum_{j=1}^{2n+1}y_j\right)^2\right]\right\}. \qquad (5.11)$$

Rewrite the term involving $(\sum y_j)^2$, using

$$\sum_{j=1}^{n}(y_{2j-1}+y_{2j})\sum_{k=j}^{n}(y_{2k}+y_{2k+1})$$

$$=\frac{1}{2}\left\{\left(\sum_{j=1}^{2n+1}y_j\right)^2+\sum_{l=1}^{n}y_{2l}^2-\sum_{l=1}^{n+1}y_{2l-1}^2\right\} \qquad (5.12)$$

to obtain for the term in brackets

$$\prod_{j=1}^{n+1}(y_{2j-1}^2-1)+\prod_{j=1}^{n}(y_{2j}^2-1)\left[1-\left(\sum_{j=1}^{2n+1}y_j\right)^2\right]$$

$$=\prod_{j=1}^{n+1}(y_{2j-1}^2-1)+\prod_{j=1}^{n}(y_{2j}^2-1)\left[\sum_{l=1}^{n}(y_{2l}^2-1)\right.$$

$$\left.-\sum_{l=1}^{n+1}(y_{2l-1}^2-1)-2\sum_{j=1}^{n}(y_{2j-1}+y_{2j})\sum_{k=j}^{n}(y_{2k}+y_{2k+1})\right]. \qquad (5.13)$$

Then use the identity

$$X_1X_3\cdots X_{2n+1}+X_2X_4\cdots X_{2n}\left[\sum_{l=1}^{n}X_{2l}-\sum_{l=1}^{n+1}X_{2l-1}\right]$$

$$=-\sum_{j=1}^{n}(X_{2j-1}-X_{2j})\sum_{k=j}^{n}(X_{2k}-X_{2k+1})$$

$$\times\prod_{l_1=1}^{j-1}X_{2l_1}\prod_{l_2=j}^{k-1}X_{2l_2+1}\prod_{l_3=k+1}^{n}X_{2l_3} \qquad (5.14)$$

with

$$X_l=y_l^2-1 \qquad (5.15)$$

to obtain

$$h_{2n+1}-(g_{2n+1}-g''_{2n+1})=(-)^{n+1}\int_1^\infty dy_1\cdots dy_{2n+1}$$

$$\times\prod_{j=1}^{2n+1}\left[\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right]\prod_{j=1}^{2n}\frac{1}{y_j+y_{j+1}}$$

$$\times\sum_{j=1}^{n}(y_{2j-1}+y_{2j})\sum_{k=j}^{n}(y_{2k}+y_{2k+1})\left\{-2\prod_{l=1}^{n}(y_{2l}^2-1)\right.$$

$$-(y_{2j-1}-y_{2j})(y_{2k}-y_{2k+1})$$

$$\left.\times\prod_{l_1=1}^{j-1}(y_{2l_1}^2-1)\prod_{l_2=j}^{k-1}(y_{2l_2+1}^2-1)\prod_{l_3=k+1}^{n}(y_{2l_3}^2-1)\right\}. \qquad (5.16)$$

The first term in brackets in (5.16) gives the term $2G^2H$ in (5.10) and we note that after we expand the product

$$(y_{2j-1}-y_{2j})(y_{2k}-y_{2k+1})$$

$$=y_{2j-1}(y_{2k}-y_{2k+1})-y_{2j}y_{2k}+y_{2j}y_{2k+1} \qquad (5.17)$$

that the first term on the right-hand side vanishes when used in the double sum over j and k in (5.16). Moreover we may rewrite $y_{2j}y_{2k}$ using

$$\left[\sum_{l=2j}^{2k}y_l\right]^2$$

$$=y_{2j}y_{2k}+y_{2j}\sum_{l=j}^{k-1}(y_{2l}+y_{2l+1})+y_{2k}\sum_{l=j+1}^{k}(y_{2l-1}+y_{2l})$$

$$+\sum_{l_1=j}^{n}(y_{2l_1}+y_{2l_1+1})\sum_{l_2=l_1+1}^{k}(y_{2l_2-1}+y_{2l_2})$$

$$+\sum_{l_1=j}^{n}(y_{2l_1}+y_{2l_1+1})\sum_{l_2=j}^{l_1}(y_{2l_2-1}+y_{2l_2}) \qquad (5.18)$$

and obtain

$$h_{2n+1}-(g_{2n+1}-g''_{2n+1})=(-1)^{n+1}\int_1^\infty dy_1\cdots dy_{2n+1}$$

$$\times\prod_{j=1}^{2n+1}\left[\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right]\prod_{j=1}^{2n}\frac{1}{y_j+y_{j+1}}$$

$$\times\sum_{j=1}^{n}(y_{2j-1}+y_{2j})\sum_{k=j}^{n}(y_{2k}+y_{2k+1})\left\{-2\prod_{l=1}^{n}(y_{2l}^2-1)\right.$$

$$+\prod_{l_1=1}^{j-1}(y_{2l_1}^2-1)\prod_{l_2=j}^{k-1}(y_{2l_2+1}^2-1)\prod_{l_3=k+1}^{n}(y_{2l_3}^2-1)$$

$$\times\left[-y_{2j}y_{2k+1}+\left(\sum_{l=2j}^{2k}y_l\right)^2-y_{2j}\sum_{l=j}^{k-1}(y_{2l}+y_{2l+1})-y_{2k}\right.$$

$$\times\sum_{l=j+1}^{k}(y_{2l-1}+y_{2l})-\sum_{l_1=j}^{n}(y_{2l_1}+y_{2l_1+1})\sum_{l_2=l_1+1}^{k}(y_{2l_2-1}+y_{2l_2})$$

$$\left.\left.-\sum_{l_1=j}^{n}(y_{2l_1}+y_{2l_1+1})\sum_{l_2=j}^{l_1}(y_{2l_2-1}+y_{2l_2})\right]\right\}. \qquad (5.19)$$

We note that the terms involving $y_{2j}\sum_{l=j}^{k-1}(y_{2l}+y_{2l+1})$ and $y_{2k}\sum_{l=j+1}^{k}(y_{2l-1}+y_{2l})$ are equal and we eliminate y_{2j} and y_{2k+1} using

$$y_{2j}=\sum_{l=2j}^{2k}y_l-\sum_{l=j}^{k}(y_{2l+1}+y_{2l}). \qquad (5.20a)$$

and

$$y_{2k+1}=\sum_{l=2k+1}^{2n+1}y_l-\sum_{l=k+1}^{n}(y_{2l}+y_{2l+1}). \qquad (5.20b)$$

Therefore, upon combining terms we find that

$$h_{2n+1}-(g_{2n+1}-g''_{2n+1})=\int_1^\infty dy\cdots dy_{2n+1}$$

$$\times\prod_{j=1}^{2n+1}\left[\frac{\exp(-ty_j)}{(y_j^2-1)^{1/2}}\left(\frac{y_j-1}{y_j+1}\right)^\nu\right]\prod_{j=1}^{2n}\frac{1}{y_j+y_{j+1}}$$

$$\times\sum_{j=1}^{n}(-1)^{j-1}(y_{2j-1}+y_{2j})\sum_{k=j}^{n}(-1)^{n-k}(y_{2k}+y_{2k+1})$$

$$\times\left\{2(-1)^{k-j+1}\prod_{l=1}^{n}(y_{2l}^2-1)+\prod_{l_1=1}^{j-1}(y_{2l_1}^2-1)\prod_{l_2=j}^{k-1}(y_{2l_2+1}^2-1)\right.$$

$$\times\prod_{l_3=k+1}^{n}(y_{2l_3}^2-1)\left[(-1)^{k-j}\left(\sum_{l=2j}^{2k}y_l\right)^2-(-1)^{k-j}\sum_{l_1=2j}^{2k}y_{l_1}\right.$$

$$\times\sum_{l_2=2k+1}^{2n}y_{l_2}-\sum_{l_1=j}^{n}(-1)^{l_1-j}(y_{2l_1}+y_{2l_1+1})$$

$$\times \sum_{l_2=l_1+1}^{k} (-1)^{l_2-l_1}(y_{2l_2-1}+y_{2l_2})(-1)^{k-l_2}\Big]\Big\}, \qquad (5.21)$$

from which (5.10) follows. Hence, Theorem 4 is established.

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